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HERMITE-HADAMARD-FEJER TYPE INEQUALITIES FOR GA-CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, firstly we have established Hermite–Hadamard-Fejér inequality for GA-convex functions in fractional integral forms. Secondly, an integral identity and some Hermite-Hadamard-Fejer type integral inequalities for GA-convex functions in fractional integral forms have been obtained.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality [2].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In [1], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative,integrable and symmetric to $(a + b)/2$.

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2) see [3, 12, 13, 14].

Definition 1. [9, 10]. *A function $f : I \subseteq (0, \infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if*

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)\,f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

In [8], Latif et al. established the following inequality which is the weighted generalization of Hermite-Hadamard inequality for GA-convex functions as follows:

Theorem 2. *Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a GA-convex function and $a, b \in I$ with $a < b$. Let g:$[a, b] \to [0, \infty)$ be continuous positive mapping and geometrically symmetric to $\sqrt{ab}$. Then*

$$f\left(\sqrt{ab}\right)\int_a^b \frac{g\left(x\right)}{x}dx \leq \int_a^b \frac{f\left(x\right)g\left(x\right)}{x}dx \leq \frac{f\left(a\right)+f\left(b\right)}{2}\int_a^b \frac{g\left(x\right)}{x}dx. \tag{1.3}$$

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We will now give definitions of the right-hand side and left-hand side Hadamard fractional integrals which are used throughout this paper.

Definition 2. [7]. *Let $f \in L[a,b]$. The right-hand side and left-hand side Hadamard fractional integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $b > a \geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t},\ x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t},\ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [4, 5, 6, 15, 16].

In [5], İşcan represented Hermite–Hadamard's inequalities for GA-convex functions in fractional integral forms as follows.

Theorem 3. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$ where $a, b \in I$ with $a < b$. If f is a GA-convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.4}$$

with $\alpha > 0$.

Lemma 1. [11, 16]. *For $0 < \alpha \leq 1$ and $0 \leq a < b$, we have*

$$|a^{\alpha} - b^{\alpha}| \leq (b-a)^{\alpha}.$$

In this paper, we firstly represented Hermite-Hadamard-Fejér inequality for GA-convex function in fractional integral forms which is the weighted generalization of Hermite-Hadamard inequality for GA-convex functions (1.4). Secondly, we obtained some new inequalities connected with the right-hand side of Hermite-Hadamard-Fejér type integral inequality for GA-convex function in fractional integrals.

2. Main results

Throughout this section, let $\|g\|_{\infty} = \sup_{t\in[a,b]}|g(x)|$, for the continuous function $g : [a,b] \to \mathbb{R}$.

Lemma 2. *If $g : [a,b] \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is integrable and geometrically symmetric with respect to $\sqrt{ab}$ (i.e. $g\left(\frac{ab}{x}\right) = g(x)$ holds for all $x \in [a,b]$) with $a < b$, then*

$$J_{a+}^{\alpha}g(b) = J_{b-}^{\alpha}g(a) = \frac{1}{2}\left[J_{a+}^{\alpha}g(b) + J_{b-}^{\alpha}g(a)\right]$$

with $\alpha > 0$.

Proof. Since g is geometrically symmetric with respect to $\sqrt{ab}$, we have $g(ab/x) = g(x)$, for all $x \in [a,b]$. Hence, in the following integral if we setting $x = ab/t$ and

$dx = -\left(ab/t^2\right) dt$ we have

$$\begin{aligned} J_{a+}^{\alpha} g(b) &= \frac{1}{\Gamma(\alpha)} \int_a^b \left(\ln \frac{b}{x}\right)^{\alpha-1} g(x) \frac{dx}{x} \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b \left(\ln \frac{t}{a}\right)^{\alpha-1} g\left(\frac{ab}{t}\right) \frac{dt}{t} \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b \left(\ln \frac{t}{a}\right)^{\alpha-1} g\left(t\right) \frac{dt}{t} = J_{b-}^{\alpha} g(a). \end{aligned}$$

This completes the proof. □

Theorem 4. *Let $f : [a,b] \subseteq \mathbb{R}_+ \to \mathbb{R}$ be GA-convex function with $a < b$ and $f \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is nonnegative,integrable and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals hold*

$$\begin{aligned} f\left(\sqrt{ab}\right)\left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] &\leq \left[J_{a+}^{\alpha}\left(fg\right)(b) + J_{b-}^{\alpha}\left(fg\right)(a)\right] \\ &\leq \frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] \end{aligned} \quad (2.1)$$

with $\alpha > 0$.

Proof. Since f is a GA-convex function on $[a,b]$, we have for all $t \in [0,1]$

$$f\left(\sqrt{ab}\right) = f\left(\sqrt{a^t b^{1-t}.a^{1-t}b^t}\right) \leq \frac{f\left(a^t b^{1-t}\right) + f\left(a^{1-t}b^t\right)}{2}. \quad (2.2)$$

Multiplying both sides of (2.2) by $2t^{\alpha-1} g\left(a^{1-t}b^t\right)$ then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned} & 2f\left(\sqrt{ab}\right)\int_0^1 t^{\alpha-1} g\left(a^{1-t}b^t\right) dt \\ \leq & \int_0^1 t^{\alpha-1}\left[f\left(a^t b^{1-t}\right) + f\left(a^{1-t}b^t\right)\right] g\left(a^{1-t}b^t\right) dt \\ = & \int_0^1 t^{\alpha-1} f\left(a^t b^{1-t}\right) g\left(a^{1-t}b^t\right) dt + \int_0^1 t^{\alpha-1} f\left(a^{1-t}b^t\right) g\left(a^{1-t}b^t\right) dt. \end{aligned}$$

Setting $x = a^{1-t}b^t$, and $dx = a^{1-t}b^t \ln\left(\frac{b}{a}\right) dt$ gives

$$\begin{aligned} & \frac{2}{\left(\ln \frac{b}{a}\right)^{\alpha}} f\left(\sqrt{ab}\right)\int_a^b \left(\ln \frac{x}{a}\right)^{\alpha-1} g\left(x\right)\frac{dx}{x} \\ \leq & \frac{1}{\left(\ln \frac{b}{a}\right)^{\alpha}}\left\{\int_a^b \left(\ln \frac{x}{a}\right)^{\alpha-1} f\left(\frac{ab}{x}\right) g\left(x\right)\frac{dx}{x} + \int_a^b \left(\ln \frac{x}{a}\right)^{\alpha-1} f\left(x\right) g\left(x\right)\frac{dx}{x}\right\} \\ = & \frac{1}{\left(\ln \frac{b}{a}\right)^{\alpha}}\left\{\int_a^b \left(\ln \frac{b}{x}\right)^{\alpha-1} f\left(x\right) g\left(\frac{ab}{x}\right)\frac{dx}{x} + \int_a^b \left(\ln \frac{x}{a}\right)^{\alpha-1} f\left(x\right) g\left(x\right)\frac{dx}{x}\right\} \\ = & \frac{1}{\left(\ln \frac{b}{a}\right)^{\alpha}}\left\{\int_a^b \left(\ln \frac{b}{x}\right)^{\alpha-1} f\left(x\right) g\left(x\right)\frac{dx}{x} + \int_a^b \left(\ln \frac{x}{a}\right)^{\alpha-1} f\left(x\right) g\left(x\right)\frac{dx}{x}\right\}. \end{aligned}$$

Therefore, by Lemma 2 we have

$$\frac{\Gamma(\alpha)}{\left(\ln \frac{b}{a}\right)^{\alpha}} f\left(\sqrt{ab}\right)\left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] \leq \frac{\Gamma(\alpha)}{\left(\ln \frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}\left(fg\right)(b) + J_{b-}^{\alpha}\left(fg\right)(a)\right]$$

and the first inequality is proved.

For the proof of the second inequality in (2.1) we first note that if f is a GA-convex function, then, for all $t \in [0,1]$, it yields

$$f\left(a^t b^{1-t}\right) + f\left(a^{1-t} b^t\right) \leq f(a) + f(b). \tag{2.3}$$

Then multiplying both sides of (2.3) by $t^{\alpha-1} g\left(a^{1-t} b^t\right)$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned} & \int_0^1 t^{\alpha-1} f\left(a^t b^{1-t}\right) g\left(a^{1-t} b^t\right) dt + \int_0^1 t^{\alpha-1} f\left(a^{1-t} b^t\right) g\left(a^{1-t} b^t\right) dt \\ \leq & \quad [f(a)+f(b)] \int_0^1 t^{\alpha-1} g\left(a^{1-t} b^t\right) dt \end{aligned}$$

i.e.

$$\begin{aligned} \frac{\Gamma(\alpha)}{\left(\ln \frac{b}{a}\right)^\alpha} \left[J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)\right] \quad \leq & \quad \frac{\Gamma(\alpha)}{\left(\ln \frac{b}{a}\right)^\alpha} \left(\frac{f(a)+f(b)}{2}\right) \\ & \times \left[J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)\right] \end{aligned}$$

The proof is completed. □

Remark 1. *In Theorem 4,*

(1) if we take $\alpha = 1$, then inequality (2.1) becomes inequality (1.3) of Theorem 2,

(2) if we take $g(x) = 1$, then inequality (2.1) becomes inequality (1.4) of Theorem 3.

Lemma 3. *Let $f : [a,b] \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and $f' \in L[a,b]$. If $g : [a,b] \rightarrow \mathbb{R}$ is integrable and geometrically symmetric with respect to $\sqrt{ab}$ then the following equality for fractional integrals holds*

$$\begin{aligned} & \left(\frac{f(a)+f(b)}{2}\right) \left[J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)\right] - \left[J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)\right] \\ = & \quad \frac{1}{\Gamma(\alpha)} \int_a^b \left[\int_a^t \left(\ln \frac{b}{s}\right)^{\alpha-1} g(s) \frac{ds}{s} - \int_t^b \left(\ln \frac{s}{a}\right)^{\alpha-1} g(s) \frac{ds}{s}\right] f'(t) dt \end{aligned} \tag{2.4}$$

with $\alpha > 0$.

Proof. It suffices to note that

$$\begin{aligned} I &= \int_a^b \left[\int_a^t \left(\ln \frac{b}{s}\right)^{\alpha-1} g(s) \frac{ds}{s} - \int_t^b \left(\ln \frac{s}{a}\right)^{\alpha-1} g(s) \frac{ds}{s}\right] f'(t) dt \\ &= \int_a^b \left(\int_a^t \left(\ln \frac{b}{s}\right)^{\alpha-1} g(s) \frac{ds}{s}\right) f'(t) dt + \int_a^b \left(-\int_t^b \left(\ln \frac{s}{a}\right)^{\alpha-1} g(s) \frac{ds}{s}\right) f'(t) dt \\ &= I_1 + I_2. \end{aligned}$$

By integration by parts and Lemma 2 we get

$$\begin{aligned} I_1 &= \left(\int_a^t \left(\ln \frac{b}{s}\right)^{\alpha-1} g(s) \frac{ds}{s}\right) f(t) \Big|_a^b - \int_a^b \left(\ln \frac{b}{t}\right)^{\alpha-1} g(t) f(t) \frac{dt}{t} \\ &= \left(\int_a^b \left(\ln \frac{b}{s}\right)^{\alpha-1} g(s) \frac{ds}{s}\right) f(b) - \int_a^b \left(\ln \frac{b}{t}\right)^{\alpha-1} (fg)(t) \frac{dt}{t} \\ &= \Gamma(\alpha) \left[f(b) J_{a+}^\alpha g(b) - J_{a+}^\alpha (fg)(b)\right] \\ &= \Gamma(\alpha) \left[\frac{f(b)}{2} \left[J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)\right] - J_{a+}^\alpha (fg)(b)\right] \end{aligned}$$

and similarly

$$\begin{aligned} I_2 &= \left(-\int_t^b \left(\ln \frac{s}{a}\right)^{\alpha-1} g(s)\frac{ds}{s}\right) f(t)\Bigg|_a^b - \int_a^b \left(\ln \frac{t}{a}\right)^{\alpha-1} g(t)f(t)\frac{dt}{t} \\ &= \left(\int_a^b \left(\ln \frac{s}{a}\right)^{\alpha-1} g(s)\frac{ds}{s}\right) f(a) - \int_a^b \left(\ln \frac{t}{a}\right)^{\alpha-1} (fg)(t)\frac{dt}{t} \\ &= \Gamma(\alpha)\left[\frac{f(a)}{2}\left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] - J_{b-}^{\alpha}\left(fg\right)(a)\right]. \end{aligned}$$

Thus, we can write

$$I = I_1 + I_2 = \Gamma(\alpha)\left\{ \begin{array}{c} \left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] \\ -\left[J_{a+}^{\alpha}\left(fg\right)(b) + J_{b-}^{\alpha}\left(fg\right)(a)\right] \end{array} \right\}.$$

Multiplying the both sides by $(\Gamma(\alpha))^{-1}$ we obtain (2.4) which completes the proof. □

Theorem 5. *Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|$ is GA-convex on $[a,b]$ and $g : [a,b] \rightarrow \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequality for fractional integrals holds*

$$\begin{aligned} &\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] - \left[J_{a+}^{\alpha}\left(fg\right)(b) + J_{b-}^{\alpha}\left(fg\right)(a)\right]\right| \\ \leq\ & \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[C_1(\alpha)\left|f'(a)\right| + C_2(\alpha)\left|f'(b)\right|\right], \end{aligned} \tag{2.5}$$

where

$$C_1(\alpha) = \int_0^{\frac{1}{2}} \left[(1-u)^{\alpha} - u^{\alpha}\right]\left[(1-u)\, a^{1-u} b^{u} + u a^{u} b^{1-u}\right] du$$

and

$$C_2(\alpha) = \int_0^{\frac{1}{2}} \left[(1-u)^{\alpha} - u^{\alpha}\right]\left[u a^{1-u} b^{u} + (1-u)\, a^{u} b^{1-u}\right] du$$

with $\alpha > 0$.

Proof. From Lemma 3 we have

$$\begin{aligned} &\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] - \left[J_{a+}^{\alpha}\left(fg\right)(b) + J_{b-}^{\alpha}\left(fg\right)(a)\right]\right| \\ \leq\ & \frac{1}{\Gamma(\alpha)}\int_a^b \left|\int_a^t \left(\ln \frac{b}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} - \int_t^b \left(\ln \frac{s}{a}\right)^{\alpha-1} g(s)\frac{ds}{s}\right| \left|f'(t)\right| dt. \end{aligned}$$

setting $t = a^{1-u}b^u$ and $dt = a^{1-u}b^u \ln\left(\frac{b}{a}\right) du$ gives

$$\begin{aligned} &\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] - \left[J_{a+}^{\alpha}\left(fg\right)(b) + J_{b-}^{\alpha}\left(fg\right)(a)\right]\right| \\ \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^1 \left|\int_a^{a^{1-u}b^u} \left(\ln \frac{b}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} - \int_{a^{1-u}b^u}^b \left(\ln \frac{s}{a}\right)^{\alpha-1} g(s)\frac{ds}{s}\right| \\ & \times \left|f'(a^{1-u}b^u)\right| a^{1-u}b^u \ln\left(\frac{b}{a}\right) du \end{aligned} \tag{2.6}$$

Since $g:[a,b]\rightarrow\mathbb{R}$ is geometrically symmetric with respect to $\sqrt{ab}$ we write

$$\begin{aligned}\int_{a^{1-u}b^{u}}^{b}\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s} &= \int_{a}^{a^{u}b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(\frac{ab}{s})\frac{ds}{s}\\ &= \int_{a}^{a^{u}b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\end{aligned}$$

then we have

$$\begin{aligned}&\left|\int_{a}^{a^{1-u}b^{u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}-\int_{a^{1-u}b^{u}}^{b}\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|\\ =&\left|\int_{a^{1-u}b^{u}}^{a^{u}b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|\\ \leq&\begin{cases}\int_{a^{1-u}b^{u}}^{a^{u}b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}|g(s)|\frac{ds}{s} & u\in\left[0,\frac{1}{2}\right]\\ \int_{a^{u}b^{1-u}}^{a^{1-u}b^{u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}|g(s)|\frac{ds}{s} & u\in\left[\frac{1}{2},1\right]\end{cases}\\ \leq&\ \|g\|_{\infty}\begin{cases}\int_{a^{1-u}b^{u}}^{a^{u}b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s} & u\in\left[0,\frac{1}{2}\right]\\ \int_{a^{u}b^{1-u}}^{a^{1-u}b^{u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s} & u\in\left[\frac{1}{2},1\right]\end{cases}\\ =&\ \|g\|_{\infty}\frac{\left(\ln\frac{b}{a}\right)^{\alpha}}{\alpha}\begin{cases}(1-u)^{\alpha}-u^{\alpha} & u\in\left[0,\frac{1}{2}\right]\\ u^{\alpha}-(1-u)^{\alpha} & u\in\left[\frac{1}{2},1\right]\end{cases}\end{aligned}\tag{2.7}$$

Since $|f'|$ is GA-convex on $[a,b]$, we know that for $u\in[0,1]$

$$\left|f'(a^{1-u}b^{u})\right|\leq(1-u)\left|f'(a)\right|+u\left|f'(b)\right|,\tag{2.8}$$

A combination (2.6), (2.7) and (2.8)

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}\left(fg\right)(b)+J_{b-}^{\alpha}\left(fg\right)(a)\right]\right|\\ \leq&\ \frac{1}{\Gamma(\alpha)}\int_{0}^{1}\left|\int_{a}^{a^{1-u}b^{u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}-\int_{a^{1-u}b^{u}}^{b}\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|\\ &\times\left|f'(a^{1-u}b^{u})\right|a^{1-u}b^{u}\ln\left(\frac{b}{a}\right)du.\\ \leq&\ \frac{1}{\Gamma(\alpha)}\int_{0}^{\frac{1}{2}}\left(\|g\|_{\infty}\frac{\left(\ln\frac{b}{a}\right)^{\alpha}}{\alpha}\left[(1-u)^{\alpha}-u^{\alpha}\right]\right)\\ &\times\left((1-u)\left|f'(a)\right|+u\left|f'(b)\right|\right)a^{1-u}b^{u}\ln\left(\frac{b}{a}\right)du\\ &+\frac{1}{\Gamma(\alpha)}\int_{\frac{1}{2}}^{1}\left(\|g\|_{\infty}\frac{\left(\ln\frac{b}{a}\right)^{\alpha}}{\alpha}\left[u^{\alpha}-(1-u)^{\alpha}\right]\right)\\ &\times\left((1-u)\left|f'(a)\right|+u\left|f'(b)\right|\right)a^{1-u}b^{u}\ln\left(\frac{b}{a}\right)du\\ \leq&\ \frac{\ln^{\alpha+1}\left(\frac{b}{a}\right)\|g\|_{\infty}}{\Gamma(\alpha+1)}\left\{\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]\left((1-u)\left|f'(a)\right|+u\left|f'(b)\right|\right)a^{1-u}b^{u}du\right.\\ &\left.+\int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right]\left((1-u)\left|f'(a)\right|+u\left|f'(b)\right|\right)a^{1-u}b^{u}du\right\}\end{aligned}\tag{2.9}$$

Since

$$\begin{aligned}&\int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right](1-u)\,a^{1-u}b^{u}du\\ =\;&\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]ua^{u}b^{1-u}du\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}&\int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right]ua^{1-u}b^{u}du\\ =\;&\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right](1-u)\,a^{u}b^{1-u}du\end{aligned}\tag{2.11}$$

Hence, if we use (2.10) and (2.11) in (2.9), we obtain

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}\left(fg\right)(b)+J_{b-}^{\alpha}\left(fg\right)(a)\right]\right|\\ \leq\;&\frac{\ln^{\alpha+1}\left(\frac{b}{a}\right)\|g\|_{\infty}}{\Gamma(\alpha+1)}\left\{\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]\left[(1-u)\,a^{1-u}b^{u}+ua^{u}b^{1-u}\right]du\left|f'(a)\right|\right.\\ &\left.+\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]\left[ua^{1-u}b^{u}+(1-u)\,a^{u}b^{1-u}\right]du\left|f'(b)\right|\right\}.\end{aligned}$$

This completes the proof. □

Corollary 1. *In Theorem 5;*

(1) If we take $\alpha=1$ *we have the following Hermite-Hadamard-Fejer type inequality for GA-convex function which is related the right-hand side of* (1.3)*:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\int_{a}^{b}\frac{g(x)}{x}dx-\int_{a}^{b}\frac{f(x)\,g(x)}{x}dx\right|\\ \leq\;&\|g\|_{\infty}\frac{\ln^{2}\left(\frac{b}{a}\right)}{2}\left[C_{1}(1)\left|f'(a)\right|+C_{2}(1)\left|f'(b)\right|\right],\end{aligned}$$

(2) If we take $g(x)=1$ *we have following Hermite-Hadamard type inequality for GA-convex function in fractional integral forms which is related the right-hand side of* (1.4)*:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|\\ \leq\;&\frac{\ln\left(\frac{b}{a}\right)}{2}\left[C_{1}(\alpha)\left|f'(a)\right|+C_{2}(\alpha)\left|f'(b)\right|\right],\end{aligned}$$

(3) If we take $\alpha=1$ *and* $g(x)=1$ *we have the following Hermite-Hadamard type inequality for GA-convex function*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln\frac{b}{a}}\int_{a}^{b}\frac{f(x)}{x}dx\right|\leq\frac{\ln\left(\frac{b}{a}\right)}{2}\left[C_{1}(1)\left|f'(a)\right|+C_{2}(1)\left|f'(b)\right|\right].$$

Theorem 6. *Let* $f:I\subseteq\mathbb{R}_{+}\rightarrow\mathbb{R}$ *be a differentiable mapping on* I° *and* $f'\in L[a,b]$ *with* $a<b$. *If* $|f'|^{q}, q\geq 1$, *is GA-convex on* $[a,b]$ *and* $g:[a,b]\rightarrow\mathbb{R}$ *is continuous and geometrically symmetric with respect to* $\sqrt{ab}$, *then the following inequalities for*

fractional integrals hold:

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}\left(fg\right)(b)+J_{b-}^{\alpha}\left(fg\right)(a)\right]\right|$$
$$\leq \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[\left(1-\frac{1}{2^{\alpha}}\right)\left(\frac{2}{\alpha+1}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left[C_3(\alpha)\left|f'(a)\right|^q+C_4(\alpha)\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

where

$$C_3(\alpha)=\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]\left[(1-u)\left(a^{1-u}b^u\right)^q+u\left(a^ub^{1-u}\right)^q\right]du,$$
$$C_4(\alpha)=\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]\left[u\left(a^{1-u}b^u\right)^q+(1-u)\left(a^ub^{1-u}\right)^q\right]du,$$

with $\alpha>0$.

Proof. Similarly proof of Theorem 5, using Lemma 3, (2.6), (2.7), power mean inequality and GA-convexity of $|f'|^q$ we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}\left(fg\right)(b)+J_{b-}^{\alpha}\left(fg\right)(a)\right]\right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^1\left|\int_{a^{1-u}b^u}^{a^ub^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|\left|f'(a^{1-u}b^u)\right|a^{1-u}b^u\ln\left(\frac{b}{a}\right)du$$
$$\leq \frac{\ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\int_0^1\left|\int_{a^{1-u}b^u}^{a^ub^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|du\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1\left|\int_{a^{1-u}b^u}^{a^ub^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|\left|f'(a^{1-u}b^u)\right|^q\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}}$$
$$\leq \frac{\|g\|_{\infty}^{1-\frac{1}{q}}\ln^{\alpha\left(1-\frac{1}{q}\right)+1}\left(\frac{b}{a}\right)}{\alpha^{1-\frac{1}{q}}\Gamma(\alpha)}\left[\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]du+\int_{\frac{1}{2}}^1\left[u^{\alpha}-(1-u)^{\alpha}\right]du\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1\left|\int_{a^{1-u}b^u}^{a^ub^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|\left[(1-u)\left|f'(a)\right|^q+u\left|f'(b)\right|^q\right]\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}}$$
$$= \frac{\|g\|_{\infty}^{1-\frac{1}{q}}\ln^{\alpha\left(1-\frac{1}{q}\right)+1}\left(\frac{b}{a}\right)}{\alpha^{1-\frac{1}{q}}\Gamma(\alpha)}\left[\left(1-\frac{1}{2^{\alpha}}\right)\left(\frac{2}{\alpha+1}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_0^1\left|\int_{a^{1-u}b^u}^{a^ub^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|(1-u)\left(a^{1-u}b^u\right)^q du\right)\left|f'(a)\right|^q\right.$$
$$\left.+\left(\int_0^1\left|\int_{a^{1-u}b^u}^{a^ub^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|u\left(a^{1-u}b^u\right)^q du\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}$$
$$\leq \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[\left(1-\frac{1}{2^{\alpha}}\right)\left(\frac{2}{\alpha+1}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left[\left(\begin{array}{c}\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right](1-u)\left(a^{1-u}b^u\right)^q du+\\ \int_{\frac{1}{2}}^1\left[u^{\alpha}-(1-u)^{\alpha}\right](1-u)\left(a^{1-u}b^u\right)^q du\end{array}\right)\left|f'(a)\right|^q\right.$$
$$\left.+\left(\begin{array}{c}\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]u\left(a^{1-u}b^u\right)^q du+\\ \int_{\frac{1}{2}}^1\left[u^{\alpha}-(1-u)^{\alpha}\right]u\left(a^{1-u}b^u\right)^q du\end{array}\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}$$

$$
\begin{aligned}
&= \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[\left(1-\frac{1}{2^\alpha}\right)\left(\frac{2}{\alpha+1}\right)\right]^{1-\frac{1}{q}} \\
&\times\left[\left(\int_0^{\frac{1}{2}}\left[(1-u)^\alpha-u^\alpha\right]\left[(1-u)\left(a^{1-u}b^u\right)^q+u\left(a^u b^{1-u}\right)^q\right]du\right)\left|f'(a)\right|^q\right. \\
&\left.+\left(\int_0^{\frac{1}{2}}\left[(1-u)^\alpha-u^\alpha\right]\left[u\left(a^{1-u}b^u\right)^q+(1-u)\left(a^u b^{1-u}\right)^q\right]du\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}
\end{aligned}
$$

This completes the proof. □

Corollary 2. *In Theorem 6;*

(1) If we take $\alpha=1$ we have the following Hermite-Hadamard-Fejer type inequality for GA-convex function which is related the right-hand side of (1.3)*:*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x}dx-\int_a^b\frac{f(x)g(x)}{x}dx\right| \\
&\le \|g\|_\infty \ln^2\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left[C_3(1)\left|f'(a)\right|^q+C_4(1)\left|f'(b)\right|^q\right]^{\frac{1}{q}},
\end{aligned}
$$

(2) If we take $g(x)=1$ we have following Hermite-Hadamard type inequality for GA-convex function in fractional integral forms which is related the right-hand side of (1.4)*:*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right| \\
&\le \frac{\ln\left(\frac{b}{a}\right)}{2}\left[\left(1-\frac{1}{2^\alpha}\right)\left(\frac{2}{\alpha+1}\right)\right]^{1-\frac{1}{q}}\left[C_3(\alpha)\left|f'(a)\right|^q+C_4(\alpha)\left|f'(b)\right|^q\right]^{\frac{1}{q}},
\end{aligned}
$$

(3) If we take $\alpha=1$ and $g(x)=1$ we have the following Hermite-Hadamard type inequality for GA-convex function

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln\frac{b}{a}}\int_a^b\frac{f(x)}{x}dx\right| \\
&\le \ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left[C_3(1)\left|f'(a)\right|^q+C_4(1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}
$$

Theorem 7. *Let $f: I\subseteq\mathbb{R}_+\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with $a<b$. If $|f'|^q, q>1$, is GA-convex on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals hold:*

(i)

$$
\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^\alpha g(b)+J_{b-}^\alpha g(a)\right]-\left[J_{a+}^\alpha(fg)(b)+J_{b-}^\alpha(fg)(a)\right]\right| \\
&\le \frac{\|g\|_\infty \ln^{\alpha+1-\frac{2}{q}}\left(\frac{b}{a}\right)}{q^{\frac{2}{q}}\Gamma(\alpha+1)}\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}} \\
&\times\left[\left(b^q-qa^q\ln\frac{b}{a}-a^q\right)\left|f'(a)\right|^q+\left(a^q+qb^q\ln\frac{b}{a}-b^q\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}} \quad (2.12)
\end{aligned}
$$

with $\alpha>0$.

(ii)

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}\left(fg\right)(b)+J_{b-}^{\alpha}\left(fg\right)(a)\right]\right| \\
\leq\ & \frac{\|g\|_{\infty}\ln^{\alpha+1-\frac{2}{q}}\left(\frac{b}{a}\right)}{q^{\frac{2}{q}}\Gamma(\alpha+1)}\left[\frac{1}{\alpha p+1}\right]^{\frac{1}{p}} \\
&\times\left[\left(b^{q}-qa^{q}\ln\frac{b}{a}-a^{q}\right)\left|f'(a)\right|^{q}+\left(a^{q}+qb^{q}\ln\frac{b}{a}-b^{q}\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}} \quad (2.13)
\end{aligned}$$

for $0<\alpha\leq 1$. *Where* $1/p+1/q=1$.

Proof. (i) Using Lemma 3, (2.6), (2.7) and Hölder's inequality and GA-convexity of $|f'|^{q}$ we have

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}\left(fg\right)(b)+J_{b-}^{\alpha}\left(fg\right)(a)\right]\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_{0}^{1}\left|\int_{a^{1-u}b^{u}}^{a^{u}b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|\left|f'(a^{1-u}b^{u})\right|a^{1-u}b^{u}\ln\left(\frac{b}{a}\right)du \\
\leq\ & \frac{\ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\int_{0}^{1}\left|\int_{a^{1-u}b^{u}}^{a^{u}b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|^{p}du\right]^{\frac{1}{p}} \\
&\times\left[\int_{0}^{1}\left|f'(a^{1-u}b^{u})\right|^{q}\left(a^{1-u}b^{u}\right)^{q}du\right]^{\frac{1}{q}} \\
\leq\ & \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]^{p}du+\int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right]^{p}du\right]^{\frac{1}{p}} \\
&\times\left[\int_{0}^{1}\left[(1-u)\left|f'(a)\right|^{q}+u\left|f'(b)\right|^{q}\right]\left(a^{1-u}b^{u}\right)^{q}du\right]^{\frac{1}{q}} \\
=\ & \frac{\|g\|_{\infty}\ln^{\alpha+1-\frac{2}{q}}\left(\frac{b}{a}\right)}{q^{\frac{2}{q}}\Gamma(\alpha+1)}\left[\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]^{p}du+\int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right]^{p}du\right]^{\frac{1}{p}} \\
&\times\left[\left(b^{q}-qa^{q}\ln\frac{b}{a}-a^{q}\right)\left|f'(a)\right|^{q}+\left(a^{q}+qb^{q}\ln\frac{b}{a}-b^{q}\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}} \quad (2.14) \\
\leq\ & \frac{\|g\|_{\infty}\ln^{\alpha+1-\frac{2}{q}}\left(\frac{b}{a}\right)}{q^{\frac{2}{q}}\Gamma(\alpha+1)}\left[\int_{0}^{\frac{1}{2}}(1-u)^{\alpha p}-u^{\alpha p}du+\int_{\frac{1}{2}}^{1}u^{\alpha p}-(1-u)^{\alpha p}du\right]^{\frac{1}{p}} \\
&\times\left[\left(b^{q}-qa^{q}\ln\frac{b}{a}-a^{q}\right)\left|f'(a)\right|^{q}+\left(a^{q}+qb^{q}\ln\frac{b}{a}-b^{q}\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}} \\
=\ & \frac{\|g\|_{\infty}\ln^{\alpha+1-\frac{2}{q}}\left(\frac{b}{a}\right)}{q^{\frac{2}{q}}\Gamma(\alpha+1)}\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}} \\
&\times\left[\left(b^{q}-qa^{q}\ln\frac{b}{a}-a^{q}\right)\left|f'(a)\right|^{q}+\left(a^{q}+qb^{q}\ln\frac{b}{a}-b^{q}\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}
\end{aligned}$$

Here we use

$$\left[(1-t)^{\alpha}-t^{\alpha}\right]^{p}\leq(1-t)^{\alpha p}-t^{\alpha p}$$

for $t\in[0,1/2]$ and

$$\left[t^{\alpha}-(1-t)^{\alpha}\right]^{p}\leq t^{\alpha p}-(1-t)^{\alpha p}$$

for $t \in [1/2, 1]$, which follows from

$$(A-B)^q \leq A^q - B^q,$$

for any $A \geq B \geq 0$ and $q \geq 1$. Hence the inequality (2.12) is proved.

(ii) The inequality (2.13) is easily proved using (2.14) and Lemma 1. $\square$

Corollary 3. *In Theorem 7;*

(i) In (2.12);

(1) If we take $\alpha = 1$ we have the following Hermite-Hadamard-Fejer type inequality for GA-convex function which is related the right-hand side of (1.3)*:*

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x) g(x)}{x} dx \right| \\
\leq\ & \frac{\|g\|_\infty \ln^{2-\frac{2}{q}}\left(\frac{b}{a}\right)}{2q^{\frac{2}{q}}} \left[\frac{2}{p+1} \left(1 - \frac{1}{2^p}\right)\right]^{\frac{1}{p}} \\
&\times \left[\left(b^q - q a^q \ln \frac{b}{a} - a^q \right) |f'(a)|^q + \left(a^q + q b^q \ln \frac{b}{a} - b^q \right) |f'(b)|^q \right]^{\frac{1}{q}},
\end{aligned}$$

(2) If we take $g(x) = 1$ we have following Hermite-Hadamard type inequality for GA-convex function in fractional integral forms which is related the right-hand side of (1.4)*:*

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2\left(\ln \frac{b}{a}\right)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \\
\leq\ & \frac{\ln^{1-\frac{2}{q}}\left(\frac{b}{a}\right)}{2q^{\frac{2}{q}}} \left[\frac{2}{\alpha p+1} \left(1 - \frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}} \\
&\times \left[\left(b^q - q a^q \ln \frac{b}{a} - a^q \right) |f'(a)|^q + \left(a^q + q b^q \ln \frac{b}{a} - b^q \right) |f'(b)|^q \right]^{\frac{1}{q}},
\end{aligned}$$

(3) If we take $\alpha = 1$ and $g(x) = 1$ we have the following Hermite-Hadamard type inequality for GA-convex function

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{1}{\ln \frac{b}{a}} \int_a^b \frac{f(x)}{x} dx \right| \\
\leq\ & \frac{\ln^{1-\frac{2}{q}}\left(\frac{b}{a}\right)}{2q^{\frac{2}{q}}} \left[\frac{2}{p+1} \left(1 - \frac{1}{2^p}\right)\right]^{\frac{1}{p}} \\
&\times \left[\left(b^q - q a^q \ln \frac{b}{a} - a^q \right) |f'(a)|^q + \left(a^q + q b^q \ln \frac{b}{a} - b^q \right) |f'(b)|^q \right]^{\frac{1}{q}}.
\end{aligned}$$

(ii) In (2.13);

(1) If we take $\alpha = 1$ we have the following Hermite-Hadamard-Fejer type inequality for GA-convex function which is related the right-hand side of (1.3)*:*

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x) g(x)}{x} dx \right| \leq \frac{\|g\|_\infty \ln^{2-\frac{2}{q}}\left(\frac{b}{a}\right)}{2q^{\frac{2}{q}}} \left[\frac{1}{p+1} \right]^{\frac{1}{p}} \\
&\times \left[\left(b^q - q a^q \ln \frac{b}{a} - a^q \right) |f'(a)|^q + \left(a^q + q b^q \ln \frac{b}{a} - b^q \right) |f'(b)|^q \right]^{\frac{1}{q}},
\end{aligned}$$

(2) If we take $g(x) = 1$ we have following Hermite-Hadamard type inequality for GA-convex function in fractional integral forms which is related the right-hand side

of (1.4)*:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|\leq\frac{\ln^{1-\frac{2}{q}}\left(\frac{b}{a}\right)}{2q^{\frac{2}{q}}}\left[\frac{1}{\alpha p+1}\right]^{\frac{1}{p}}$$
$$\times\left[\left(b^{q}-qa^{q}\ln\frac{b}{a}-a^{q}\right)\left|f'(a)\right|^{q}+\left(a^{q}+qb^{q}\ln\frac{b}{a}-b^{q}\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}},$$

(3) If we take $\alpha=1$ *and* $g(x)=1$ *we have the following Hermite-Hadamard type inequality for GA-convex function*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln\frac{b}{a}}\int_{a}^{b}\frac{f(x)}{x}dx\right|\leq\frac{\ln^{1-\frac{2}{q}}\left(\frac{b}{a}\right)}{2q^{\frac{2}{q}}}\left[\frac{1}{p+1}\right]^{\frac{1}{p}}$$
$$\times\left[\left(b^{q}-qa^{q}\ln\frac{b}{a}-a^{q}\right)\left|f'(a)\right|^{q}+\left(a^{q}+qb^{q}\ln\frac{b}{a}-b^{q}\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}.$$

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