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REFINING LAH-RIBARIĆ INTEGRAL INEQUALITY FOR DIVISIONS OF MEASURABLE SPACE

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Abstract. In this paper we establish some refinements of Lah-Ribarić inequality for the general Lebesgue integral on divisions of measurable space. Applications for discrete inequalities and weighted means of positive numbers are also given. Some examples related to Hermite-Hadamard inequality for convex functions are provided as well.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For the μ-integrable positive μ-a.e. weight w consider the Lebesgue space

$$L_w(\Omega,\mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, w(t)\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$ etc...

We say that the family of measurable sets $F_n(\Omega) = \{\Omega_i\}_{i\in\{1,\dots,n\}}$ is a n-division for Ω if $\Omega = \bigcup_{i=1}^n \Omega_i$ and $\Omega_i\cap\Omega_j = \emptyset$ for any $i, j \in \{1, ..., n\}$ with $i \neq j$ and $\mu(\Omega_i) > 0$ for any $i \in \{1, ..., n\}$. In this situation, if $f \in L_w(\Omega,\mu)$ then $f \in L_w(\Omega_i,\mu)$ for any $i \in \{1, ..., n\}$ and $\int_\Omega f w d\mu = \sum_{i=1}^n \int_{\Omega_i} f w d\mu$. Also, $\int_\Omega w d\mu = \sum_{i=1}^n \int_{\Omega_i} w d\mu$ with $\int_{\Omega_i} w d\mu > 0$ for any $i \in \{1, ..., n\}$.

For a given $n \geq 2$ we denote by $\mathfrak{D}_n(\Omega)$ the set of all n-divisions of Ω and consider the functional $\psi(\Phi, w, f, \cdot) : \mathfrak{D}_n(\Omega) \to \mathbb{R}$ defined by

$$\psi(\Phi, f, w, F_n(\Omega)) := \frac{1}{\int_\Omega w d\mu} \sum_{i=1}^n \Phi\left(\frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu}\right) \int_{\Omega_i} w d\mu. \tag{1.1}$$

The following result has been obtained in [14].

Theorem 1. *Let $\Phi : [m, M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m, M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega,\mu)$. Then for any $F_n(\Omega) \in \mathfrak{D}_n(\Omega)$ with $\int_{\Omega_i} w d\mu > 0$ for any $i \in \{1, ..., n\}$ we have*

$$\frac{\int_\Omega (\Phi \circ f)\, w d\mu}{\int_\Omega w d\mu} \geq \psi(\Phi, f, w, F_n(\Omega)) \geq \Phi\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right), \tag{1.2}$$

where $n \geq 2$.

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For a nonempty finite family of indices J and positive weights w_j, $j \in J$ we denote $W_J := \sum_{j\in J} w_j$. If $\Phi : [m, M] \to \mathbb{R}$ is a convex function and $x_j \in [m, M]$, $j \in J$ then Jensen's inequality states that

$$\frac{1}{W_J}\sum_{j\in J} w_j \Phi(x_j) \geq \Phi\left(\frac{1}{W_J}\sum_{j\in J} w_j x_j\right).$$

Assume that, for $n \geq 2$, the family J of indices containing more than n elements and $F_n(J) = \{J_i\}_{i\in\{1,...,n\}}$ is a n-division for J, namely $J = \bigcup_{i=1}^n J_i$ and $J_i \cap J_j = \emptyset$ for any $i, j \in \{1, ..., n\}$ with $i \neq j$.

For a given $n \geq 2$ we denote by $\mathfrak{D}_n(J)$ the set of all n-divisions of J and consider the functional $\psi(\Phi, f, \cdot) : \mathfrak{D}_n(J) \to \mathbb{R}$ defined by

$$\psi(\Phi, f, w, F_n(J)) := \frac{1}{W_J}\sum_{i=1}^n W_{J_i}\Phi\left(\frac{1}{W_{J_i}}\sum_{j\in J_i} w_j x_j\right).$$

From the inequality (1.2) for the discrete measure we have

$$\frac{1}{W_J}\sum_{j\in J} w_j \Phi(x_j) \geq \frac{1}{W_J}\sum_{i=1}^n W_{J_i}\Phi\left(\frac{1}{W_{J_i}}\sum_{j\in J_i} w_j x_j\right) \geq \Phi\left(\frac{1}{W_J}\sum_{j\in J} w_j x_j\right), \tag{1.3}$$

for any $F_n(J) \in \mathfrak{D}_n(J)$.

The following reverse of Jensen's inequality is known in the literature as *Lah-Ribarić inequality* [20]:

$$\frac{\int_\Omega (\Phi\circ f)\, w d\mu}{\int_\Omega w d\mu} \leq \frac{1}{M-m}\left[\left(M - \frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right)\Phi(m) + \left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} - m\right)\Phi(M)\right] \tag{1.4}$$

provided $\Phi : [m, M] \to \mathbb{R}$ is a convex function, $f : \Omega \to [m, M]$ is a μ-measurable function and such that f, $\Phi \circ f \in L_w(\Omega, \mu)$.

For other results and applications related to Ky Fan's inequality, the arithmetic mean-geometric mean inequality, the generalized triangle inequality, the f-Divergence measure etc., see [1], [3]-[16], [17]-[19] and [22]-[23].

Motivated by the above results we establish in this paper some refinements of Lah-Ribarić inequality for the general Lebesgue integral on divisions of measurable space. Applications for discrete inequalities and weighted means of positive numbers are also given. Some examples related to Hermite-Hadamard inequality for convex functions are provided as well.

2. The Results

Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ and for $a, b \in I$ with $a < b$ consider the function $\Delta(\Phi; a, b, \cdot) : \mathbb{R} \to \mathbb{R}$ defined by

$$\Delta(\Phi; a, b, t) = \frac{(b-t)\Phi(a) + (t-a)\Phi(b)}{b-a}.$$

This is the straight line that connects the points $(a, \Phi(a))$ and $(b, \Phi(b))$.

The following lemma holds:

Lemma 1. *Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b, c, d \in I$ with $a < c < d < b$. Then*

$$\Phi(t) \le \Delta(\Phi; c, d, t) \le \Delta(\Phi; a, b, t) \tag{2.1}$$

for any $t \in [c, d]$.

Proof. By the convexity of Φ we have for any $t \in [c, d]$ that

$$\begin{aligned} \Delta(\Phi; c, d, t) - \Phi(t) &= \frac{(d-t)\Phi(c) + (t-c)\Phi(d)}{d-c} - \Phi(t) \\ &= \frac{(d-t)\Phi(c) + (t-c)\Phi(d)}{d-c} - \Phi\left(\frac{(d-t)c + (t-c)d}{d-c}\right) \\ &\ge 0. \end{aligned}$$

We observe that for $t \in [a, b]$,

$$y = \frac{(b-t)\Phi(a) + (t-a)\Phi(b)}{b-a}$$

is the equation of the segment joining the points $(a, \Phi(a))$ and $(b, \Phi(b))$ while

$$y = \frac{(d-t)\Phi(c) + (t-c)\Phi(d)}{d-c},\ t \in [c, d]$$

is the equation of the segment joining the points $(c, \Phi(c))$ and $(d, \Phi(d))$.

Since the function Φ is convex on I the segment on the smaller interval $[c, d]$ is under the segment on the larger interval $[a, b]$ containing $[c, d]$.

These prove the desired inequality (2.1). □

For a division $F_n(\Omega) = \{\Omega_i\}_{i \in \{1,...,n\}} \in \mathfrak{D}_n(\Omega)$ and the measurable essentially bounded function $f : \Omega \to \mathbb{R}$ we denote $M_i := \operatorname{essup}_{x \in \Omega_i} f(x) < \infty$ and $m_i := \operatorname{essinf}_{x \in \Omega_i} f(x) > -\infty$. We also consider

$$M := \operatorname*{essup}_{x \in \Omega} f(x) < \infty \text{ and } m := \operatorname*{essinf}_{x \in \Omega} f(x) > -\infty.$$

Obviously, $M \ge M_i$ and $m \le m_i$ for any $i \in \{1, ..., n\}$.

We assume in what follows that $M_i > m_i$ for any $i \in \{1, ..., n\}$.

We define the functional

$$\begin{aligned} &\sigma(\Phi, f, w, F_n(\Omega)) \\ &:= \frac{1}{\int_\Omega w d\mu} \sum_{i=1}^n \left(\int_{\Omega_i} w d\mu\right) \Delta\left(\Phi; m_i, M_i, \frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu}\right) \\ &= \frac{1}{\int_\Omega w d\mu} \sum_{i=1}^n \left(\int_{\Omega_i} w d\mu\right) \\ &\times \frac{1}{M_i - m_i}\left[\left(M_i - \frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu}\right)\Phi(m_i) + \left(\frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu} - m_i\right)\Phi(M_i)\right]. \end{aligned} \tag{2.2}$$

Observe also that

$$\begin{aligned} &\Delta\left(\Phi; m, M, \frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right) \\ &= \frac{1}{M - m}\left[\left(M - \frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right)\Phi(m) + \left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} - m\right)\Phi(M)\right]. \end{aligned}$$

We have the following refinement of Lah-Ribarić inequality:

Theorem 2. *Let* $\Phi : [m, M] \to \mathbb{R}$ *be a convex function,* $f : \Omega \to [m, M]$ *a* μ*-measurable function such that* $f,\ \Phi \circ f \in L_w(\Omega, \mu)$*. Then for any* $F_n(\Omega) = \{\Omega_i\}_{i\in\{1,\dots,n\}} \in \mathfrak{D}_n(\Omega)$ *we have*

$$\frac{\int_\Omega w(\Phi \circ f)\, d\mu}{\int_\Omega w d\mu} \le \sigma(\Phi, f, w, F_n(\Omega)) \le \Delta\left(\Phi; m, M, \frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right). \tag{2.3}$$

Proof. From the second inequality (2.1) we have for

$$t = \frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu} \in [m_i, M_i],\ i \in \{1, \dots, n\}$$

that

$$\begin{aligned} &\Delta\left(\Phi; m_i, M_i, \frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu}\right) \\ &\le \Delta\left(\Phi; m, M, \frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu}\right) \\ &= \frac{1}{M-m}\left[\left(M - \frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu}\right)\Phi(m) + \left(\frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu} - m\right)\Phi(M)\right] \end{aligned} \tag{2.4}$$

for any $i \in \{1, \dots, n\}$.

If we multiply by $\int_{\Omega_i} w d\mu > 0$ and sum over i from 1 to n we get

$$\begin{aligned} &\sum_{i=1}^n \left(\int_{\Omega_i} w d\mu\right) \Delta\left(\Phi; m_i, M_i, \frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu}\right) \\ &\qquad \le \frac{1}{M-m}\left[\left(M \sum_{i=1}^n \int_{\Omega_i} w d\mu - \sum_{i=1}^n \int_{\Omega_i} f w d\mu\right)\Phi(m)\right. \\ &\qquad\qquad \left. + \left(\sum_{i=1}^n \int_{\Omega_i} f w d\mu - m \sum_{i=1}^n \int_{\Omega_i} w d\mu\right)\Phi(M)\right] \end{aligned}$$

that is equivalent to the second inequality in (2.3).

For μ-almost every $x \in \Omega_i$ we have $f(x) \in [m_i, M_i]$ and then by the first inequality in (2.1) we have

$$\Phi(f(x)) \le \Delta(\Phi; m_i, M_i, f(x))$$

namely,

$$\Phi(f(x)) \le \frac{1}{M_i - m_i}\left[(M_i - f(x))\Phi(m_i) + (f(x) - m_i)\Phi(M_i)\right] \tag{2.5}$$

μ-almost every $x \in \Omega_i$ and for any $i \in \{1, \dots, n\}$.

If we multiply by $w \geq 0$ μ-almost everywhere and integrate on Ω_i we get

$$\int_{\Omega_i} w\left(\Phi\circ f\right)d\mu \leq \frac{1}{M_i - m_i} \times\left[\left(M_i\int_{\Omega_i} wd\mu - \int_{\Omega_i} fwd\mu\right)\Phi\left(m_i\right) + \left(\int_{\Omega_i} fwd\mu - m_i\int_{\Omega_i} wd\mu\right)\Phi\left(M_i\right)\right] = \frac{\int_{\Omega_i} wd\mu}{M_i - m_i}\left[\left(M_i - \frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\right)\Phi\left(m_i\right) + \left(\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu} - m_i\right)\Phi\left(M_i\right)\right] \tag{2.6}$$

for any $i \in \{1, ..., n\}$.

Now, if we sum the inequality (2.6) over i from 1 to n we get the first inequality in (2.3). □

The following lemma holds:

Lemma 2. *Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b, c, d \in I$ with $a < c < d < b$. Then*

$$0 \leq \left[\Delta\left(\Phi; c, d, t\right) - \Phi\left(t\right)\right]\left(d-c\right) \leq \left[\Delta\left(\Phi; a, b, t\right) - \Phi\left(t\right)\right]\left(b-a\right) \tag{2.7}$$

for any $t \in [c, d]$.

Proof. We observe that for any $t \in (c, d)$ we also have

$$\begin{aligned}\Delta\left(\Phi; c, d, t\right) - \Phi\left(t\right) &= \frac{\left(d-t\right)\Phi\left(c\right) + \left(t-c\right)\Phi\left(d\right)}{d-c} - \Phi\left(t\right) \\ &= \frac{\left(d-t\right)\Phi\left(c\right) + \left(t-c\right)\Phi\left(d\right) - \left(d-c\right)\Phi\left(t\right)}{d-c} \\ &= \frac{\left(d-t\right)\Phi\left(c\right) + \left(t-c\right)\Phi\left(d\right) - \left(d-t+t-c\right)\Phi\left(t\right)}{d-c} \\ &= \frac{\left(t-c\right)\left(\Phi\left(d\right) - \Phi\left(t\right)\right) - \left(d-t\right)\left(\Phi\left(t\right) - \Phi\left(c\right)\right)}{d-c} \\ &= \frac{\left(t-c\right)\left(d-t\right)}{d-c}\left(\frac{\Phi\left(d\right) - \Phi\left(t\right)}{d-t} - \frac{\Phi\left(t\right) - \Phi\left(c\right)}{t-c}\right)\end{aligned}$$

giving that

$$\left[\Delta\left(\Phi; c, d, t\right) - \Phi\left(t\right)\right]\left(d-c\right) = \left(t-c\right)\left(d-t\right)\left(\frac{\Phi\left(d\right) - \Phi\left(t\right)}{d-t} - \frac{\Phi\left(t\right) - \Phi\left(c\right)}{t-c}\right). \tag{2.8}$$

Similarly we have

$$\left[\Delta\left(\Phi; a, b, t\right) - \Phi\left(t\right)\right]\left(b-a\right) = \left(t-a\right)\left(b-t\right)\left(\frac{\Phi\left(b\right) - \Phi\left(t\right)}{b-t} - \frac{\Phi\left(t\right) - \Phi\left(a\right)}{t-a}\right), \tag{2.9}$$

for any $t \in I$.

It is know that, since $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function, then for any $\alpha \in I$ the function $\psi : I \setminus \{\alpha\} \to \mathbb{R}$,

$$\psi\left(s\right) := \frac{\Phi\left(s\right) - \Phi\left(\alpha\right)}{s-\alpha}$$

is monotonic nondecreasing on $I \setminus \{\alpha\}$.

Then for $t \in (c,d)$ we have

$$\frac{\Phi(d)-\Phi(t)}{d-t} \leq \frac{\Phi(b)-\Phi(t)}{b-t}$$

and

$$\frac{\Phi(t)-\Phi(c)}{t-c} = \frac{\Phi(c)-\Phi(t)}{c-t} \geq \frac{\Phi(a)-\Phi(t)}{a-t} = \frac{\Phi(t)-\Phi(a)}{t-a}$$

giving that

$$\frac{\Phi(d)-\Phi(t)}{d-t} - \frac{\Phi(t)-\Phi(c)}{t-c} \leq \frac{\Phi(b)-\Phi(t)}{b-t} - \frac{\Phi(t)-\Phi(a)}{t-a} \tag{2.10}$$

for any $t \in (c,d)$.

We also have

$$0 \leq (t-c)(d-t) \leq (t-a)(b-t) \tag{2.11}$$

for any $t \in (c,d)$.

Therefore, by (2.10) and (2.11) we get

$$\begin{aligned}(t-c)(d-t)&\left(\frac{\Phi(d)-\Phi(t)}{d-t} - \frac{\Phi(t)-\Phi(c)}{t-c}\right)\\ &\leq (t-a)(b-t)\left(\frac{\Phi(b)-\Phi(t)}{b-t} - \frac{\Phi(t)-\Phi(a)}{t-a}\right)\end{aligned} \tag{2.12}$$

for any $t \in (c,d)$.

If $t = c$ then (2.7) becomes

$$0 \leq \Delta(\Phi; a, b, c) - \Phi(c)$$

namely

$$0 \leq \frac{(b-c)\Phi(a) + (c-a)\Phi(b)}{b-a} - \Phi(c)$$

that is also obvious by the convexity of Φ.

The case $t = d$ is similar and the details are omitted. □

The following result also holds:

Theorem 3. *Let $\Phi : [m,M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m,M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega,\mu)$. Then for any $F_n(\Omega) = \{\Omega_i\}_{i\in\{1,\dots,n\}} \in \mathfrak{D}_n(\Omega)$ we have*

$$\begin{aligned}0 \leq \frac{1}{(M-m)\int_\Omega w d\mu}&\left[\sum_{i=1}^n \left(\int_{\Omega_i}(M_i - f) w d\mu\right)\Phi(m_i)\right.\\ &+\sum_{i=1}^n \left(\int_{\Omega_i}(f-m_i) w d\mu\right)\Phi(M_i)\\ &\left.-\sum_{i=1}^n (M_i - m_i)\Phi\left(\frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu}\right)\int_{\Omega_i} w d\mu\right]\\ &\leq \Delta\left(\Phi; m, M, \frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right) - \psi(\Phi, f, w, F_n(\Omega)),\end{aligned} \tag{2.13}$$

where $\psi(\Phi, f, w, F_n(\Omega))$ is defined by (1.1).

Proof. From the inequality (2.7) we have for

$$t=\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\in[m_i,M_i]\,,\ i\in\{1,...,n\}$$

that

$$0\le\left[\Delta\left(\Phi;m_i,M_i,\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\right)-\Phi\left(\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\right)\right](M_i-m_i)$$
$$\le\left[\Delta\left(\Phi;m,M,\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\right)-\Phi\left(\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\right)\right](M-m) \tag{2.14}$$

for any $i\in\{1,...,n\}$.

This inequality is equivalent to

$$0\le\left(M_i-\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\right)\Phi(m_i)+\left(\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}-m_i\right)\Phi(M_i)$$
$$-\Phi\left(\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\right)(M_i-m_i)$$
$$\le\left(M-\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\right)\Phi(m)+\left(\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}-m\right)\Phi(M)$$
$$-(M-m)\,\Phi\left(\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\right) \tag{2.15}$$

for any $i\in\{1,...,n\}$.

If we multiply this inequality by $\int_{\Omega_i} wd\mu>0$ we get

$$0\le\left(\int_{\Omega_i}(M_i-f)\,wd\mu\right)\Phi(m_i)+\left(\int_{\Omega_i}(f-m_i)\,wd\mu\right)\Phi(M_i)$$
$$-(M_i-m_i)\,\Phi\left(\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\right)\int_{\Omega_i} wd\mu$$
$$\le\left(M\int_{\Omega_i} wd\mu-\int_{\Omega_i} fwd\mu\right)\Phi(m)+\left(\int_{\Omega_i} fwd\mu-m\int_{\Omega_i} wd\mu\right)\Phi(M)$$
$$-(M-m)\int_{\Omega_i} wd\mu\Phi\left(\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\right) \tag{2.16}$$

for any $i\in\{1,...,n\}$.

Now, if we sum the inequality (2.16) over i from 1 to n we get

$$0\leq\sum_{i=1}^{n}\left(\int_{\Omega_i}(M_i-f)\,wd\mu\right)\Phi(m_i)+\sum_{i=1}^{n}\left(\int_{\Omega_i}(f-m_i)\,wd\mu\right)\Phi(M_i) \tag{2.17}$$
$$-\sum_{i=1}^{n}(M_i-m_i)\,\Phi\left(\frac{\int_{\Omega_i}fwd\mu}{\int_{\Omega_i}wd\mu}\right)\int_{\Omega_i}wd\mu$$
$$\leq\left(M\sum_{i=1}^{n}\int_{\Omega_i}wd\mu-\sum_{i=1}^{n}\int_{\Omega_i}fwd\mu\right)\Phi(m)$$
$$+\left(\sum_{i=1}^{n}\int_{\Omega_i}fwd\mu-m\sum_{i=1}^{n}\int_{\Omega_i}wd\mu\right)\Phi(M)$$
$$-(M-m)\sum_{i=1}^{n}\int_{\Omega_i}wd\mu\Phi\left(\frac{\int_{\Omega_i}fwd\mu}{\int_{\Omega_i}wd\mu}\right)$$
$$=\left(M\int_{\Omega}wd\mu-\int_{\Omega}fwd\mu\right)\Phi(m)+\left(\int_{\Omega}fwd\mu-m\int_{\Omega}wd\mu\right)\Phi(M)$$
$$-(M-m)\,\psi(\Phi,f,w,F_n(\Omega))\int_{\Omega}wd\mu,$$

which is equivalent to the desired result (2.13). □

The following result also holds:

Theorem 4. *Let $\Phi:[m,M]\to\mathbb{R}$ be a convex function, $f:\Omega\to[m,M]$ a μ-measurable function such that $f,\ \Phi\circ f\in L_w(\Omega,\mu)$. Then for any $F_n(\Omega)=\{\Omega_i\}_{i\in\{1,...,n\}}\in\mathfrak{D}_n(\Omega)$ we have*

$$0\leq\frac{1}{(M-m)\int_{\Omega}wd\mu}\left[\sum_{i=1}^{n}\Phi(m_i)\left(\int_{\Omega_i}(M_i-f)\,wd\mu\right)\right. \tag{2.18}$$
$$\left.+\sum_{i=1}^{n}\Phi(M_i)\int_{\Omega_i}(f-m_i)\,wd\mu-\sum_{i=1}^{n}(M_i-m_i)\int_{\Omega_i}w(\Phi\circ f)\,d\mu\right]$$
$$\leq\Delta\left(\Phi;m,M,\frac{\int_{\Omega}fwd\mu}{\int_{\Omega}wd\mu}\right)-\int_{\Omega}w(\Phi\circ f)\,d\mu.$$

Proof. For μ-almost every $x\in\Omega_i$ we have $f(x)\in[m_i,M_i]$, $i\in\{1,...,n\}$ and then by the inequality (2.7) we get

$$0\leq[\Delta(\Phi;m_i,M_i,f(x))-\Phi(f(x))](M_i-m_i) \tag{2.19}$$
$$\leq[\Delta(\Phi;m,M,f(x))-\Phi(f(x))](M-m)$$

for μ-almost every $x\in\Omega_i$.

This is equivalent to

$$\begin{aligned}0&\leq(M_i-f(x))\,\Phi(m_i)+(f(x)-m_i)\,\Phi(M_i)-\Phi(f(x))(M_i-m_i)\\&\leq(M-f(x))\,\Phi(m)+(f(x)-m)\,\Phi(M)-\Phi(f(x))(M-m)\end{aligned}$$

for μ-almost every $x\in\Omega_i$ and every $i\in\{1,...,n\}$.

If we multiply by $w \geq 0$ μ-almost everywhere and integrate on Ω_i we get

$$
\begin{aligned}
0 &\leq \Phi(m_i)\left(\int_{\Omega_i}(M_i - f)\, w d\mu\right) + \Phi(M_i)\int_{\Omega_i}(f - m_i)\, w d\mu \\
&- (M_i - m_i)\int_{\Omega_i} w(\Phi \circ f)\, d\mu \\
&\leq \left(M\int_{\Omega_i} w d\mu - \int_{\Omega_i} f w d\mu\right)\Phi(m) + \left(\int_{\Omega_i} f w d\mu - m\int_{\Omega_i} w d\mu\right)\Phi(M) \\
&- (M - m)\int_{\Omega_i} w(\Phi \circ f)\, d\mu
\end{aligned}
$$

for every $i \in \{1, ..., n\}$.

If we sum over i from 1 to n we get

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n}\Phi(m_i)\left(\int_{\Omega_i}(M_i - f)\, w d\mu\right) + \sum_{i=1}^{n}\Phi(M_i)\int_{\Omega_i}(f - m_i)\, w d\mu \\
&- \sum_{i=1}^{n}(M_i - m_i)\int_{\Omega_i} w(\Phi \circ f)\, d\mu \\
&\leq \left(M\sum_{i=1}^{n}\int_{\Omega_i} w d\mu - \sum_{i=1}^{n}\int_{\Omega_i} f w d\mu\right)\Phi(m) \\
&+ \left(\sum_{i=1}^{n}\int_{\Omega_i} f w d\mu - m\sum_{i=1}^{n}\int_{\Omega_i} w d\mu\right)\Phi(M) \\
&- (M - m)\sum_{i=1}^{n}\int_{\Omega_i} w(\Phi \circ f)\, d\mu,
\end{aligned}
$$

which is equivalent to (2.18). □

3. Discrete Inequalities

Assume that, for $n \geq 2$, we have a family J of indices containing more than n elements and $F_n(J) = \{J_i\}_{i \in \{1,...,n\}}$ is a n-division for J, namely $J = \bigcup_{i=1}^{n} J_i$ and $J_i \cap J_j = \emptyset$ for any $i, j \in \{1, ..., n\}$ with $i \neq j$.

Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function, $\{x_j\}_{j \in J} \subset I$ and put $m := \min_{j \in J}\{x_j\}$ and $M := \max_{j \in J}\{x_j\}$. Also let $m_{J_i} := \min_{j \in Ji}\{x_j\}$ and $M_{J_i} = \max_{j \in Ji}\{x_j\}$ and assume that $m_{J_i} < M_{J_i}$ for $i \in \{1, ..., n\}$. For a nonempty finite family of indices J and positive weights w_j, $j \in J$ we denote $W_J := \sum_{j \in J} w_j$.

Consider the discrete version of the functional (2.2)

$$
\begin{aligned}
(3.1)\qquad & \sigma(\Phi, x, w, F_n(J)) \\
&:= \frac{1}{W_J}\sum_{i=1}^{n} W_{J_i}\Delta\left(\Phi; m_{J_i}, M_{J_i}, \frac{\sum_{j \in J_i} w_j x_j}{W_{J_i}}\right) \\
&= \frac{1}{W_J}\sum_{i=1}^{n}\frac{W_{J_i}}{M_{J_i} - m_{J_i}} \\
&\times\left[\left(M_{J_i} - \frac{\sum_{j \in J_i} w_j x_j}{W_{J_i}}\right)\Phi(m_i) + \left(\frac{\sum_{j \in J_i} w_j x_j}{W_{J_i}} - m_{J_i}\right)\Phi(M_{J_i})\right].
\end{aligned}
$$

If we write the inequality (2.3) for the *discrete measure* we get

$$\frac{1}{W_J}\sum_{j\in J}w_j\Phi\left(x_j\right)\leq\sigma\left(\Phi,x,w,F_n\left(J\right)\right)\leq\Delta\left(\Phi;m,M,\frac{1}{W_J}\sum_{j\in J}w_jx_j\right). \tag{3.2}$$

From (2.13) we have

$$\begin{aligned} 0 &\leq \frac{1}{(M-m)W_J}\left[\sum_{i=1}^{n}\left(\sum_{j\in J_i}\left(M_{J_i}-x_j\right)w_j\right)\Phi\left(m_{J_i}\right)\right. \\ &\quad+\sum_{i=1}^{n}\left(\sum_{j\in J_i}\left(x_j-m_{J_i}\right)w_j\right)\Phi\left(M_{J_i}\right) \\ &\quad\left.-\sum_{i=1}^{n}\left(M_{J_i}-m_{J_i}\right)\Phi\left(\frac{\sum_{j\in J_i}x_jw_j}{W_{J_i}}\right)W_{J_i}\right] \\ &\leq\Delta\left(\Phi;m,M,\frac{1}{W_J}\sum_{j\in J}w_jx_j\right)-\psi\left(\Phi,x,w,F_n\left(J\right)\right), \end{aligned} \tag{3.3}$$

where

$$\psi\left(\Phi,f,w,F_n\left(J\right)\right):=\frac{1}{W_J}\sum_{i=1}^{n}W_{J_i}\Phi\left(\frac{1}{W_{J_i}}\sum_{j\in J_i}w_jx_j\right).$$

From (2.18) we also have

$$\begin{aligned} 0 &\leq \frac{1}{(M-m)W_J}\left[\sum_{i=1}^{n}\Phi\left(m_{J_i}\right)\left(\sum_{j\in J_i}\left(M_{J_i}-x_j\right)w_j\right)\right. \\ &\quad\left.+\sum_{i=1}^{n}\Phi\left(M_{J_i}\right)\sum_{j\in J_i}\left(x_j-m_{J_i}\right)w_j-\sum_{i=1}^{n}\left(M_{J_i}-m_{J_i}\right)\sum_{j\in J_i}w_j\Phi\left(x_j\right)\right] \\ &\leq\Delta\left(\Phi;m,M,\frac{1}{W_J}\sum_{j\in J}w_jx_j\right)-\frac{1}{W_J}\sum_{j\in J}w_j\Phi\left(x_j\right). \end{aligned} \tag{3.4}$$

If we write the above inequalities for the positive numbers $x_i>0, i\in\{1,...,n\}$ and the convex power function $\Phi(t)=t^p$, $p\in(-\infty,0)\cup(1,\infty)$ we have

$$\begin{aligned} &\frac{1}{W_J}\sum_{j\in J}w_jx_j^p \\ &\leq\frac{1}{W_J}\sum_{i=1}^{n}\frac{W_{J_i}}{M_{J_i}-m_{J_i}} \\ &\times\left[\left(M_{J_i}-\frac{\sum_{j\in J_i}w_jx_j}{W_{J_i}}\right)m_i^p+\left(\frac{\sum_{j\in J_i}w_jx_j}{W_{J_i}}-m_{J_i}\right)M_{J_i}^p\right] \\ &\leq\frac{1}{M-m}\left[\left(M-\frac{1}{W_J}\sum_{j\in J}w_jx_j\right)m^p+\left(\frac{1}{W_J}\sum_{j\in J}w_jx_j-m\right)M^p\right], \end{aligned} \tag{3.5}$$

$$
\begin{aligned}
(3.6) \qquad 0 &\leq \frac{1}{(M-m)\,W_J}\left[\sum_{i=1}^{n}\left(\sum_{j\in J_i}(M_{J_i}-x_j)\,w_j\right)m_{J_i}^p\right.\\
&+\sum_{i=1}^{n}\left(\sum_{j\in J_i}(x_j-m_{J_i})\,w_j\right)M_{J_i}^p\\
&\left.-\sum_{i=1}^{n}(M_{J_i}-m_{J_i})\left(\sum_{j\in J_i}x_jw_j\right)^p W_{J_i}^{1-p}\right]\\
&\leq \frac{1}{M-m}\left[\left(M-\frac{1}{W_J}\sum_{j\in J}w_jx_j\right)m^p+\left(\frac{1}{W_J}\sum_{j\in J}w_jx_j-m\right)M^p\right]\\
&-\frac{1}{W_J}\sum_{i=1}^{n}W_{J_i}^{1-p}\left(\sum_{j\in J_i}w_jx_j\right)^p
\end{aligned}
$$

and

$$
\begin{aligned}
(3.7) \qquad 0 &\leq \frac{1}{(M-m)\,W_J}\left[\sum_{i=1}^{n}m_{J_i}^p\left(\sum_{j\in J_i}(M_{J_i}-x_j)\,w_j\right)\right.\\
&\left.+\sum_{i=1}^{n}M_{J_i}^p\sum_{j\in J_i}(x_j-m_{J_i})\,w_j-\sum_{i=1}^{n}(M_{J_i}-m_{J_i})\sum_{j\in J_i}w_jx_j^p\right]\\
&\leq \frac{1}{M-m}\left[\left(M-\frac{1}{W_J}\sum_{j\in J}w_jx_j\right)m^p+\left(\frac{1}{W_J}\sum_{j\in J}w_jx_j-m\right)M^p\right]\\
&-\frac{1}{W_J}\sum_{j\in J}w_jx_j^p.
\end{aligned}
$$

4. Some Inequalities Related to HH-Inequality

It is clear that all inequalities from Section 2 can be written for univariate functions $f:[a,b]\subset\mathbb{R}\to[m,M]$ and the functional defined in (2.2).

We are, however, interested here in the particular case that is related to the celebrated *Hermite-Hadamard inequality*

$$\Phi\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b\Phi(t)\,dt\leq\frac{\Phi(a)+\Phi(b)}{2},$$

where $\Phi:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$.

Let $\Phi:[m,M]\to\mathbb{R}$ be a convex function and $f:[a,b]\to[m,M]$ an integrable function. Consider the division of the interval $[a,b]$ given by

$$d_n: a=x_0<x_{1<}\ldots<x_{n-1}<x_n=b,\ n\geq 2.$$

If we take $\Omega=[a,b]$ and $\Omega_1=[a,x_1]$, $\Omega_i=(x_i,x_{i+1}]$ for $i\in\{1,\ldots,n-1\}$ then $\Omega=\bigcup_{i=1}^n\Omega_i$ and $\Omega_i\cap\Omega_j=\emptyset$ for any $i,\,j\in\{1,\ldots,n\}$ with $i\neq j$.

By making use of (2.2) for this division and $f:[a,b]\subset\mathbb{R}\to[a,b]$, $f(x)=x$, we can consider the functional

$$\sigma(\Phi,d_n):=\frac{1}{b-a}\sum_{i=1}^{n}(x_{i+1}-x_i)\frac{\Phi(x_i)+\Phi(x_{i+1})}{2}. \tag{4.1}$$

If we use the inequality (2.3) we have

$$\frac{1}{b-a}\int_a^b\Phi(t)\,dt\le\frac{1}{b-a}\sum_{i=1}^{n}(x_i-x_{i-1})\frac{\Phi(x_i)+\Phi(x_{i-1})}{2}\le\frac{\Phi(a)+\Phi(b)}{2}. \tag{4.2}$$

This inequality was obtained by the author in 1994 in [2], see also [15, p. 22].

From (2.13) we have

$$\begin{aligned}0&\le\frac{1}{(b-a)^2}\sum_{i=1}^{n}(x_i-x_{i-1})^2\left[\frac{\Phi(x_i)+\Phi(x_{i-1})}{2}-\Phi\left(\frac{x_{i-1}+x_i}{2}\right)\right]\\&\le\frac{\Phi(a)+\Phi(b)}{2}-\frac{1}{b-a}\sum_{i=1}^{n}\Phi\left(\frac{x_i+x_{i-1}}{2}\right)(x_i-x_{i-1}),\end{aligned} \tag{4.3}$$

while from (2.18) we have

$$\begin{aligned}0&\le\frac{1}{(b-a)^2}\sum_{i=1}^{n}(x_i-x_{i-1})^2\\&\times\left[\frac{\Phi(x_i)+\Phi(x_{i-1})}{2}-\frac{1}{x_i-x_{i-1}}\int_{x_{i-1}}^{x_i}\Phi(x)\,dx\right]\\&\le\frac{\Phi(a)+\Phi(b)}{2}-\frac{1}{b-a}\int_a^b\Phi(t)\,dt.\end{aligned} \tag{4.4}$$

If we take in (4.3) and (4.4) $\Phi(t)=\frac{1}{t}$, $t\in[a,b]\subset(0,\infty)$ then we get the inequalities

$$\frac{1}{2(b-a)^2}\sum_{i=1}^{n}\frac{(x_i-x_{i-1})^4}{x_ix_{i-1}(x_i+x_{i-1})}\le\frac{a+b}{2ab}-\frac{2}{b-a}\sum_{i=1}^{n}\frac{x_i-x_{i-1}}{x_i+x_{i-1}}, \tag{4.5}$$

and

$$\begin{aligned}0&\le\frac{1}{(b-a)^2}\sum_{i=1}^{n}(x_i-x_{i-1})^2\left[\frac{L(x_i,x_{i-1})-H(x_i,x_{i-1})}{L(x_i,x_{i-1})H(x_i,x_{i-1})}\right]\\&\le\frac{L(a,b)-H(a,b)}{L(a,b)H(a,b)},\end{aligned} \tag{4.6}$$

where

$$H(\alpha,\beta):=\frac{2\alpha\beta}{\alpha+\beta}$$

is the *harmonic mean* while

$$L(\alpha,\beta):=\frac{\alpha-\beta}{\ln\alpha-\ln\beta},\alpha\neq\beta$$

is the *logarithmic mean*.

If we take in (4.3) and (4.4) $\Phi(t) = -\ln t$, $t \in [a,b] \subset (0,\infty)$ then we get the inequalities

$$1 \leq \prod_{i=1}^{n} \left(\frac{A(x_{i-1}, x_i)}{G(x_{i-1}, x_i)} \right)^{\frac{(x_i - x_{i-1})^2}{(b-a)^2}} \leq \frac{\prod_{i=1}^{n} \left(A(x_{i-1}, x_i)\right)^{\frac{(x_i - x_{i-1})}{b-a}}}{G(a,b)}, \tag{4.7}$$

and

$$1 \leq \prod_{i=1}^{n} \left(\frac{I(x_{i-1}, x_i)}{G(x_{i-1}, x_i)} \right)^{\frac{(x_i - x_{i-1})^2}{(b-a)^2}} \leq \frac{I(a,b)}{G(a,b)}, \tag{4.8}$$

where

$$G(\alpha, \beta) := \sqrt{\alpha\beta}$$

is the *geometric mean* while

$$I(\alpha, \beta) := \frac{1}{e} \left(\frac{\beta^{\beta}}{\alpha^{\alpha}} \right)^{\frac{1}{\beta - \alpha}}, \alpha \neq \beta$$

is the *identric mean*.

Now, consider the *p-logarithmic mean* defined by

$$L_p(\alpha, \beta) := \left(\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\alpha - \beta)} \right)^{1/p}$$

where $p \in \mathbb{R} \setminus \{-1, 0\}$.

From (4.3) and (4.4) we have for $p \in (-\infty, 0) \cup (1, \infty) \setminus \{-1\}$

$$\begin{aligned} 0 &\leq \frac{1}{(b-a)^2} \sum_{i=1}^{n} (x_i - x_{i-1})^2 \left[A\left(x_i^p, x_{i-1}^p\right) - A^p(x_{i-1}, x_i) \right] \\ &\leq A(a^p, b^p) - \frac{1}{b-a} \sum_{i=1}^{n} A^p(x_{i-1}, x_i)(x_i - x_{i-1}), \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{(b-a)^2} \sum_{i=1}^{n} (x_i - x_{i-1})^2 \left[A\left(x_i^p, x_{i-1}^p\right) - L_p^p(x_{i-1}, x_i) \right] \\ &\leq A(a^p, b^p) - L_p^p(a,b). \end{aligned} \tag{4.10}$$

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