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SOME INEQUALITIES RELATED TO η-CONVEX FUNCTIONS

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ABSTRACT. We introduce the concept of η-convex functions as generalization of convex functions. Some basic inequalities related to η-convex functions are proved. Also we investigate the famous Hermite-Hadamard, Fejer, Jensen and Slater type inequalities for this class of functions. Furthermore some inequalities related to differentiable η-convex functions are obtained as well.

1. INTRODUCTION

Almost no mathematician in applied mathematics, especially in nonlinear programing and optimization theory, can ignore the significant role of convex sets and convex functions. Furthermore, the elegance in shape and property of convex sets and functions makes it attractive to study this branch of mathematical analysis. On the other hand it should be noticed that in new problems related to convexity, generalized notions for convex sets and functions are required to reach favorite and applicable results. In the last 60 years many efforts have gone on generalization of notion of convexity. In our opinion the following classification in generalization of convex functions holds:

(1) Works that change the form of defining convex functions to a generalized form such as quasi-convex [5], pseudo-convex [16], strongly convex [19], logarithmically convex [18], approximately convex [11], delta-convex [20], h-convex [24], φ-convex [6], λ-convex [7], midconvex functions [13] and others [1], [17], [22], [3] etc.

(2) Works that extend the domain set of convex functions such as E-convex functions [25], α-convex functions, all works on convex functions from $\mathbb{R}^n$ to $\mathbb{R}$ [4], invex functions [10] etc.

(3) Works that extend the range set of convex functions such as works on functions with range in vector spaces [12], all kind of multivalued convex functions [2, 14, 15, 26] etc.

Motivated by works done in (1) we introduce in this paper the concept of η-*convex functions* as generalization of convex functions. First we show some basic results as inequalities related to η-convex functions. Second we investigate the famous Hermite-Hadamard, Fejer, Jensen and Slater type inequalities for this class of functions. Finally some inequalities related to differentiable η-convex functions are obtained as well.

2. BASIC RESULTS

Through this paper let I be an interval in real line $\mathbb{R}$. Also consider $\eta : A\times A \to B$ for appropriate $A, B \subseteq \mathbb{R}$.

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Definition 1. *A function $f : I \to \mathbb{R}$ is called convex with respect to η (briefly η-convex), if*

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y)),$$

for all $x, y \in I$ and $t \in [0, 1]$.

In the above definition if we set $\eta(x, y) = x - y$, then we recapture the classic definition of a convex function.

Example 1. *a. Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} -x, & x \geq 0; \\ x, & x < 0. \end{cases}$$

and define a bifunction η as $\eta(x, y) = -x - y$, for all $x, y \in \mathbb{R}^- = (-\infty, 0]$.

It is not hard to check that f is a η-convex function but not a convex one.

b. Define the function $f : \mathbb{R} \to \mathbb{R}^+$ by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ 1, & x > 1. \end{cases}$$

and define

$$\eta(x, y) = \begin{cases} x + y, & x \leq y; \\ 2(x + y), & x > y. \end{cases}$$

for all $x, y \in \mathbb{R}^+ = [0, +\infty)$. Then f is η-convex but is not convex.

The following results are η-convex versions of some basic theorems and propositions related to convex functions.

Proposition 1. *If $f : [a, b] \to \mathbb{R}$ is η-convex, then*

$$\max_{x \in [a,b]} f(x) \leq \max\{f(b), f(b) + \eta(f(a), f(b))\}.$$

Proof. For any $x \in [a, b]$ we have $x = ta + (1-t)b$ for some $t \in [0, 1]$, which implies that

$$f(x) \leq f(b) + t\eta(f(a), f(b)) \leq \max\{f(b), f(b) + \eta(f(a), f(b))\}.$$

Since x is arbitrary, so

$$\max_{x \in [a,b]} f(x) \leq \max\{f(b), f(b) + \eta(f(a), f(b))\}$$

and the statement is proved. □

Example 2. *In Example 1 a., consider $a = 2, b = -2$. Then*

$$0 = \max_{x \in [-2,2]} f(x) \leq \max\{-2, -2 + \eta(-2, -2)\} = \max\{-2, -2 + 4\} = 2.$$

In Example 1 b., consider $a = -1, b = 2$. Then

$$1 = \max_{x \in [0,2]} f(x) \leq \max\{1, 1 + \eta(0, 1)\} = \max\{1, 2\} = 2.$$

Definition 2. [21] *A function $f : I \to \mathbb{R}$ has a local minimum at $x_0 \in I$, if there is a neighborhood $N_r(x_0) \subset I$ such that $f(x_0) \leq f(x)$ for all $x \in N_r(x_0)$.*

We have:

Proposition 2. *If $f : I \to \mathbb{R}$ is η-convex and attains a local minimum at $x_0 \in I$, then $\eta(f(x), f(x_0)) \geq 0$, for any $x \in I$.*

Proof. Suppose that f has a local minimum at $x_0 \in I$. For any $x \in I$ we can find $t > 0$ sufficiently small such that $tx + (1-t)x_0 \in N_r(x_0)$. So we reach to the conclusion by the following inequality:

$$f(x_0) \leq f(tx + (1-t)x_0) \leq f(x_0) + t\eta(f(x), f(x_0)).$$

□

The following characterization of η-convexity holds:

Theorem 1. *A function $f : I \to \mathbb{R}$ is η-convex if and only if for any $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$,*

$$\det \begin{pmatrix} 1 & x_1 & \eta(f(x_1), f(x_3)) \\ 1 & x_2 & f(x_2) - f(x_3) \\ 1 & x_3 & 0 \end{pmatrix} \geq 0 \tag{2.1}$$

and

$$f(x_1) \leq f(x_3) + \eta(f(x_1), f(x_3)). \tag{2.2}$$

Proof. Suppose that f is a η-convex function. Consider arbitrary $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$. So there is a $t \in (0,1)$ such that $x_2 = tx_1 + (1-t)x_3$, namely $t = \frac{x_2 - x_3}{x_1 - x_3}$. From η-convexity of f we have

$$f(x_2) \leq f(x_3) + \frac{x_2 - x_3}{x_1 - x_3}\eta(f(x_1), f(x_3))$$

or

$$(x_3 - x_1)[f(x_3) - f(x_2)] + (x_3 - x_2)\eta(f(x_1), f(x_3)) \geq 0,$$

which is equivalent to above determinant being nonnegative.

Also for $t = 1$,

$$f(x_1) \leq f(x_3) + \eta(f(x_1), f(x_3))$$

and for $t = 0$,

$$f(x_3) \leq f(x_3).$$

For the inverse implications, consider $x, y \in I$ with $x < y$. Choosing any $t \in (0,1)$ we have $x < tx + (1-t)y < y$ and so

$$\det \begin{pmatrix} 1 & x & \eta(f(x), f(y)) \\ 1 & tx + (1-t)y & f(tx + (1-t)y) - f(y) \\ 1 & y & 0 \end{pmatrix} \geq 0.$$

By expanding this determinant we reach to the inequality

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y))$$

for any $t \in (0,1)$.

From assumption we have

$$f(x) \leq f(y) + \eta(f(x), f(y))$$

that gives η-convexity for $t = 1$. Also $f(y) \leq f(y)$ gives η-convexity of f for $t = 0$. □

Theorem 2. *For a function $f : I \to \mathbb{R}$ the following assertions are equivalent.*
(a) f is η-convex function;
(b) For any $x, y, z \in I$ with $x < y < z$ we have

$$\frac{\eta(f(x), f(z))}{x - z} \leq \frac{f(y) - f(z)}{y - z} \quad \text{and} \quad f(x) \leq f(y) + \eta(f(x), f(y)). \tag{2.3}$$

Proof. Suppose that f is η-convex and $x, y, z \in I$ with $x < y < z$, then there is a $t \in (0, 1)$ such that $y = tx + (1 - t)z$. So we have $t = \frac{y-z}{x-z}$. Also

$$f(y) \leq f(z) + t\eta(f(x), f(z))$$

or

$$f(y) - f(z) \leq \frac{y - z}{x - z}\eta(f(x), f(z)).$$

Hence

$$\frac{\eta(f(x), f(z))}{x - z} \leq \frac{f(y) - f(z)}{y - z}.$$

For the inverse implications, consider $x, y \in I$ with $x < y$. It is clear that for any $t \in (0, 1)$, $x < tx + (1 - t)y < y$. It follows that

$$\frac{\eta(f(x), f(y))}{x - y} \leq \frac{f(tx + (1 - t)y) - f(y)}{tx + (1 - t)y - y}$$

that is equivalent to

$$\frac{\eta(f(x), f(y))}{x - y} \leq \frac{f(tx + (1 - t)y) - f(y)}{t(x - y)}.$$

Therefore

$$f(tx + (1 - t)y) \leq f(y) + t\eta(f(x), f(y))$$

for any $x, y \in I$ with $x < y$ and $t \in (0, 1)$. So f is η-convex. □

With the same argument as Theorem 2 we have also:

Theorem 3. *For a function $f : I \to \mathbb{R}$ the following assertions are equivalent.*
(a) f is η-convex function.
(b) For any $x, y, z \in I$ with $x < y < z$ we have

$$\frac{f(y) - f(x)}{y - x} \leq \frac{\eta(f(z), f(x))}{z - x} \quad \text{and} \quad f(x) \leq f(y) + \eta(f(x), f(y)). \tag{2.4}$$

The following particular case is of interest:

Corollary 1. *Any η-convex function with $\eta(x, y) = -\eta(y, x)$, is convex.*

Proof. Consider $x, y, z \in I$ such that $x < y < z$. From Theorem 2 and 3 we have

$$\frac{\eta(f(x), f(z))}{x - z} \leq \frac{f(y) - f(z)}{y - z} \text{ and } \frac{f(y) - f(x)}{y - x} \leq \frac{\eta(f(x), f(z))}{x - z}.$$

So, for any $x, y, z \in I$ with $x < y < z$, we have

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(y)}{z - y}.$$

The last inequality is equivalent to convexity of f (see [21], Chapter 1). □

3. Main Results

The first result of this section shows that a η-convex function with a bounded bifunction η from above, satisfies a Lipschitz condition. We need two definitions:

Definition 3. [21] *A function $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ if corresponding to any $\varepsilon>0$ there exists a $\delta>0$ such that for any collection $\{a_i,b_i\}_1^n$ of disjoint open intervals of $[a,b]$ with $\sum_1^n(b_i-a_i)<\delta$, $\sum_1^n|f(b_i)-f(a_i)|<\varepsilon$.*

Definition 4. [21] *A function $f:[a,b]\to\mathbb{R}$ is said to satisfy Lipschitz condition on $[a,b]$ if there is a constant K so that for any two points $x,y\in[a,b]$, $|f(x)-f(y)|\le K|x-y|$.*

The first result is as follows:

Theorem 4. *Suppose that $f:I\to\mathbb{R}$ is a η-convex function and η is bounded from above on $f(I)\times f(I)$. Then f satisfies a Lipschitz condition on any closed interval $[a,b]$ contained in the interior I° of I. Hence, f is absolutely continuous on $[a,b]$ and continuous on I°.*

Proof. Let M_η be the upper bound of η on $f(I)\times f(I)$. Consider closed interval $[a,b]$ in I° and choose $\varepsilon>0$ such that $[a-\varepsilon,b+\varepsilon]$ belongs to I. Suppose that x,y are distinct points of $[a,b]$. Set $z=y+\frac{\varepsilon}{|y-x|}(y-x)$ and $t=\frac{|y-x|}{\varepsilon+|y-x|}$. So it is not hard to see that $z\in[a-\varepsilon,b+\varepsilon]$ and $y=tz+(1-t)x$. Then

$$f(y)\le f(x)+t\eta(f(z),f(x))\le f(x)+tM_\eta.$$

This implies that

$$f(y)-f(x)\le tM_\eta=\frac{|y-x|}{\varepsilon+|y-x|}M_\eta\le\frac{|y-x|}{\varepsilon}M_\eta=K\left|y-x\right|,$$

where $K=\frac{M_\eta}{\varepsilon}$.

Also if we change the place of x,y in above argument we have $f(x)-f(y)\le K\left|y-x\right|$. Therefore $\mid f(y)-f(x)\mid\le K\left|y-x\right|$.

It follows that if we choose $\delta<\varepsilon/k$, then f is absolutely continuous. Finally since $[a,b]$ is arbitrary on I°, then f is continuous on I°. □

Proposition 3. *Any η-convex function $f:[a,b]\to\mathbb{R}$ with respect to a bifunction η bounded from above on $f([a,b])\times f([a,b])$, has lower and upper bounds.*

Proof. Suppose that M_η is upper bound of η on $f([a,b])\times f([a,b])$. Consider any $x=ta+(1-t)b\in[a,b]$ with $t\in[0,1]$. In fact, we have

$$\begin{aligned}f(x)&=f(ta+(1-t)b)\le f(b)+t\eta(f(a),f(b))\\&\le\max\{f(b),f(b)+\eta(f(a),f(b))\}\le max\{f(b),f(b)+M_\eta\}.\end{aligned}$$

Now set $M=\max\{f(b),f(b)+M_\eta\}$.

For lower bound of f consider an arbitrary point in the form $\frac{a+b}{2} - t$ in $[a, b]$. Then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{a+b}{4} + \frac{t}{2} + \frac{a+b}{4} - \frac{t}{2}\right) \\ &= f\big(\frac{1}{2}\left(\frac{a+b}{2} + t\right) + \frac{1}{2}\left(\frac{a+b}{2} - t\right)\big) \\ &\leq f\left(\frac{a+b}{2} - t\right) + \frac{1}{2}\eta\big(f\left(\frac{a+b}{2} + t\right), f\left(\frac{a+b}{2} - t\right)\big) \\ &\leq f\left(\frac{a+b}{2} - t\right) + \frac{M_\eta}{2}. \end{aligned}$$

Now consider $m = f(\frac{a+b}{2}) - \frac{M_\eta}{2}$, and the statement is proved. □

As a consequence of Theorem 4, a η-convex function $f : [a, b] \to \mathbb{R}$ with respect to a bifunction η bounded from above on $f([a, b]) \times f([a, b])$, is integrable.

Theorem 5. *Suppose that $f : [a, b] \to \mathbb{R}$ is a η-convex function such that η is bounded from above on $f([a, b]) \times f([a, b])$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) - \frac{1}{2}M_\eta &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{1}{2}\big[f(a) + f(b)\big] + \frac{1}{4}\big[\eta(f(a), f(b)) + \eta(f(b), f(a))\big] \\ &\leq \frac{f(a) + f(b)}{2} + \frac{M_\eta}{2}, \end{aligned}$$

where M_η is upper bound of η.

Proof. For the right side of inequality consider an arbitrary point $x = ta + (1 - t)b$ with $t \in [0, 1]$. So $f(x) \leq f(b) + t\eta(f(a), f(b))$ with $t = \frac{x-b}{a-b}$. It follows that

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)dx &\leq \frac{1}{b-a}\int_a^b \big[f(b) + \frac{x-b}{a-b}\eta(f(a), f(b))\big]dx \\ &= \frac{1}{b-a}\left(f(b)(b-a) + \frac{\eta(f(a), f(b))}{b-a}.\frac{(b-a)^2}{2}\right) \\ &= f(b) + \frac{1}{2}\eta(f(a), f(b)). \end{aligned}$$

Also we have the inequality

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq f(a) + \frac{1}{2}\eta\,(f(b), f(a)).$$

Therefore we get

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)dx &\leq \min\Big\{f(b) + \frac{1}{2}\eta(f(a), f(b)), f(a) + \frac{1}{2}\eta(f(b), f(a))\Big\} \\ &\leq \frac{1}{2}\big[f(a) + f(b)\big] + \frac{1}{4}\big[\eta(f(a), f(b)) + \eta(f(b), f(a))\big] \\ &\leq \frac{1}{2}\big[f(a) + f(b)\big] + \frac{1}{2}M_\eta. \end{aligned}$$

For the left side of inequality, η-convexity of f implies that

$$\begin{aligned}
&f(\frac{a+b}{2})\\
&= f\Big(\frac{a+b}{4} - \frac{t(b-a)}{4} + \frac{a+b}{4} + \frac{t(b-a)}{4}\Big)\\
&= f\Big(\frac{1}{2}\left(\frac{a+b-t(b-a)}{2}\right) + \frac{1}{2}\left(\frac{a+b+t(b-a)}{2}\right)\Big)\\
&\leq f\Big(\frac{a+b+t(b-a)}{2}\Big)\\
&+ \frac{1}{2}\eta\Big(f\left(\frac{a+b-t(b-a)}{2}\right), f\left(\frac{a+b+t(b-a)}{2}\right)\Big)\\
&\leq f\Big(\frac{a+b+t(b-a)}{2}\Big) + \frac{1}{2}M_\eta
\end{aligned}$$

for all $t \in [0,1]$. So

$$f\Big(\frac{a+b+t(b-a)}{2}\Big) \geq f\left(\frac{a+b}{2}\right) - \frac{1}{2}M_\eta.$$

Also with the same argument we have

$$f\Big(\frac{a+b-t(b-a)}{2}\Big) \geq f\left(\frac{a+b}{2}\right) - \frac{1}{2}M_\eta.$$

Finally using the change of variable we have

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)dx &= \frac{1}{b-a}\Big[\int_a^{\frac{a+b}{2}} f(x)dx + \int_{\frac{a+b}{2}}^b f(x)dx\Big]\\
&= \frac{1}{2}\int_0^1 \left[f\left(\frac{a+b-t(b-a)}{2}\right) + f\left(\frac{a+b+t(b-a)}{2}\right)\right]dt\\
&\geq \frac{1}{2}\int_0^1 \left[2f\left(\frac{a+b}{2}\right) - M_\eta\right]dt = f\left(\frac{a+b}{2}\right) - \frac{1}{2}M_\eta.
\end{aligned}$$

□

Remark 1. *Note that:*
(1) According to Theorem 5, if we consider $\eta(x,y) = x - y$ then we have the classic Hermite-Hadamard inequality for convex function f

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

For various related inequalities see the monograph online [8].
(2) We also remark that the following statements hold:

(i) Let $f : I \to \mathbb{R}$ be an integrable function and $\eta : f(I) \times f(I) \to \mathbb{R}$ be a bifunction bounded from above with M_η as its upper bound. Suppose that for any $a, b \in I$ with $a < b$,

$$f\left(\frac{a+b}{2}\right) - \frac{1}{2}M_\eta \leq \frac{1}{b-a}\int_a^b f(x)dx.$$

Then for any $a, b \in I$ with $a < b$ there exists a $t \in (0,1)$ such that

$$f(ta + (1-t)b) \geq f\left(\frac{a+b}{2}\right) - \frac{1}{2}M_\eta.$$

(ii) Let $f : I \to \mathbb{R}$ be an integrable function and $\eta : f(I) \times f(I) \to \mathbb{R}$ be a bifunction bounded from above with M_η as its upper bound. Suppose that for any $a, b \in I$ with $a < b$,

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} + \frac{1}{2}M_\eta.$$

Then for any $a, b \in I$ with $a < b$ there exists a $t \in (0,1)$ such that

$$f(ta+(1-t)b) \leq \frac{f(a)+f(b)}{2} + \frac{1}{2}t\big(\eta(f(a),f(b)) + \eta(f(b),f(a))\big).$$

Hermite-Hadamard-Fejer inequality is an interesting inequality related to convex functions. The η-convex version of this inequality is considered in two parts below.

We need the following definition:

Definition 5. *A function $g : [a,b] \to \mathbb{R}$ is said to be symmetric with respect to $\frac{a+b}{2}$ on $[a,b]$ if*

$$g(x) = g(a+b-x), \ \text{for any } a \leq x \leq b.$$

Theorem 6 (Hermite-Hadamard-Fejer Right Inequality)**.** *Suppose that $f : [a,b] \to \mathbb{R}$ is a η-convex function such that η is bounded from above on $f([a,b]) \times f([a,b])$. Also suppose that $g : [a,b] \to \mathbb{R}^+$ is integrable and symmetric with respect to $\frac{a+b}{2}$. Then*

$$\int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx + \frac{\eta(f(a),f(b)) + \eta(f(b),f(a))}{2(b-a)}\int_a^b (b-x)g(x)dx. \tag{3.1}$$

Proof. From η-convexity of f, using the change of variable and the fact that g is symmetric with respect to $\frac{a+b}{2}$, we get two inequalities.

First

$$\begin{aligned} &\int_a^b f(x)g(x)dx \\ &\leq (b-a)\int_0^1 \Big[f(b) + t\eta(f(a),f(b))\Big] g(ta+(1-t)b)dt \\ &= (b-a)\Bigg[\int_0^1 f(b)g(ta+(1-t)b)dt \\ &+\eta(f(a),f(b))\int_0^1 tg(ta+(1-t)b)dt\Bigg]. \end{aligned} \tag{3.2}$$

Second

$$\begin{aligned} &\int_a^b f(x)g(x)dx \\ &\leq (b-a)\int_0^1 \Big[f(a) + t\eta(f(b),f(a))\Big] g((1-t)a+tb)dt \\ &= (b-a)\Bigg[\int_0^1 f(a)g(ta+(1-t)b)dt \\ &+\eta(f(b),f(a))\int_0^1 tg(ta+(1-t)b)dt\Bigg]. \end{aligned} \tag{3.3}$$

Finally if we add (3.2) and (3.3) we obtain

$$\begin{aligned}
&2\int_a^b f(x)g(x)dx \\
&\leq (b-a)\big(f(a)+f(b)\big)\int_0^1 g(ta+(1-t)b)dt \\
&+(b-a)\big(\eta(f(a),f(b))+\eta(f(b),f(a))\big)\int_0^1 tg(ta+(1-t)b)dt.
\end{aligned}$$

So, the change of variable $x=ta+(1-t)b$ implies that

$$\begin{aligned}
\int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}&\int_a^b g(x)dx \\
&+\frac{\eta(f(a),f(b))+\eta(f(b),f(a))}{2(b-a)}\int_a^b (b-x)g(x)dx.
\end{aligned}$$

□

Theorem 7 (Hermite-Hadamard-Fejer Left Inequality)**.** *Suppose that $f:[a,b]\to\mathbb{R}$ is a η-convex function such that η is bounded from above on $f([a,b])\times f([a,b])$. Also suppose that $g:[a,b]\to\mathbb{R}^+$ is integrable and symmetric with respect to $\frac{a+b}{2}$. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx-\frac{1}{2}\int_a^b \eta\big(f(a+b-x),f(x)\big)g(x)dx \leq \int_a^b f(x)g(x)dx. \tag{3.4}$$

Proof. From η-convexity of f we have

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &= f\Big(\frac{ta-ta+a+b+tb-tb}{2}\Big) \\
&= f\Big(\frac{ta+(1-t)b+tb+(1-t)a}{2}\Big) \\
&\leq f(tb+(1-t)a)+\frac{1}{2}\eta\big(f(ta+(1-t)b),f(tb+(1-t)a)\big).
\end{aligned}$$

Also with the change of variable $x=tb+(1-t)a$ we have

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \\
&= f(\frac{a+b}{2})\int_0^1 g\big(tb+(1-t)a\big)\big(b-a\big)dt \\
&\leq \int_0^1 f\big(tb+(1-t)a\big)g\big(tb+(1-t)a\big)\big(b-a\big)dt \\
&+\frac{1}{2}\int_0^1 \eta\Big(f(ta+(1-t)b),f(tb+(1-t)a)\Big)g\big(tb+(1-t)a\big)\big(b-a\big)dt \\
&= \int_a^b f(x)g(x)dx+\frac{1}{2}\int_a^b \eta\big(f(a+b-x),f(x)\big)g(x)dx.
\end{aligned}$$

□

Corollary 2. *Withe the assumption of Theorem 6 we have:*

(i) if $g(x) \equiv 1$, then we have the Hermite-Hadamard type inequalities:

$$f\left(\frac{a+b}{2}\right)(b-a) - \frac{1}{2}\int_a^b \eta\big(f(a+b-x), f(x)\big)dx \tag{3.5}$$
$$\leq \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} + \frac{\eta(f(a), f(b)) + \eta(f(b), f(a))}{4}.$$

(ii) if we consider M_η as the upper bound of η, then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} &\leq \int_a^b f(x)g(x)dx \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx + \frac{M_\eta}{b-a}\int_a^b (b-x)g(x)dx. \end{aligned}$$

(iii) if we set $\eta(x,y) = x - y$, then classic form of Hermite-Hadamard-Fejer inequality can be obtained as

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx.$$

Furthermore if we consider (i), (ii) and (iii) together, then we reach to Hermite-Hadamard inequality mentioned in Remark 1 (1).

We will use the following relations in the proof of Theorem 8 which is Jensen type inequality for η-convex functions.

Let $f : I \to \mathbb{R}$ be a η-convex function. For $x_1, x_2 \in I$ and $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 + \alpha_2 = 1$ we have

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leq f(x_2) + \alpha_1 \eta(f(x_1), f(x_2)).$$

For $n > 2$ and $i = 1, 2, ..., n$ consider $x_i \in I$ and $\alpha_i \in [0,1]$. Define $T_i = \sum_{j=1}^{i} \alpha_j$ and choose α_i such that $T_n = 1$. So

$$f\big(\sum_{i=1}^{n} \alpha_i x_i\big) = f\Big(\big(T_{n-1}\sum_{i=1}^{n-1}\frac{\alpha_i}{T_{n-1}}x_i\big) + \alpha_n x_n\Big) \tag{3.6}$$
$$\leq f(x_n) + T_{n-1}\eta\Big(f\big(\sum_{i=1}^{n-1}\frac{\alpha_i}{T_{n-1}}x_i\big), f(x_n)\Big).$$

We need the following definition:

Definition 6. *The function η is said to be:*

(i) nondecreasing on first variable if $x \leq y$ implies that $\eta(x,z) \leq \eta(y,z)$, for all $x, y, z \in \mathbb{R}$.

(ii) nonnegative sublinear on first variable if $\eta(\gamma x + y, z) \leq \gamma\eta(x,z) + \eta(y,z)$, for all $x, y, z \in \mathbb{R}$ and $\gamma \geq 0$.

The following version of weighted Jensen's discrete inequality holds:

Theorem 8. *Consider functions $f : I \to \mathbb{R}$ and $\eta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that η is nondecreasing and nonnegative sublinear on first variable. Also define*

$$\eta_f(x_i, x_{i+1}, \cdots, x_n) = \eta\big(\eta_f(x_i, x_{i+1}, \cdots, x_{n-1}), f(x_n)\big)$$

and $\eta_f(x) = f(x)$ for all $x \in I$. Then f is η-convex iff for any $n \geq 2$,

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} T_i \eta_f(x_i, x_{i+1}, \dots, x_n), \tag{3.7}$$

where $T_i = \sum\limits_{j=1}^{i} \alpha_j$ for $i = 1, 2, ..., n$ such that $T_n = 1$.

Proof. Suppose that f is η-convex. Since η is nondecreasing and nonnegative sublinear on first variable then from (3.6) it follows that

$$\begin{aligned}
&f\big(\sum_{i=1}^{n} \alpha_i x_i\big)\\
&\leq f(x_n) + T_{n-1}\eta\Big(f\big(\sum_{i=1}^{n-1} \frac{\alpha_i}{T_{n-1}} x_i\big), f(x_n)\Big)\\
&\leq f(\leq x_n) + T_{n-1}\eta\Big(f\big(\frac{T_{n-2}}{T_{n-1}} \sum_{i=1}^{n-2} \frac{\alpha_i}{T_{n-2}} x_i + \frac{\alpha_{n-1}}{T_{n-1}} x_{n-1}\big), f(x_n)\Big)\\
&\leq f(x_n) + T_{n-1}\eta\bigg(f(x_{n-1}) + \frac{T_{n-2}}{T_{n-1}}\eta\Big(f\big(\sum_{i=1}^{n-2} \frac{\alpha_i}{T_{n-2}} x_i\big), f(x_{n-1})\Big), f(x_n)\bigg)\\
&\leq f(x_n) + T_{n-1}\eta(f(x_{n-1}), f(x_n))\\
&+ T_{n-2}\eta\bigg(\eta\Big(f\big(\sum_{i=1}^{n-2} \frac{\alpha_i}{T_{n-2}} x_i\big), f(x_{n-1})\Big), f(x_n)\bigg)\\
&\leq ... \leq f(x_n) + T_{n-1}\eta(f(x_{n-1}), f(x_n))\\
&+ T_{n-2}\eta\bigg(\eta\Big(f(x_{n-2}), f\big(x_{n-1})\big), f(x_n)\Big)\\
&+ ... + T_1\eta\bigg(\eta\Big(\cdots\eta\big(\eta(f(x_1), f(x_2)), f(x_3)\big)\cdots\big)\Big), f(x_{n-1})\Big), f(x_n)\bigg)\\
&= f(x_n) + T_{n-1}\eta_f(x_{n-1}, x_n) + T_{n-2}\eta_f(x_{n-2}, x_{n-1}, x_n) + ...\\
&+ T_1\eta_f(x_1, x_2, \cdots, x_{n-1}, x_n)\\
&= \sum_{i=1}^{n} T_i \eta_f(x_i, x_{i+1}, \cdots, x_n).
\end{aligned}$$

For the inverse implication consider $n = 2$ in (3.7). We omit the details. □

Example 3. *Consider $f(x) = x^2$ and $\eta(x, y) = x(1 + 2y)$ for $x, y \in \mathbb{R}^+$. The function η is nondecreasing nonnegative sublinear in first variable and f is η-convex since $(\alpha_1 x_1 + \alpha_2 x_2)^2 \leq x_2^2 + \alpha_1 x_1^2(1 + 2x_2^2)$, for $x_1, x_2 \in \mathbb{R}^+$ and $\alpha_1, \alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 = 1$. Now for $x_1, x_2, \cdots, x_n \in \mathbb{R}^+$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ with $\sum\limits_{i=1}^{n} \alpha_i = 1$ according to Theorem 8, we have*

$$\big(\sum_{i=1}^{n} \alpha_i x_i\big)^2 \leq \sum_{i=1}^{n} T_i \big[x_i^2(1 + 2x_{i+1}^2)(1 + 2x_{i+2}^2)...(1 + 2x_n^2)\big]. \tag{3.8}$$

4. The Case of Differentiable Functions

The case of differentiable functions is of interest as well.

Theorem 9. *Suppose that $f:[a,b]\to\mathbb{R}$ is a differentiable η-convex function on (a,b) and that η is measurable on $f([a,b])\times f([a,b])$. Then we have*

$$f'(y)\left(\frac{a+b}{2}-y\right)\le\frac{1}{b-a}\int_a^b\eta(f(x),f(y))dx \tag{4.1}$$

for every $y\in(a,b)$, in particular

$$\int_a^b\eta\left(f(x),f\left(\frac{a+b}{2}\right)\right)dx\ge 0.$$

Proof. From the definition of η-convex functions we have

$$\frac{f(tx+(1-t)y)-f(y)}{t}\le\eta(f(x),f(y)),$$

for $t\in(0,1]$. Taking the limit over $t\to 0+$ we get

$$f'(y)(x-y)\le\eta(f(x),f(y)) \tag{4.2}$$

for any $x\in[a,b]$ and any $y\in(a,b)$.

Since η is measurable on $f([a,b])\times f([a,b])$, then the integral

$$\int_a^b\eta(f(x),f(y))dx$$

exists for any $y\in(a,b)$. Integrating (4.2) over x on $[a,b]$ and dividing by $b-a$ we deduce (4.1). □

Remark 2. *If $a>0$, then from (4.1) we have the inequalities*

$$f'\left(\sqrt{ab}\right)\left(\frac{a+b}{2}-\sqrt{ab}\right)\le\frac{1}{b-a}\int_a^b\eta(f(x),f(\sqrt{ab})dx \tag{4.3}$$

and

$$\frac{1}{2}f'\left(\frac{2}{\frac{1}{a}+\frac{1}{b}}\right)\frac{(b-a)^2}{a+b}\le\frac{1}{b-a}\int_a^b\eta\left(f(x),f\left(\frac{2}{\frac{1}{a}+\frac{1}{b}}\right)\right)dx. \tag{4.4}$$

The dual result also holds.

Theorem 10. *Suppose that $f:[a,b]\to\mathbb{R}$ is a η-convex function and η is bounded from above on $f([a,b])\times f([a,b])$. Then we have*

$$\int_a^b f(y)\,dy\le(x-a)f(a)+(b-x)f(b)+\int_a^b\eta(f(x),f(y))dy \tag{4.5}$$

for any $x\in[a,b]$ and, in particular

$$\frac{1}{b-a}\int_a^b f(y)\,dy\le\frac{f(a)+f(b)}{2}+\frac{1}{b-a}\int_a^b\eta\left(f\left(\frac{a+b}{2}\right),f(y)\right)dy. \tag{4.6}$$

Proof. Since the function f is absolutely continuous, then it is differentiable almost everywhere on $[a,b]$ and , as above, we have

$$f'(y)(x-y)\le\eta(f(x),f(y)) \tag{4.7}$$

for any $x\in[a,b]$ and almost every $y\in(a,b)$.

Since η is bounded from above on $f([a,b])\times f([a,b])$, then the integral $\int_a^b \eta(f(x),f(y))dy$ exists for any $x\in[a,b]$.

Integrating in (4.7) over y on the interval $[a,b]$ we get

$$\int_a^b f'(y)(x-y)\,dy \le \int_a^b \eta(f(x),f(y))dy \tag{4.8}$$

for any $x\in[a,b]$.

Integrating by parts, we also have

$$\int_a^b f'(y)(x-y)\,dy = \int_a^b f(y)\,dy-(x-a)f(a)-(b-x)f(b)$$

and by (4.8) we get the desired result (4.5). □

Corollary 3. *With the assumptions of Theorem 10 we have the double integral inequality*

$$\frac{1}{b-a}\int_a^b f(y)\,dy \le \frac{f(a)+f(b)}{2}+\frac{1}{(b-a)^2}\int_a^b\int_a^b \eta(f(x),f(y))dxdy.$$

The proof follows by (4.5) integrating over $x\in[a,b]$.

Remark 3. *The case $\eta(x,y)=x-y$ taken in the above inequalities will produce some know Hermite-Hadamard related inequalities, see* [8]*. The details are not presented here.*

The following Jensen type inequalities may be stated as well.

Theorem 11. *Suppose that $f:[a,b]\to\mathbb{R}$ is a differentiable η-convex function on (a,b), $x_i\in[a,b],\alpha_i\ge 0$, $i\in\{1,...,n\}$ and $\sum_{i=1}^n\alpha_i=1$. Then for any $y\in(a,b)$ we have*

$$f'(y)\left(\sum_{i=1}^n \alpha_i x_i - y\right)\le \sum_{i=1}^n \alpha_i\eta(f(x_i),f(y)). \tag{4.9}$$

In particular, we have

$$\sum_{i=1}^n \alpha_i\eta\left(f(x_i),f\left(\sum_{i=1}^n\alpha_i x_i\right)\right)\ge 0. \tag{4.10}$$

Also, for any $x\in[a,b]$ we have

$$x\sum_{i=1}^n \alpha_i f'(x_i)-\sum_{i=1}^n \alpha_i f'(x_i)x_i \le \sum_{i=1}^n \alpha_i\eta(f(x),f(x_i)). \tag{4.11}$$

Moreover, if

$$\frac{\sum_{j=1}^n \alpha_j f'(x_j)x_j}{\sum_{j=1}^n \alpha_j f'(x_j)}\in[a,b], \tag{4.12}$$

then

$$\sum_{i=1}^n \alpha_i\eta\left(f\left(\frac{\sum_{j=1}^n \alpha_j f'(x_j)x_j}{\sum_{j=1}^n \alpha_j f'(x_j)}\right),f(x_i)\right)\ge 0 \tag{4.13}$$

Proof. If we use the inequality (4.2), then we get

$$f'(y)(x_i - y) \leq \eta(f(x_i), f(y)) \tag{4.14}$$

for any $x_i \in [a,b]$ and any $y \in (a,b)$.

If we multiply (4.14) by $\alpha_i \geq 0$, $i \in \{1,...,n\}$ and sum over i from 1 to n we get the desired inequality (4.9).

From (4.2) we also have

$$f'(x_i)(x - x_i) \leq \eta(f(x), f(x_i)) \tag{4.15}$$

for any $x_i \in [a,b]$ and any $x \in (a,b)$.

If we multiply (4.15) by $\alpha_i \geq 0$, $i \in \{1,...,n\}$ and sum over i from 1 to n we get

$$\sum_{i=1}^{n} \alpha_i f'(x_i)(x - x_i) \leq \sum_{i=1}^{n} \alpha_i \eta(f(x), f(x_i)),$$

which is equivalent to (4.11). □

Remark 4. *We observe that a sufficient condition for (4.12) to hold is that f is nondecreasing (nonincreasing) on the whole interval $[a,b]$.*

Remark 5. *In the case that $\eta(x,y) = x - y$ we get from (4.13) Slater's inequality* [23]

$$\left(\frac{\sum_{j=1}^{n} \alpha_j f'(x_j) x_j}{\sum_{j=1}^{n} \alpha_j f'(x_j)} \right) \geq \sum_{i=1}^{n} \alpha_i f(x_i), \tag{4.16}$$

provided that (4.12) is satisfied.

References

[1] S. Abramovich, Convexity, subadditivity and generalized Jensen's inequality. *Ann. Funct. Anal.* **4** (2013), no. 2, 183–194

[2] A. Aleman, *On some generalizations of convex sets and convex functions*, Anal. Numer. Theor. Approx. **14** (1985) 1–6.

[3] G. A. Anastassiou, Basic and s-convexity Ostrowski and Grüss type inequalities involving several functions. *Commun. Appl. Anal.* **17** (2013), no. 2, 189–212.

[4] C. R. Bector and C. Singh, *B-Vex functions*, J. Optim. Theory. Appl. **71** (2) (1991) 237–253.

[5] B. Definetti, *Sulla stratificazioni convesse*, Ann. Math. Pura. Appl. **30** (1949) 173–183.

[6] S. S. Dragomir, Inequalities of Hermite-Hadamard Type for φ-Convex Functions, Preprint *RGMIA Res. Rep. Coll.* **16**(2013), Art. 87, pp. 14. [Online http://rgmia.org/papers/v16/v16a87.pdf].

[7] S. S. Dragomir, Inequalities of Hermite-Hadamard type for λ-convex functions on linear spaces, Preprint *RGMIA Res. Rep. Coll.* **17**(2014), Art. 13, pp.18. [Online http://rgmia.org/papers/v17/v17a13.pdf].

[8] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications,* RGMIA Monographs, Victoria University, 2000. [Online http://rgmia.org/monographs/hermite_hadamard.html].

[9] L. Fejer, *Uberdie Fourierreihen*, II, Math. Naturwise. Anz Ungar. Akad. Wiss. **24** (1906) 369–390.

[10] M. A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. **80** (1981) 545–550.

[11] D. H. Hyers and S. M. Ulam, *Approximately convex functions*, Proc. Amer. Math. Soc. **3** (1952), 821–828.

[12] I. Hsu and R. G. Kuller, *Convexity of vector-valued functions*, Proc. Amer. Math. Soc. **46** (1974), 363–366.

[13] J. L. W. V. Jensen, *On konvexe funktioner og uligheder mellem middlvaerdier*, Nyt. Tidsskr. Math. B. **16** (1905) 49-69.

[14] D. Kuroiwa, *Convexity for set-valued maps*, Appl. Math. Lett. **9** (1996) 97–101.
[15] D. T. Luc, *Theory of Vector Optimization*, Springer-Verlag, Berlin, 1989.
[16] O. L. Mangasarian, *Pseudo-Convex functions*, SIAM Journal on Control, **3** (1965) 281–290.
[17] M. E. Özdemir, M. Avci and H. Kavurmaci, Hermite-Hadamard-type inequalities via (α, m)-convexity. Comput. Math. Appl. 61 (2011), no. 9, 2614–2620
[18] J. E. Pečarić , F. Proschan and Y. L. Tong, *Convex functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.
[19] B. T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. Dokl. **7** (1966) 72–75.
[20] T. Rajba, On strong delta-convexity and Hermite-Hadamard type inequalities for delta-convex functions of higher order. *Math. Inequal. Appl.* **18** (2015), no. 1, 267–293
[21] A. W. Robert and D. E. Varbeg, *Convex Functions*, Academic Press, (1973).
[22] E. Set, M. E. Özdemir and S. S. Dragomir, On Hadamard-type inequalities involving several kinds of convexity. *J. Inequal. Appl.* **2010**, Art. ID 286845, 12 pp.
[23] M.S. Slater, *A companion inequality to Jensen's inequality*, J. Approx. Theory, **32**(1981), 160-166.
[24] S. Varošanec, On h-convexity. *J. Math. Anal. Appl.* **326** (2007), no. 1, 303–311.
[25] X. M. Yang, *E-convex sets, E-convex functions and E-convex programming*, J. Optim. Theory. Appl. **109** (2001), 699-704.
[26] X. M. Yang and S. Y. Liu, *Three kinds of generalized convexity*, J. Optim. Theory. Appl. **86** (1995) 501–513.

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