Received 15/01/15

FURTHER BOUNDS FOR ČEBYŠEV FUNCTIONAL FOR POWER SERIES IN BANACH ALGEBRAS VIA GRÜSS-LUPAŞ TYPE INEQUALITIES FOR p-NORMS

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Abstract. Some Grüss-Lupaş type inequalities for p-norms of sequences in Banach algebras are obtained. Moreover, if $f(\lambda)=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ is a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$ and $x,y\in\mathcal{B}$, a Banach algebra, with $xy=yx$, then we also establish some new upper bounds for the norm of the *Čebyšev type difference*

$$f(\lambda)f(\lambda xy)-f(\lambda x)f(\lambda y),\lambda\in D(0,R).$$

These results build upon the earlier results obtained by the authors. Applications for some fundamental functions such as the *exponential function* and the *resolvent function* are provided as well.

1. Introduction

In 1935, G. Grüss [47] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integral means integrals as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \le\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma) \tag{1.1}$$

where $f,g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and satisfying the assumption

$$\varphi\le f(x)\le\Phi,\gamma\le g(x)\le\Gamma$$

for each $x\in[a,b]$ where $\varphi,\Phi,\gamma,\Gamma$ are given real constants.

Moreover the constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For a simple proof of (1.1) as well as for some other integral inequalities of Grüss' type see the Chapter X of the recent book [53] by Mitrinović, Pečarić and Fink.

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardzewski [4] established the following discrete version of Grüss' inequality, see also [53, Ch. X]:

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Banach algebras, Power series, Exponential function, Resolvent function, Norm inequalities.

Theorem 1. *Let $a = (a_1, ..., a_n), b = (b_1, ..., b_n)$ be two n-tuples of real numbers such that $r \le a_i \le R$ and $s \le b_i \le S$ for $i = 1, ..., n$. Then one has the inequality:*

$$\left| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} b_i \right| \le \frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) (R - r)(S - s) \tag{1.2}$$

when $[x]$ is the integer part of $x, x \in \mathbb{R}$.

A weighted version of Grüss' discrete inequality was proved by J. E. Pečarić in 1979, see for instance [53, Ch. X]:

Theorem 2. *Let a, b be two monotonic n-tuples and p a positive one. Then*

$$\left| \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \right| \le |a_n - a_1| |b_n - b_1| \max_{1 \le k \le n-1} \left(\frac{P_k \bar{P}_{k+1}}{P_n^2} \right) \tag{1.3}$$

where $P_n := \sum_{i=1}^{n} p_i$, $\bar{P}_{k+1} = P_n - P_{k+1}$.

In 1981 , A. Lupaş [53, Ch. X] proved some similar results for the first difference of a as follows :

Theorem 3. *Let a, b two monotonic n-tuples in the same sense and p a positive n-tuple. Then*

$$\begin{aligned} &\min_{1 \le i \le n-1} |a_{i+1} - a_i| \min_{1 \le i \le n-1} |b_{i+1} - b_i| \left[\frac{1}{P_n} \sum_{i=1}^{n} i^2 p_i - \left(\frac{1}{P_n} \sum_{i=1}^{n} i p_i \right)^2 \right] \\ &\le \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \\ &\le \max_{1 \le i \le n-1} |a_{i+1} - a_i| \max_{1 \le i \le n-1} |b_{i+1} - b_i| \left[\frac{1}{P_n} \sum_{i=1}^{n} i^2 p_i - \left(\frac{1}{P_n} \sum_{i=1}^{n} i p_i \right)^2 \right]. \end{aligned} \tag{1.4}$$

If there exists the numbers $\bar{a}, \bar{a}_1, r, r_1, (rr_1 > 0)$ such that $a_k = \bar{a} + kr$ and $b_k = \bar{a}_1 + kr_1$, then in (1.4) the equality holds.

For some generalizations of Gruss' inequality for isotonic linear functionals defined on certain spaces of mappings see Chapter X of the book [53] where further references are given .

For related results, see [1]-[33], [39]-[45] and [48]-[64].

2. Some Facts on Banach Algebras

In order to extend the above results for Banach algebras, we need some preliminary facts as follows:

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \le \|a\| \|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.*

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by Inv$\mathcal{B}$. If $a, b \in$Inv$\mathcal{B}$ then $ab \in$Inv$\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in$Inv$\mathcal{B}$;
(ii) $\{a \in \mathcal{B}\text{: } \|1 - b\| < 1\} \subset$Inv$\mathcal{B}$;
(iii) Inv$\mathcal{B}$ is an *open subset* of $\mathcal{B}$;
(iv) The map Inv$\mathcal{B} \ni a \longmapsto a^{-1} \in$Inv$\mathcal{B}$ is continuous.

For simplicity, we denote $\lambda 1$, where $\lambda \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by λ. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{\lambda \in \mathbb{C} : \ \lambda - a \in \text{Inv}\mathcal{B}\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to$Inv$\mathcal{B}$, $R_a(\lambda) := (\lambda - a)^{-1}$. For each $\lambda, \gamma \in \rho(a)$ we have the identity

$$R_a(\gamma) - R_a(\lambda) = (\lambda - \gamma) R_a(\lambda) R_a(\gamma).$$

We also have that $\sigma(a) \subset \{\lambda \in \mathbb{C} : \ |\lambda| \le \|a\|\}$. The *spectral radius* of a is defined as $\nu(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$.

If a, b are *commuting* elements in $\mathcal{B}$, i.e. $ab = ba$, then

$$\nu(ab) \le \nu(a)\nu(b) \text{ and } \nu(a + b) \le \nu(a) + \nu(b).$$

Let f be an analytic functions on the open disk $D(0, R)$ given by the *power series* $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j \ (|\lambda| < R)$. If $\nu(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|a^j\right\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathcal{B}$ denoted exp and defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \text{ for each } a \in \mathcal{B}.$$

If $\mathcal{B}$ is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for a and b from $\mathcal{B}$

$$\exp(a + b) = \exp(a)\exp(b).$$

In a general Banach algebra $\mathcal{B}$ it is difficult to determine the elements in the range of the exponential map $\exp(\mathcal{B})$, i.e. the element which have a "*logarithm*". However, it is easy to see that if a is an element in B such that $\|1 - a\| < 1$, then a is in $\exp(\mathcal{B})$. That follows from the fact that if we set

$$b = -\sum_{n=1}^{\infty} \frac{1}{n} (1 - a)^n,$$

then the series converges absolutely and, as in the scalar case, substituting this series into the series expansion for $\exp(b)$ yields $\exp(b) = a$.

It is known that if x and y are commuting, i.e. $xy = yx$, then the exponential function satisfies the property

$$\exp(x)\exp(y) = \exp(y)\exp(x) = \exp(x+y).$$

Also, if x is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tx)\,dt = x^{-1}\left[\exp(bx) - \exp(ax)\right].$$

Moreover, if x and y are commuting and $y - x$ is invertible, then

$$\begin{aligned}\int_0^1 \exp((1-s)x + sy)\,ds &= \int_0^1 \exp(s(y-x))\exp(x)\,ds \\ &= \left(\int_0^1 \exp(s(y-x))\,ds\right)\exp(x) \\ &= (y-x)^{-1}\left[\exp(y-x) - I\right]\exp(x) \\ &= (y-x)^{-1}\left[\exp(y) - \exp(x)\right].\end{aligned}$$

Inequalities for functions of operators in Hilbert spaces may be found in the papers [15], [14] and in the recent monographs [34], [35], [46] and the references therein.

The following inequality of Grüss-Lupaş type in Banach algebras holds [38]:

Theorem 4. *Let $\mathcal{B}$ be a Banach algebra over $\mathbb{K}$ $(=\mathbb{R},\mathbb{C})$, $a_i, b_i \in \mathcal{B}$ and $\alpha_i \in \mathbb{K}$ $(i = 1, ..., n)$. Then we have the inequality:*

$$\begin{aligned}&\left\|\sum_{i=1}^n \alpha_i \sum_{i=1}^n \alpha_i a_i b_i - \sum_{i=1}^n \alpha_i a_i \sum_{i=1}^n \alpha_i b_i\right\| \\ &\le \max_{1\le j\le n-1}\|a_{j+1} - a_j\| \max_{1\le j\le n-1}\|b_{j+1} - b_j\| \\ &\times \left[\sum_{i=1}^n |\alpha_i| \sum_{i=1}^n i^2 |\alpha_i| - \left(\sum_{i=1}^n i|\alpha_i|\right)^2\right].\end{aligned} \tag{2.1}$$

The inequality (2.1) is sharp in the sense that the multiplicative constant $C = 1$ in the right membership can not be replaced by a smaller one.

Let α_n be nonzero complex numbers and let

$$R := \frac{1}{\limsup |\alpha_n|^{\frac{1}{n}}}.$$

Clearly $0 \le R \le \infty$, but we consider only the case $0 < R \le \infty$.

Denote by:

$$D(0,R) = \begin{cases} \{\lambda \in \mathbb{C} : |\lambda| < R\}, & \text{if } R < \infty \\ \mathbb{C}, & \text{if } R = \infty, \end{cases}$$

consider the functions:

$$\lambda \mapsto f(\lambda) : D(0,R) \to \mathbb{C}, f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n$$

and

$$\lambda \mapsto f_A(\lambda) : D(0,R) \to \mathbb{C}, f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n.$$

Let $\mathcal{B}$ be a unital Banach algebra and 1 its unity. Denote by

$$B(0,R) = \begin{cases} \{x \in \mathcal{B} : \|x\| < R\}, & \text{if } R < \infty \\ \mathcal{B}, & \text{if } R = \infty. \end{cases}$$

We associate to f the map:

$$x \mapsto \widetilde{f}(x) : B(0,R) \to \mathcal{B}, \widetilde{f}(x) := \sum_{n=0}^{\infty} \alpha_n x^n.$$

Obviously, $\widetilde{f}$ is correctly defined because the series $\sum_{n=0}^{\infty} \alpha_n x^n$ is absolutely convergent, since $\sum_{n=0}^{\infty} \|\alpha_n x^n\| \le \sum_{n=0}^{\infty} |\alpha_n| \|x\|^n$.

In addition, we assume that $s_2 := \sum_{n=0}^{\infty} n^2 |\alpha_n| < \infty$. Let $s_0 := \sum_{n=0}^{\infty} |\alpha_n| < \infty$ and $s_1 := \sum_{n=0}^{\infty} n |\alpha_n| < \infty$.

With the above assumptions we have that [36]:

Theorem 5. *Let $\lambda \in \mathbb{C}$ such that $\max\{|\lambda|, |\lambda|^2\} < R < \infty$ and let $x, y \in \mathcal{B}$ with $\|x\|, \|y\| \le 1$ and $xy = yx$. Then:*

(i) *We have*

$$\left\| \widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y) \right\| \le \sqrt{2}\psi \min\{\|x-1\|, \|y-1\|\} f_A\left(|\lambda|^2\right) \tag{2.2}$$

where:

$$\psi^2 := s_0 s_2 - s_1^2. \tag{2.3}$$

(ii) *We also have*

$$\begin{aligned} &\left\| \widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y) \right\| \\ &\le \sqrt{2} \min\{\|x-1\|, \|y-1\|\} f_A(|\lambda|) \\ &\times \left\{ f_A(|\lambda|) \left[|\lambda| f_A'(|\lambda|) + |\lambda|^2 f_A''(|\lambda|) \right] - \left[|\lambda| f_A'(|\lambda|) \right]^2 \right\}^{1/2}. \end{aligned} \tag{2.4}$$

For other similar results, see [36], [37] and [38]

Motivated by the above results we establish in this paper other similar inequalities for the norm of the *Čebyšev difference*

$$\widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y)$$

via some Grüss'-Lupaş type inequality for p-norms with $p \ge 1$, where λ is a complex number and the vectors x, y belong to the Banach algebra $\mathcal{B}$. Applications for some fundamental functions such as the *exponential function* and the *resolvent function* are provided as well.

3. A Discrete Inequality of Grüss Type for 1-Norm

The following inequality of Grüss type that complements the result from (2.1) holds.

Theorem 6. *Let $\mathcal{B}$ be a Banach algebra over $\mathbb{K}$ $(=\mathbb{R},\mathbb{C})$, $a_i, b_i \in \mathcal{B}$ and $\alpha_i \in \mathbb{K}$ $(i = 1, ..., n)$. Then we have the inequality:*

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \leq \frac{1}{2} \left[\left(\sum_{i=1}^{n} |\alpha_i| \right)^2 - \sum_{i=1}^{n} |\alpha_i|^2 \right] \sum_{i=1}^{n-1} \|\Delta a_i\| \sum_{i=1}^{n-1} \|\Delta b_i\| , \tag{3.1}$$

where $\Delta a_i := a_{i+1} - a_i$ $(i = 1, ..., n-1)$ and $\Delta b_i := b_{i+1} - b_i$ $(i = 1, ..., n-1)$ are the usual forward differences.
The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. Let us start with the following identity in Banach algebras which can be proved by direct computation

$$\sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i = \frac{1}{2} \sum_{i,j=1}^{n} \alpha_i \alpha_j (a_j - a_i)(b_j - b_i) = \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j (a_j - a_i)(b_j - b_i) .$$

As $i < j$, we can write

$$a_j - a_i = \sum_{k=i}^{j-1} (a_{k+1} - a_k) = \sum_{k=i}^{j-1} \Delta a_k$$

and

$$b_j - b_i = \sum_{l=i}^{j-1} (b_{l+1} - b_l) = \sum_{l=i}^{j-1} \Delta b_l.$$

Using the generalized triangle inequality, we have successively:

$$\begin{aligned} &\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \\ &= \left\| \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \sum_{k=i}^{j-1} \Delta a_k \sum_{l=i}^{j-1} \Delta b_l \right\| \\ &\leq \sum_{1 \leq i < j \leq n} |\alpha_i| |\alpha_j| \left\| \sum_{k=i}^{j-1} \Delta a_k \right\| \left\| \sum_{l=i}^{j-1} \Delta b_l \right\| \\ &\leq \sum_{1 \leq i < j \leq n} |\alpha_i| |\alpha_j| \sum_{k=i}^{j-1} \|\Delta a_k\| \sum_{l=i}^{j-1} \|\Delta b_l\| =: A. \end{aligned} \tag{3.2}$$

It is obvious for all $1 \leq i < j \leq n-1$, we have that

$$\sum_{k=i}^{j-1} \|\Delta a_k\| \leq \sum_{k=1}^{n-1} \|\Delta a_k\|$$

and

$$\sum_{l=i}^{j-1} \|\Delta b_l\| \leq \sum_{l=1}^{n-1} \|\Delta b_l\|$$

and then

$$A \leq \sum_{k=1}^{n-1} \|\Delta a_k\| \sum_{l=1}^{n-1} \|\Delta b_l\| \sum_{1\leq i<j\leq n} |\alpha_i| |\alpha_j| . \tag{3.3}$$

Now, let us observe that

$$\begin{aligned}\sum_{1\leq i<j\leq n} |\alpha_i| |\alpha_j| &= \frac{1}{2}\left[\sum_{i,j=1}^{n} |\alpha_i| |\alpha_j| - \sum_{i=j} |\alpha_i| |\alpha_j|\right] \\ &= \frac{1}{2}\left[\sum_{i=1}^{n} |\alpha_i| \sum_{j=1}^{n} |\alpha_j| - \sum_{i=1}^{n} |\alpha_i|^2\right] \\ &= \frac{1}{2}\left[\left(\sum_{i=1}^{n} |\alpha_i|\right)^2 - \sum_{i=1}^{n} |\alpha_i|^2\right] .\end{aligned} \tag{3.4}$$

Using (3.2)-(3.4), we deduce the desired inequality (3.1).

To prove the sharpness of the constant $\frac{1}{2}$, let us assume that (3.1) holds with a constant $c > 0$. That is,

$$\begin{aligned}&\left\|\sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i\right\| \\ &\leq c\left[\left(\sum_{i=1}^{n} |\alpha_i|\right)^2 - \sum_{i=1}^{n} |\alpha_i|^2\right] \sum_{i=1}^{n-1} \|\Delta a_i\| \sum_{i=1}^{n-1} \|\Delta b_i\|\end{aligned} \tag{3.5}$$

for all a_i, b_i, α_i $(i = 1, ..., n)$ as above and $n \geq 2$.

Choose in (3.1) $n = 2$ and compute

$$\begin{aligned}\sum_{i=1}^{2} \alpha_i \sum_{i=1}^{2} \alpha_i a_i b_i - \sum_{i=1}^{2} \alpha_i a_i \sum_{i=1}^{2} \alpha_i b_i &= \frac{1}{2} \sum_{i,j=1}^{2} \alpha_i \alpha_j (a_i - a_j)(b_i - b_j) \\ &= \sum_{1\leq i<j\leq 2} \alpha_i \alpha_j (a_i - a_j)(b_i - b_j) \\ &= \alpha_1 \alpha_2 (a_1 - a_2)(b_1 - b_2) .\end{aligned}$$

Also,

$$\sum_{1\leq i<j\leq 2} |\alpha_i| |\alpha_j| \sum_{i=1}^{1} \|\Delta a_i\| \sum_{i=1}^{1} \|\Delta b_i\| = |\alpha_1| |\alpha_2| \|a_1 - a_2\| \|b_1 - b_2\| .$$

Substituting in (3.5), we obtain

$$|\alpha_1| |\alpha_2| \|a_1 - a_2\| \|b_1 - b_2\| \leq 2c |\alpha_1| |\alpha_2| \|a_1 - a_2\| \|b_1 - b_2\| .$$

If we assume that $\alpha_1, \alpha_2 > 0$, $a_1 \neq a_2$, $b_1 \neq b_2$, then we obtain $c \geq \frac{1}{2}$, which proves the sharpness of the constant $\frac{1}{2}$. □

Remark 1. *Let $\mathcal{B}$ be a Banach algebra over $\mathbb{K}$ $(=\mathbb{R},\mathbb{C})$, $a_i \in \mathcal{B}$ and $\alpha_i \in \mathbb{K}$ $(i = 1, ..., n)$. Then we have the inequality:*

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i^2 - \left(\sum_{i=1}^{n} \alpha_i a_i \right)^2 \right\| \leq \frac{1}{2} \left[\left(\sum_{i=1}^{n} |\alpha_i| \right)^2 - \sum_{i=1}^{n} |\alpha_i|^2 \right] \left(\sum_{i=1}^{n-1} \|\Delta a_i\| \right)^2 . \tag{3.6}$$

The constant $\frac{1}{2}$ is best possible.

The following corollary holds.

Corollary 1. *Under the above assumptions for a_i, b_i $(i = 1, ..., n)$, we have the inequality*

$$\left\| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} b_i \right\| \leq \frac{1}{2} \left(1 - \frac{1}{n} \right) \sum_{i=1}^{n-1} \|\Delta a_i\| \sum_{i=1}^{n-1} \|\Delta b_i\| , \tag{3.7}$$

and the constant $\frac{1}{2}$ is sharp.

In particular, we have

$$\left\| \frac{1}{n} \sum_{i=1}^{n} a_i^2 - \left(\frac{1}{n} \sum_{i=1}^{n} a_i \right)^2 \right\| \leq \frac{1}{2} \left(1 - \frac{1}{n} \right) \left(\sum_{i=1}^{n-1} \|\Delta a_i\| \right)^2 .$$

4. An Inequality of Grüss Type for p-Norm

The following result that provides a version for the p-norm, $p > 1$ of the forward difference also holds.

Theorem 7. *Let $\mathcal{B}$ be a Banach algebra over $\mathbb{K}$ $(=\mathbb{R},\mathbb{C})$, $a_i, b_i \in \mathcal{B}$ and $\alpha_i \in \mathbb{K}$ $(i = 1, ..., n)$. Then we have the inequality:*

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \leq \sum_{1 \leq j < i \leq n} (i - j) |\alpha_i| |\alpha_j| \left(\sum_{k=1}^{n-1} \|\Delta a_k\|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta b_k\|^q \right)^{\frac{1}{q}} , \tag{4.1}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The constant $C = 1$ in the right hand side of (3.1) *is sharp in the sense that it cannot be replaced by a smaller one.*

Proof. From the proof of Theorem 6 we have

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \leq \sum_{1 \leq j < i \leq n} |\alpha_i| |\alpha_j| \sum_{k=j}^{i-1} \|\Delta a_k\| \sum_{l=j}^{i-1} \|\Delta b_l\| =: A. \tag{4.2}$$

Using Hölder's discrete inequality, we can state that

$$\sum_{k=j}^{i-1} \|\Delta a_k\| \leq (i-j)^{\frac{1}{q}} \left(\sum_{k=j}^{i-1} \|\Delta a_k\|^p \right)^{\frac{1}{p}}$$

and

$$\sum_{k=j}^{i-1} \|\Delta b_k\| \leq (i-j)^{\frac{1}{p}} \left(\sum_{k=j}^{i-1} \|\Delta b_k\|^q \right)^{\frac{1}{q}}$$

where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and then, by multiplication, we have

$$A \leq \sum_{1 \leq j < i \leq n} |\alpha_i| |\alpha_j| (i-j) \left(\sum_{k=j}^{i-1} \|\Delta a_k\|^p \right)^{\frac{1}{p}} \left(\sum_{k=j}^{i-1} \|\Delta b_k\|^q \right)^{\frac{1}{q}}. \tag{4.3}$$

As

$$\sum_{k=j}^{i-1} \|\Delta a_k\|^p \leq \sum_{k=1}^{n-1} \|\Delta a_k\|^p$$

and

$$\sum_{k=j}^{i-1} \|\Delta b_k\|^q \leq \sum_{k=1}^{n-1} \|\Delta b_k\|^q,$$

for all $1 \leq j < i \leq n$, then by (4.2) and (4.3), we get the desired inequality (4.1).

To prove the sharpness of the constant, let us assume that (4.1) holds with a constant $C > 0$. That is,

$$\begin{aligned} &\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \\ &\leq C \sum_{1 \leq j < i \leq n} (i-j) |\alpha_i| |\alpha_j| \left(\sum_{k=1}^{n-1} \|\Delta a_k\|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta b_k\|^q \right)^{\frac{1}{q}}. \end{aligned} \tag{4.4}$$

Choose $n = 2$. Then

$$\left\| \sum_{i=1}^{2} \alpha_i \sum_{i=1}^{2} \alpha_i a_i b_i - \sum_{i=1}^{2} \alpha_i a_i \sum_{i=1}^{2} \alpha_i b_i \right\| = |\alpha_1| |\alpha_2| \|a_1 - a_2\| \|b_1 - b_2\|$$

and

$$\begin{aligned} &\sum_{1 \leq j < i \leq 2} (i-j) |\alpha_i| |\alpha_j| \left(\sum_{k=1}^{1} \|\Delta a_k\|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{1} \|\Delta b_k\|^q \right)^{\frac{1}{q}} \\ &= |\alpha_1| |\alpha_2| \|a_1 - a_2\| \|b_1 - b_2\|. \end{aligned}$$

Therefore, from (4.4), we obtain

$$|\alpha_1| |\alpha_2| \|a_1 - a_2\| \|b_1 - b_2\| \leq C |\alpha_1| |\alpha_2| \|a_1 - a_2\| \|b_1 - b_2\|$$

for all $a_1 \neq a_2$, $b_1 \neq b_2$, and then $C \geq 1$, which proves the sharpness of the constant. □

Remark 2. *A coarser upper bound, which can be more useful may be obtained by applying Cauchy-Schwartz's inequality:*

$$\sum_{1\leq j<i\leq n}(i-j)|\alpha_i||\alpha_j| \leq \left(\sum_{1\leq j<i\leq n}|\alpha_i||\alpha_j|\right)^{\frac{1}{2}}\left(\sum_{1\leq j<i\leq n}|\alpha_i||\alpha_j|(i-j)^2\right)^{\frac{1}{2}}$$

and taking into account that

$$\sum_{1\leq j<i\leq n}|\alpha_i||\alpha_j| = \frac{1}{2}\left[\left(\sum_{i=1}^{n}|\alpha_i|\right)^2 - \sum_{i=1}^{n}|\alpha_i|^2\right]$$

and

$$\begin{aligned}\sum_{1\leq j<i\leq n}|\alpha_i||\alpha_j|(i-j)^2 &= \frac{1}{2}\sum_{i,j=1}^{n}|\alpha_i||\alpha_j|(i-j)^2\\ &= \left[\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}i^2|\alpha_i| - \left(\sum_{i=1}^{n}i|\alpha_i|\right)^2\right].\end{aligned}$$

Thus, from (3.1), *we can state the inequality*

$$\begin{aligned}&\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_i a_i b_i - \sum_{i=1}^{n}\alpha_i a_i\sum_{i=1}^{n}\alpha_i b_i\right\| \\ &\leq \frac{\sqrt{2}}{2}\left[\left(\sum_{i=1}^{n}|\alpha_i|\right)^2 - \sum_{i=1}^{n}|\alpha_i|^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}i^2|\alpha_i| - \left(\sum_{i=1}^{n}i|\alpha_i|\right)^2\right]^{\frac{1}{2}} \\ &\times\left(\sum_{k=1}^{n-1}\|\Delta a_k\|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}\|\Delta b_k\|^q\right)^{\frac{1}{q}},\end{aligned} \tag{4.5}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

The following corollary holds.

Corollary 2. *With the above assumptions, we have*

$$\begin{aligned}&\left\|\frac{1}{n}\sum_{i=1}^{n}a_i b_i - \frac{1}{n}\sum_{i=1}^{n}a_i\cdot\frac{1}{n}\sum_{i=1}^{n}b_i\right\| \\ &\leq \frac{n^2-1}{6n}\left(\sum_{k=1}^{n-1}\|\Delta a_k\|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}\|\Delta b_k\|^q\right)^{\frac{1}{q}}\end{aligned} \tag{4.6}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

The constant $\frac{1}{6}$ *is the best possible.*

Proof. The proof follows by (3.1), putting $\alpha_i = \frac{1}{n}$ and taking into account that

$$\begin{aligned}
&\sum_{1\le j<i\le n} (i-j) \\
&= \sum_{1\le j\le 2} (2-j) + \sum_{1\le j\le 3} (3-j) + \dots + \sum_{1\le j\le n} (n-j) \\
&= 2\cdot 2 - (1+2) + 3\cdot 3 - (1+2+3) + \dots + n\cdot n - (1+2+\dots+n) \\
&= 1^2 + 2^2 + \dots + n^2 - 1 - (1+2) - (1+2+3) - \dots - (1+2+\dots+n) \\
&= \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{1}{2}\left(\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k\right) = \frac{n(n^2-1)}{6},
\end{aligned}$$

and the corollary is thus proved. □

Remark 3. *If in* (4.1) *and* (4.5) *we assume that* $p = q = 2$*, then we get the inequalities:*

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \\
&\le \sum_{1\le j<i\le n} (i-j) |\alpha_i| |\alpha_j| \left(\sum_{k=1}^{n-1} \|\Delta a_k\|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n-1} \|\Delta b_k\|^2\right)^{\frac{1}{2}}
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&\left\| \frac{1}{n}\sum_{i=1}^{n} a_i b_i - \frac{1}{n}\sum_{i=1}^{n} a_i \cdot \frac{1}{n}\sum_{i=1}^{n} b_i \right\| \\
&\le \frac{n^2-1}{6n} \left(\sum_{k=1}^{n-1} \|\Delta a_k\|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n-1} \|\Delta b_k\|^2\right)^{\frac{1}{2}},
\end{aligned} \tag{4.8}$$

respectively.

We also have the inequality

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \\
&\le \frac{\sqrt{2}}{2} \left[\left(\sum_{i=1}^{n} |\alpha_i|\right)^2 - \sum_{i=1}^{n} |\alpha_i|^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} |\alpha_i| \sum_{i=1}^{n} i^2 |\alpha_i| - \left(\sum_{i=1}^{n} i |\alpha_i|\right)^2\right]^{\frac{1}{2}} \\
&\times \left(\sum_{k=1}^{n-1} \|\Delta a_k\|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n-1} \|\Delta b_k\|^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.9}$$

In the case when $b_i = a_i$, $i \in \{1, \dots, n\}$ *we get from (4.7)*

$$\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i^2 - \left(\sum_{i=1}^{n} \alpha_i a_i\right)^2 \right\| \le \sum_{1\le j<i\le n} (i-j) |\alpha_i| |\alpha_j| \left(\sum_{k=1}^{n-1} \|\Delta a_k\|^2\right) \tag{4.10}$$

and from (4.9)

$$\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_i a_i^2-\left(\sum_{i=1}^{n}\alpha_i a_i\right)^2\right\| \tag{4.11}$$
$$\leq\frac{\sqrt{2}}{2}\left[\left(\sum_{i=1}^{n}|\alpha_i|\right)^2-\sum_{i=1}^{n}|\alpha_i|^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}i^2|\alpha_i|-\left(\sum_{i=1}^{n}i|\alpha_i|\right)^2\right]^{\frac{1}{2}}$$
$$\times\left(\sum_{k=1}^{n-1}\|\Delta a_k\|^2\right).$$

5. Inequalities for Power Series

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \lambda\in D(0,1); \tag{5.1}$$
$$g(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \lambda\in\mathbb{C};$$
$$h(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \lambda\in\mathbb{C};$$
$$l(\lambda)=\sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \lambda\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_A(\lambda)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n=\ln\frac{1}{1-\lambda},\ \lambda\in D(0,1); \tag{5.2}$$
$$g_A(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}=\cosh\lambda,\ \lambda\in\mathbb{C};$$
$$h_A(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}=\sinh\lambda,\ \lambda\in\mathbb{C};$$
$$l_A(\lambda)=\sum_{n=0}^{\infty}\lambda^n=\frac{1}{1-\lambda},\ \lambda\in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n \qquad \lambda \in \mathbb{C}, \tag{5.3}$$
$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1);$$
$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \qquad \lambda \in D(0,1);$$
$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1)$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function*.

The following new result holds:

Theorem 8. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series that is convergent on the open disk $D(0,R)$, with $R > 0$. If $x, y \in \mathcal{B}$ with $xy = yx$ and $\|x\|, \|y\| < 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ the inequality:*

$$\left\|\widetilde{f}(\lambda \cdot 1)\widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x)\widetilde{f}(\lambda y)\right\| \tag{5.4}$$
$$\leq \frac{1}{2}\frac{\|x-1\|\|y-1\|}{(1-\|x\|)(1-\|y\|)}\left[f_A^2(|\lambda|) - f_{A^2}\left(|\lambda|^2\right)\right],$$

where

$$f_{A^2}(\lambda) := \sum_{n=0}^{\infty} |\alpha_n|^2 \lambda^n \tag{5.5}$$

has the radius of convergence R^2.

Proof. From the inequality (3.1) we have

$$\left\|\sum_{i=0}^{n}\alpha_i\lambda^i\sum_{i=0}^{n}\alpha_i\lambda^i(xy)^i - \sum_{i=0}^{n}\alpha_i\lambda^i x^i\sum_{i=0}^{n}\alpha_i\lambda^i y^i\right\| \tag{5.6}$$
$$\leq \frac{1}{2}\left[\left(\sum_{i=0}^{n}|\alpha_i||\lambda|^i\right)^2 - \sum_{i=0}^{n}|\alpha_i|^2|\lambda|^{2i}\right]\sum_{j=0}^{n-1}\left\|x^{j+1}-x^j\right\|\sum_{j=0}^{n-1}\left\|y^{j+1}-y^j\right\|,$$
$$= \frac{1}{2}\left[\left(\sum_{i=0}^{n}|\alpha_i||\lambda|^i\right)^2 - \sum_{i=0}^{n}|\alpha_i|^2|\lambda|^{2i}\right]\sum_{j=0}^{n-1}\left\|x^j(x-1)\right\|\sum_{j=0}^{n-1}\left\|y^j(y-1)\right\|$$

$$
\begin{aligned}
&\leq \frac{1}{2}\|x-1\|\|y-1\|\left[\left(\sum_{i=0}^{n}|\alpha_i||\lambda|^i\right)^2-\sum_{i=0}^{n}|\alpha_i|^2|\lambda|^{2i}\right]\sum_{j=0}^{n-1}\|x\|^j\sum_{j=0}^{n-1}\|y\|^j \\
&= \frac{1}{2}\|x-1\|\|y-1\|\left[\left(\sum_{i=0}^{n}|\alpha_i||\lambda|^i\right)^2-\sum_{i=0}^{n}|\alpha_i|^2|\lambda|^{2i}\right] \\
&\times \frac{1-\|x\|^n}{1-\|x\|}\frac{1-\|y\|^n}{1-\|y\|}
\end{aligned}
$$

for any $n \geq 1$.

Since all the series whose partial sums are involved in (5.6) are convergent, then by letting $n \to \infty$ in (5.6) we deduce the desired inequality (5.4). □

Corollary 3. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series that is convergent on the open disk $D(0, R)$, with $R > 0$. If $x \in \mathcal{B}$ and $\|x\| < 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ the inequality:*

$$
\left\|\widetilde{f}(\lambda\cdot 1)\widetilde{f}\left(\lambda x^2\right)-\left[\widetilde{f}(\lambda x)\right]^2\right\| \leq \frac{1}{2}\frac{\|x-1\|^2}{(1-\|x\|)^2}\left[f_A^2(|\lambda|)-f_{A^2}\left(|\lambda|^2\right)\right]. \tag{5.7}
$$

We have the following result as well:

Theorem 9. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series that is convergent on the open disk $D(0, R)$, with $R > 0$. If $x, y \in \mathcal{B}$ with $xy = yx$ and $\|x\|, \|y\| < 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ the inequality:*

$$
\begin{aligned}
&\left\|\widetilde{f}(\lambda\cdot 1)\widetilde{f}(\lambda xy)-\widetilde{f}(\lambda x)\widetilde{f}(\lambda y)\right\| \\
&\leq \frac{\sqrt{2}}{2}\frac{\|x-1\|\|y-1\|}{(1-\|x\|^p)^{\frac{1}{p}}(1-\|y\|^q)^{\frac{1}{q}}}\left[f_A^2(|\lambda|)-f_{A^2}\left(|\lambda|^2\right)\right]^{1/2} \\
&\times\left\{f_A(|\lambda|)\left[|\lambda|f_A'(|\lambda|)+|\lambda|^2 f_A''(|\lambda|)\right]-\left[|\lambda|f_A'(|\lambda|)\right]^2\right\}^{1/2}
\end{aligned} \tag{5.8}
$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, we have

$$
\begin{aligned}
&\left\|\widetilde{f}(\lambda\cdot 1)\widetilde{f}\left(\lambda x^2\right)-\left[\widetilde{f}(\lambda x)\right]^2\right\| \\
&\leq \frac{\sqrt{2}}{2}\frac{\|x-1\|^2}{(1-\|x\|^p)^{\frac{1}{p}}(1-\|x\|^q)^{\frac{1}{q}}}\left[f_A^2(|\lambda|)-f_{A^2}\left(|\lambda|^2\right)\right]^{1/2} \\
&\times\left\{f_A(|\lambda|)\left[|\lambda|f_A'(|\lambda|)+|\lambda|^2 f_A''(|\lambda|)\right]-\left[|\lambda|f_A'(|\lambda|)\right]^2\right\}^{1/2}.
\end{aligned} \tag{5.9}
$$

Proof. Using the inequality (4.5) we have for $n \geq 1$ that

$$\left\| \sum_{i=0}^{n} \alpha_i \lambda^i \sum_{i=0}^{n} \alpha_i \lambda^i (xy)^i - \sum_{i=0}^{n} \alpha_i \lambda^i x^i \sum_{i=0}^{n} \alpha_i \lambda^i y^i \right\| \tag{5.10}$$
$$\leq \frac{\sqrt{2}}{2} \left[\left(\sum_{i=0}^{n} |\alpha_i| |\lambda|^i \right)^2 - \sum_{i=0}^{n} |\alpha_i|^2 |\lambda|^{2i} \right]^{1/2}$$
$$\times \left[\sum_{i=0}^{n} |\alpha_i| |\lambda|^i \sum_{i=0}^{n} i^2 |\alpha_i| |\lambda|^i - \left(\sum_{i=0}^{n} i |\alpha_i| |\lambda|^i \right)^2 \right]^{\frac{1}{2}}$$
$$\times \left(\sum_{j=0}^{n-1} \left\| x^{j+1} - x^j \right\|^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{n-1} \left\| y^{j+1} - y^j \right\|^q \right)^{\frac{1}{q}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Observe that

$$\sum_{j=0}^{n-1} \left\| x^{j+1} - x^j \right\|^p \leq \sum_{j=0}^{n-1} \left\| x^j (x-1) \right\|^p \leq \|x-1\|^p \sum_{j=0}^{n-1} \left\| x^j \right\|^p$$
$$\leq \|x-1\|^p \sum_{j=0}^{n-1} \|x\|^{jp} = \|x-1\|^p \frac{1 - \|x\|^{np}}{1 - \|x\|^p},$$

which implies that

$$\left(\sum_{j=0}^{n-1} \left\| x^{j+1} - x^j \right\|^p \right)^{\frac{1}{p}} \leq \|x-1\| \left(\frac{1 - \|x\|^{np}}{1 - \|x\|^p} \right)^{\frac{1}{p}}.$$

Similarly,

$$\left(\sum_{j=0}^{n-1} \left\| y^{j+1} - y^j \right\|^q \right)^{\frac{1}{q}} \leq \|y-1\| \left(\frac{1 - \|y\|^{nq}}{1 - \|y\|^q} \right)^{\frac{1}{q}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

From (5.10) we get

$$\left\| \sum_{i=0}^{n} \alpha_i \lambda^i \sum_{i=0}^{n} \alpha_i \lambda^i (xy)^i - \sum_{i=0}^{n} \alpha_i \lambda^i x^i \sum_{i=0}^{n} \alpha_i \lambda^i y^i \right\| \tag{5.11}$$
$$\leq \frac{\sqrt{2}}{2} \left[\left(\sum_{i=0}^{n} |\alpha_i| |\lambda|^i \right)^2 - \sum_{i=0}^{n} |\alpha_i|^2 |\lambda|^{2i} \right]^{1/2}$$
$$\times \left[\sum_{i=0}^{n} |\alpha_i| |\lambda|^i \sum_{i=0}^{n} i^2 |\alpha_i| |\lambda|^i - \left(\sum_{i=0}^{n} i |\alpha_i| |\lambda|^i \right)^2 \right]^{\frac{1}{2}}$$
$$\times \|x-1\| \|y-1\| \left(\frac{1 - \|x\|^{np}}{1 - \|x\|^p} \right)^{\frac{1}{p}} \left(\frac{1 - \|y\|^{nq}}{1 - \|y\|^q} \right)^{\frac{1}{q}},$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1$.

If we denote $g(u):=\sum_{n=0}^{\infty}\alpha_n u^n$, then for $|u|<R$ we have

$$\sum_{n=0}^{\infty} n\alpha_n u^n = ug'(u)$$

and

$$\sum_{n=0}^{\infty} n^2\alpha_n u^n = u\left(ug'(u)\right)'.$$

However

$$u\left(ug'(u)\right)' = ug'(u)+u^2 g''(u)$$

and then

$$\sum_{n=0}^{\infty} n^2\alpha_n u^n = ug'(u)+u^2 g''(u).$$

Therefore

$$\sum_{n=0}^{\infty} n^2|\alpha_n||\lambda|^n = |\lambda| f_A'(|\lambda|)+|\lambda|^2 f_A''(|\lambda|)$$

and

$$\sum_{n=0}^{\infty} n|\alpha_n||\lambda|^n = |\lambda| f_A'(|\lambda|)$$

for $|\lambda|<R$.

Since all the series whose partial sums are involved in (5.11) are convergent, then by letting $n\to\infty$ in (5.11) we deduce the desired inequality (5.8). □

Corollary 4. *With the assumptions of Theorem 9, we have*

$$\begin{aligned}(5.12)\qquad &\left\|\widetilde{f}(\lambda\cdot 1)\widetilde{f}(\lambda xy)-\widetilde{f}(\lambda x)\widetilde{f}(\lambda y)\right\| \\ &\le \frac{\sqrt{2}}{2}\frac{\|x-1\|\|y-1\|}{\left(1-\|x\|^2\right)^{\frac{1}{2}}\left(1-\|y\|^2\right)^{\frac{1}{2}}}\left[f_A^2(|\lambda|)-f_{A^2}\left(|\lambda|^2\right)\right]^{1/2} \\ &\times\left\{f_A(|\lambda|)\left[|\lambda| f_A'(|\lambda|)+|\lambda|^2 f_A''(|\lambda|)\right]-\left[|\lambda| f_A'(|\lambda|)\right]^2\right\}^{1/2}\end{aligned}$$

and, in particular

$$\begin{aligned}(5.13)\qquad &\left\|\widetilde{f}(\lambda\cdot 1)\widetilde{f}\left(\lambda x^2\right)-\left[\widetilde{f}(\lambda x)\right]^2\right\| \\ &\le \frac{\sqrt{2}}{2}\frac{\|x-1\|^2}{1-\|x\|^2}\left[f_A^2(|\lambda|)-f_{A^2}\left(|\lambda|^2\right)\right]^{1/2} \\ &\times\left\{f_A(|\lambda|)\left[|\lambda| f_A'(|\lambda|)+|\lambda|^2 f_A''(|\lambda|)\right]-\left[|\lambda| f_A'(|\lambda|)\right]^2\right\}^{1/2}.\end{aligned}$$

6. Some Particular Cases of Interest

Consider the function $f : D(0,1) \to \mathbb{C}$ defined by

$$f(\lambda) = (1-\lambda)^{-1} = \sum_{k=0}^{\infty} \lambda^k = f_A(\lambda).$$

Then

$$f_{A^2}(\lambda) := \sum_{n=0}^{\infty} \lambda^n = (1-\lambda)^{-1},$$

which implies that

$$f_A^2(|\lambda|) - f_{A^2}\left(|\lambda|^2\right) = \frac{2|\lambda|}{(1-|\lambda|)^2(1+|\lambda|)}, \ \ |\lambda| < 1$$

and by (5.4), we have for $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ that

$$\left\|(1-\lambda)^{-1}(1-\lambda xy)^{-1} - (1-\lambda x)^{-1}(1-\lambda y)^{-1}\right\| \leq \frac{|\lambda|\|x-1\|\|y-1\|}{(1-|\lambda|)^2(1+|\lambda|)(1-\|x\|)(1-\|y\|)}. \tag{6.1}$$

We also have for $|\lambda|, \|x\| < 1$ that

$$\left\|(1-\lambda)^{-1}\left(1-\lambda x^2\right)^{-1} - (1-\lambda x)^{-2}\right\| \leq \frac{|\lambda|\|x-1\|^2}{(1-|\lambda|)^2(1+|\lambda|)(1-\|x\|)^2}. \tag{6.2}$$

For the function $f(\lambda) = (1-\lambda)^{-1}$ we have

$$\begin{aligned} & f_A(|\lambda|)\left[|\lambda| f_A'(|\lambda|) + |\lambda|^2 f_A''(|\lambda|)\right] - \left[|\lambda| f_A'(|\lambda|)\right]^2 \\ & = \frac{1}{1-|\lambda|}\left[\frac{|\lambda|}{(1-|\lambda|)^2} + \frac{2|\lambda|^2}{(1-|\lambda|)^3}\right] - \frac{|\lambda|^2}{(1-|\lambda|)^4} \\ & = \frac{1}{(1-|\lambda|)^4}. \end{aligned}$$

From the inequality (5.8) we then have for $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ that

$$\begin{aligned} & \left\|(1-\lambda)^{-1}(1-\lambda xy)^{-1} - (1-\lambda x)^{-1}(1-\lambda y)^{-1}\right\| \\ & \leq \frac{\sqrt{2}}{2} \frac{\|x-1\|\|y-1\|}{(1-\|x\|^p)^{\frac{1}{p}}(1-\|y\|^q)^{\frac{1}{q}}}\left[\frac{2|\lambda|}{(1-|\lambda|)^2(1+|\lambda|)},\right]^{1/2} \\ & \times \left\{\frac{1}{(1-|\lambda|)^4}\right\}^{1/2}, \end{aligned}$$

which is equivalent to

$$\left\|(1-\lambda)^{-1}(1-\lambda xy)^{-1} - (1-\lambda x)^{-1}(1-\lambda y)^{-1}\right\| \leq \frac{|\lambda|^{1/2}\|x-1\|\|y-1\|}{(1-\|x\|^p)^{\frac{1}{p}}(1-\|y\|^q)^{\frac{1}{q}}(1-|\lambda|)^3(1+|\lambda|)^{1/2}}, \tag{6.3}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

If we consider the function

$$f(\lambda) = (1+\lambda)^{-1} = \sum_{k=0}^{\infty} (-1)^k \lambda^k,$$

then the inequalities (6.1)-(6.3) also holds with " $+$ " instead of " $-$ " in the left hand side expressions such as $(1-\lambda x)^{-1}$ etc.

We consider the *modified Bessel function functions of the first kind*

$$I_\nu(\lambda) := \left(\frac{1}{2}\lambda\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}\lambda^2\right)^k}{k!\Gamma(\nu+k+1)}, \ \lambda \in C$$

where Γ is the *Gamma function* and ν is a real number. An integral formula to represent I_ν is

$$I_\nu(\lambda) = \frac{1}{\pi}\int_0^\pi e^{\lambda\cos\theta}\cos(\nu\theta) - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\lambda\cosh t - \nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(\lambda) = \frac{1}{\pi}\int_0^\pi e^{\lambda\cos\theta}\cos(n\theta)\,d\theta.$$

For $n = 0$ we have

$$I_0(\lambda) = \frac{1}{\pi}\int_0^\pi e^{\lambda\cos\theta}d\theta = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}\lambda^2\right)^k}{(k!)^2}, \ \lambda \in C.$$

Now, if we consider the exponential function

$$f(\lambda) = \exp(\lambda) = \sum_{k=0}^{\infty}\frac{1}{k!}\lambda^k,$$

then for $\rho > 0$ we have

$$f_{A^2}(\rho) = \sum_{k=0}^{\infty}\frac{1}{(k!)^2}\rho^k = I_0(2\sqrt{\rho}),$$

which implies that

$$f_A^2(|\lambda|) - f_{A^2}\left(|\lambda|^2\right) = \exp(2|\lambda|) - I_0(2|\lambda|), \ \lambda \in C.$$

Making use of the inequality (5.4), we have for $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$ and $\lambda \in \mathbb{C}$ that

$$\begin{aligned} &\|\exp(\lambda(xy+1)) - \exp(\lambda(x+y))\| \\ &\le \frac{1}{2}\frac{\|x-1\|\,\|y-1\|}{(1-\|x\|)(1-\|y\|)}\left[\exp(2|\lambda|) - I_0(2|\lambda|)\right], \end{aligned} \tag{6.4}$$

In particular, we have for $\|x\| < 1$

$$\left\|\exp\left(\lambda\left(x^2+1\right)\right) - \exp(2\lambda x)\right\| \le \frac{1}{2}\frac{\|x-1\|^2}{(1-\|x\|)^2}\left[\exp(2|\lambda|) - I_0(2|\lambda|)\right] \tag{6.5}$$

for any $\lambda \in \mathbb{C}$.

For $f(\lambda) = \exp(\lambda)$ we have

$$f_A(|\lambda|)\left[|\lambda| f_A'(|\lambda|) + |\lambda|^2 f_A''(|\lambda|)\right] - \left[|\lambda| f_A'(|\lambda|)\right]^2 = |\lambda|\exp(2|\lambda|).$$

If $x,y\in\mathcal{B}$ with $xy=yx$ and $\|x\|,\|y\|<1$, then from (5.8) we have for $\lambda\in\mathbb{C}$ the inequality:

$$\begin{aligned}(6.6)\qquad &\|\exp(\lambda(xy+1))-\exp(\lambda(x+y))\|\\ &\le\frac{\sqrt{2}}{2}\frac{\|x-1\|\,\|y-1\|}{(1-\|x\|^p)^{\frac{1}{p}}(1-\|y\|^q)^{\frac{1}{q}}}|\lambda|^{1/2}\exp(|\lambda|)\left[\exp(2|\lambda|)-I_0(2|\lambda|)\right]^{1/2},\end{aligned}$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1$.

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