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SOME INEQUALITIES FOR $GG-$CONVEX FUNCTIONS

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ABSTRACT. In this paper, we establish some new integral inequalities for $GG-$convex functions by using an integral equality which is proved by the authors in [11]. We also give some new estimations for special means of real numbers.

1. INTRODUCTION

The following definition is well known in the literature and several researchers interest this useful definition.

The function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on I, if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Hermite-Hadamard inequality gives us upper and lower bounds for the mean-value of a convex function which is given as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Anderson *et. al. gave* the following definition in [4]:

Definition 1. *A function $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ is called a Mean function if*

(1) $M(x,y) = M(y,x)$,
(2) $M(x,x) = x$,
(3) $x < M(x,y) < y$, whenever $x < y$,
(4) $M(ax,ay) = aM(x,y)$ for all $a > 0$.

Based on the definition of mean function, let us recall special means (See [4])

1. Arithmetic Mean: $M(x,y) = A(x,y) = \frac{x+y}{2}$.
2. Geometric Mean: $M(x,y) = G(x,y) = \sqrt{xy}$.
3. Harmonic Mean: $M(x,y) = H(x,y) = 1/A\left(\frac{1}{x}, \frac{1}{y}\right)$.
4. Logarithmic Mean: $M(x,y) = L(x,y) = (x-y)/(\log x - \log y)$ for $x \neq y$ nd $L(x,x) = x$.
5. Identric Mean: $M(x,y) = I(x,y) = (1/e)(x^x/y^y)^{1/(x-y)}$ for $x \neq y$ nd $I(x,x) = x$.

In [4], Anderson *et. al.* also defined mean functions as the following:

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Definition 2. *Let $f : I \to (0,\infty)$ be continuous, where I is subinterval of $(0,\infty)$. Let M and N be any two Mean functions. We say f is MN-convex (concave) if*

$$f(M(x,y)) \leq (\geq) N(f(x), f(y))$$

for all $x, y \in I$.

Recall the definitions of $AG-$convex functions, $GG-$convex functions and $GA-$convex functions that are given in [2] by Niculescu:

The $AG-$convex functions (usually known as $\log-$convex functions) are those functions $f : I \to (0,\infty)$ for which

$$x, y \in I \text{ and } \lambda \in [0,1] \Longrightarrow f(\lambda x + (1-\lambda)y) \leq f(x)^{1-\lambda} f(y)^{\lambda}, \tag{1.1}$$

i.e., for which $\log f$ is convex.

The $GG-$convex functions (called in what follows multiplicatively convex functions) are those functions $f : I \to J$ (acting on subintervals of $(0,\infty)$) such that

$$x, y \in I \text{ and } \lambda \in [0,1] \Longrightarrow f\left(x^{1-\lambda} y^{\lambda}\right) \leq f(x)^{1-\lambda} f(y)^{\lambda}. \tag{1.2}$$

The class of all $GA-$convex functions is constituted by all functions $f : I \to \mathbb{R}$ (defined on subintervals of $(0,\infty)$) for which

$$x, y \in I \text{ and } \lambda \in [0,1] \Longrightarrow f\left(x^{1-\lambda} y^{\lambda}\right) \leq f(x)^{1-\lambda} f(y)^{\lambda}. \tag{1.3}$$

Besides, recall that the condition of $GG-$convexity is given in the following theorem by Anderson et. al. in [4].

Theorem 1. *Let I be an open interval of $(0,\infty)$ and let $f : I \to (0,\infty)$ be differentiable. In parts $(4)-(9)$, let $I = (0,b)$, $0 < b < \infty$.*

(1) f is $AA-$convex (concave) if and only if $f'(x)$ is increasing (decreasing).
(2) f is $AG-$convex (concave) if and only if $\frac{f'(x)}{f(x)}$ is increasing (decreasing).
(3) f is $AH-$convex (concave) if and only if $\frac{f'(x)}{f(x)^2}$ is increasing (decreasing).
(4) f is $GA-$convex (concave) if and only if $xf'(x)$ is increasing (decreasing).
(5) f is $GG-$convex (concave) if and only if $\frac{xf'(x)}{f(x)}$ is increasing (decreasing).
(6) f is $GH-$convex (concave) if and only if $\frac{xf'(x)}{f(x)^2}$ is increasing (decreasing).
(7) f is $HA-$convex (concave) if and only if $x^2 f'(x)$ is increasing (decreasing).
(8) f is $HG-$convex (concave) if and only if $\frac{x^2 f'(x)}{f(x)}$ is increasing (decreasing).
(9) f is $HH-$convex (concave) if and only if $\frac{x^2 f'(x)}{f(x)^2}$ is increasing (decreasing).

In [11], authors proved the following lemma and established new inequalities of Hermite-Hadamard type for $GA-$convex functions:

Lemma 1. *Let $f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^o and $a, b \in I^o$ with $a < b$. If $f' \in L[a,b]$, then the following identity holds:*

$$\begin{aligned} & bf(b) - af(a) - \int_a^b f(u)\,du \\ = & (\ln x - \ln a) \int_0^1 x^{2t} a^{2(1-t)} f'\left(x^t a^{1-t}\right) dt - (\ln x - \ln b) \int_0^1 x^{2t} b^{2(1-t)} f'\left(x^t b^{1-t}\right) dt \end{aligned}$$

for all $x \in [a,b]$.

For recent results, generalizations, improvements and counterparts see the papers [1]-[10] and references therein.

The main aim of this paper is to prove some new integral inequalities for $GG-$convex functions by using the above integral identity. Also some applications to special means are given.

2. MAIN RESULTS

Theorem 2. *$f : I \subset \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$ and $f' \in L[a, b]$. If $|f'(x)|$ is $GG-$convex function on $[a, b]$, then one has the following inequality:*

$$\begin{aligned} &\left| bf(b) - af(a) - \int_a^b f(u)\, du \right| \\ &\leq (\ln x - \ln a) L\left(a^2 |f'(a)|, x^2 |f'(x)|\right) \\ &\quad + (\ln b - \ln x) L\left(x^2 |f'(x)|, b^2 |f'(b)|\right) \end{aligned}$$

for all $x \in [a, b]$.

Proof. From Lemma 1 and by using the $GG-$convexity of $|f'(x)|$, we have

$$\begin{aligned} &\left| bf(b) - af(a) - \int_a^b f(u)\, du \right| \\ &\leq (\ln x - \ln a) \int_0^1 x^{2t} a^{2(1-t)} f'\left(x^t a^{1-t}\right) dt \\ &\quad + (\ln b - \ln x) \int_0^1 x^{2t} b^{2(1-t)} f'\left(x^t b^{1-t}\right) dt \\ &\leq (\ln x - \ln a) \int_0^1 x^{2t} a^{2(1-t)} \left[|f'(x)|^t |f'(a)|^{1-t}\right] dt \\ &\quad + (\ln b - \ln x) \int_0^1 x^{2t} b^{2(1-t)} \left[|f'(x)|^t |f'(b)|^{1-t}\right] dt. \end{aligned}$$

By a simple computatin, we deduce

$$\begin{aligned} &\left| bf(b) - af(a) - \int_a^b f(u)\, du \right| \\ &\leq (\ln x - \ln a) \left(\frac{x^2 |f'(x)| - a^2 |f'(a)|}{\ln(x^2 |f'(x)|) - \ln(a^2 |f'(a)|)} \right) \\ &\quad + (\ln b - \ln x) \left(\frac{b^2 |f'(b)| - x^2 |f'(x)|}{\ln(b^2 |f'(b)|) - \ln(x^2 |f'(x)|)} \right). \end{aligned}$$

Which completes the proof. □

Theorem 3. *$f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$ and $f' \in L[a,b]$. If $|f'(x)|^q$ is $GG-$convex function on $[a,b]$, then the following inequality holds:*

$$\begin{aligned} &\left| bf(b) - af(a) - \int_a^b f(u)\,du \right| \\ \leq\ & (\ln x - \ln a)\, L^{1-\frac{1}{q}}\left(a^2, x^2\right) L^{\frac{1}{q}}\left(a^2 |f'(a)|^q, x^2 |f'(x)|^q\right) \\ & + (\ln b - \ln x)\, L^{1-\frac{1}{q}}\left(x^2, b^2\right) L^{\frac{1}{q}}\left(x^2 |f'(x)|^q, b^2 |f'(b)|^q\right) \end{aligned}$$

for all $x \in [a,b]$ and $q \geq 1$.

Proof. From Lemma 1, by using the $GG-$convexity of $|f'(x)|$ and by Hölder integral inequality, we have

$$\begin{aligned} &\left| bf(b) - af(a) - \int_a^b f(u)\,du \right| \\ \leq\ & (\ln x - \ln a) \int_0^1 x^{2t} a^{2(1-t)} \left| f'\left(x^t a^{1-t}\right)\right| dt \\ & + (\ln b - \ln x) \int_0^1 x^{2t} b^{2(1-t)} \left| f'\left(x^t b^{1-t}\right)\right| dt \\ \leq\ & (\ln x - \ln a) \left(\int_0^1 x^{2t} a^{2(1-t)} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 x^{2t} a^{2(1-t)} \left[|f'(x)|^{qt} |f'(a)|^{q(1-t)} \right] dt \right)^{\frac{1}{q}} \\ & + (\ln b - \ln x) \left(\int_0^1 x^{2t} b^{2(1-t)} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 x^{2t} b^{2(1-t)} \left[|f'(x)|^{qt} |f'(b)|^{q(1-t)} \right] dt \right)^{\frac{1}{q}}. \end{aligned}$$

By making use of the necessary computation, we get

$$\begin{aligned} &\left| bf(b) - af(a) - \int_a^b f(u)\,du \right| \\ \leq\ & (\ln x - \ln a)^{\frac{1}{q}} \left(\frac{x^2 - a^2}{2} \right)^{1-\frac{1}{q}} \left(\frac{x^2 |f'(x)|^q - a^2 |f'(a)|^q}{\ln\left(x^2 |f'(x)|^q\right) - \ln\left(a^2 |f'(a)|^q\right)} \right)^{\frac{1}{q}} \\ & + (\ln b - \ln x)^{\frac{1}{q}} \left(\frac{b^2 - x^2}{2} \right)^{1-\frac{1}{q}} \left(\frac{b^2 |f'(b)|^q - x^2 |f'(x)|^q}{\ln\left(b^2 |f'(b)|^q\right) - \ln\left(x^2 |f'(x)|^q\right)} \right)^{\frac{1}{q}}. \end{aligned}$$

Which completes the proof. □

Theorem 4. *$f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$ and $f' \in L[a,b]$. If $|f'(x)|^q$ is $GG-$convex function on $[a,b]$, then the* *following inequality holds:*

$$\begin{aligned}
&\left| bf(b) - af(a) - \int_a^b f(u)\,du \right| \\
\leq\ & (\ln x - \ln a)\, L^{1-\frac{1}{q}}\left(a^{\frac{2q}{q-1}}, x^{\frac{2q}{q-1}}\right) L^{\frac{1}{q}}\left(|f'(a)|^q, |f'(x)|^q\right) \\
&+ (\ln b - \ln x)\, L^{1-\frac{1}{q}}\left(x^{\frac{2q}{q-1}}, b^{\frac{2q}{q-1}}\right) L^{\frac{1}{q}}\left(|f'(x)|^q, |f'(b)|^q\right)
\end{aligned}$$

for all $x \in [a,b]$ and $q > 1$.

Proof. Since $|f'(x)|$ is $GG-$convex function on $[a,b]$, from Lemma 1 and by using Hölder integral inequality, we can write

$$\begin{aligned}
&\left| bf(b) - af(a) - \int_a^b f(u)\,du \right| \\
\leq\ & (\ln x - \ln a) \int_0^1 x^{2t} a^{2(1-t)} \left| f'\left(x^t a^{1-t}\right)\right| dt \\
&+ (\ln b - \ln x) \int_0^1 x^{2t} b^{2(1-t)} \left| f'\left(x^t b^{1-t}\right)\right| dt \\
\leq\ & a^2 (\ln x - \ln a) \left(\int_0^1 \left(\frac{x}{a}\right)^{\frac{2qt}{q-1}} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 \left[|f'(x)|^{qt} |f'(a)|^{q(1-t)} \right] dt \right)^{\frac{1}{q}} \\
&+ b^2 (\ln b - \ln x) \left(\int_0^1 \left(\frac{x}{b}\right)^{\frac{2qt}{q-1}} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 \left[|f'(x)|^{qt} |f'(b)|^{q(1-t)} \right] dt \right)^{\frac{1}{q}}.
\end{aligned}$$

By a simple computation, we have

$$\begin{aligned}
&\left| bf(b) - af(a) - \int_a^b f(u)\,du \right| \\
\leq\ & (\ln x - \ln a) \left(\frac{x^{\frac{2q}{q-1}} - a^{\frac{2q}{q-1}}}{\ln x^{\frac{2q}{q-1}} - \ln a^{\frac{2q}{q-1}}} \right)^{1-\frac{1}{q}} \left(\frac{|f'(x)|^q - |f'(a)|^q}{\ln |f'(x)|^q - \ln |f'(a)|^q} \right)^{\frac{1}{q}} \\
&+ (\ln b - \ln x) \left(\frac{b^{\frac{2q}{q-1}} - x^{\frac{2q}{q-1}}}{\ln b^{\frac{2q}{q-1}} - \ln x^{\frac{2q}{q-1}}} \right)^{1-\frac{1}{q}} \left(\frac{|f'(b)|^q - |f'(x)|^q}{\ln |f'(b)|^q - \ln |f'(x)|^q} \right)^{\frac{1}{q}}.
\end{aligned}$$

Which completes the proof. □

Theorem 5. *$f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$ and $f' \in L[a,b]$. If $|f'(x)|^q$ is $GG-$convex function on $[a,b]$, then the* *following inequality holds:*

$$\begin{aligned}
&\left| bf(b) - af(a) - \int_a^b f(u)\,du \right| \\
\leq\ & (\ln x - \ln a)\, L^{\frac{1}{q}}\left(\left(a^2 |f'(a)|\right)^q, \left(x^2 |f'(x)|\right)^q \right) \\
&+ (\ln b - \ln x)\, L^{\frac{1}{q}}\left(\left(x^2 |f'(x)|\right)^q, \left(b^2 |f'(b)|\right)^q \right)
\end{aligned}$$

for all $x \in [a,b]$ *and* $q \geq 1$.

Proof. By a similar argument to the proof of previous theorem, since $|f'(x)|$ is $GG-$convex function on $[a,b]$, from Lemma 1 and by using a version of Hölder integral inequality, we have

$$\begin{aligned}
&\left| bf(b) - af(a) - \int_a^b f(u)\,du \right| \\
\leq\ & (\ln x - \ln a) \int_0^1 x^{2t} a^{2(1-t)} \left| f'\left(x^t a^{1-t}\right)\right| dt \\
&+ (\ln b - \ln x) \int_0^1 x^{2t} b^{2(1-t)} \left| f'\left(x^t b^{1-t}\right)\right| dt \\
\leq\ & a^2 (\ln x - \ln a) \left(\int_0^1 dt \right)^{1-\frac{1}{q}} \left(\int_0^1 \left(\frac{x}{a}\right)^{2qt} \left[|f'(x)|^{qt} |f'(a)|^{q(1-t)} \right] dt \right)^{\frac{1}{q}} \\
&+ b^2 (\ln b - \ln x) \left(\int_0^1 dt \right)^{1-\frac{1}{q}} \left(\int_0^1 \left(\frac{x}{b}\right)^{2qt} \left[|f'(x)|^{qt} |f'(b)|^{q(1-t)} \right] dt \right)^{\frac{1}{q}}.
\end{aligned}$$

By computing the above integrals, we deduce

$$\begin{aligned}
&\left| bf(b) - af(a) - \int_a^b f(u)\,du \right| \\
\leq\ & (\ln x - \ln a) \left[\frac{\left(x^2 |f'(x)|\right)^q - \left(a^2 |f'(a)|\right)^q}{\ln\left(x^2 |f'(x)|\right)^q - \ln\left(a^2 |f'(a)|\right)^q} \right]^{\frac{1}{q}} \\
&+ (\ln b - \ln x) \left[\frac{\left(b^2 |f'(b)|\right)^q - \left(x^2 |f'(x)|\right)^q}{\ln\left(b^2 |f'(b)|\right)^q - \ln\left(x^2 |f'(x)|\right)^q} \right]^{\frac{1}{q}}.
\end{aligned}$$

Which completes the proof. □

3. APPLICATIONS TO SPECIAL MEANS

Let us recall the special means of two nonnegative real numbers a, b with $a < b$:
a) The arithmetic mean:

$$A = A(a,b) := \frac{a+b}{2}, \quad a, b \geq 0,$$

b) The geometric mean:

$$G=G(a,b):=\sqrt{ab},\quad a,b\geq 0,$$

c) The harmonic mean:

$$H=H(a,b):=\frac{2ab}{a+b},\quad a,b\geq 0,$$

d) The logarithmic mean:

$$L=L(a,b):=\begin{cases} a & \text{if}\quad a=b \\ \frac{b-a}{\ln b-\ln a} & \text{if}\quad a\neq b \end{cases},\quad a,b\geq 0,$$

e) The Identric mean.

$$I=I(a,b):=\begin{cases} a & \text{if}\quad a=b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if}\quad a\neq b \end{cases},\quad a,b\geq 0,$$

f) The $p-$logarithmic mean:

$$L_p=L_p(a,b):=\begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & \text{if}\quad a\neq b \\ a & \text{if}\quad a=b \end{cases},\quad p\in\mathbb{R}\backslash\{-1,0\};\quad a,b>0.$$

The following inequality is well known in the literature:

$$H\leq G\leq L\leq I\leq A$$

It is also known that L_p is monotonically increasing over $p\in\mathbb{R}$, denoting $L_0=I$ and $L_{-1}=L$.

The following propositions holds for our main results:

Proposition 1. *Suppose that $a,b\in\mathbb{R}_+$ and $s>0$. Then, we have*

$$\begin{aligned} &\left|(b-a)L_{s+1}^{s+1}(a,b)\right| \\ \leq\ & \frac{(x-a)}{L(a,x)}L\left(a^{s+2},x^{s+2}\right)+\frac{(b-x)}{L(x,b)}L\left(x^{s+2},b^{s+2}\right) \end{aligned}\tag{3.1}$$

for all $x\in[a,b]$.

Proof. The proof is follows from Theorem 2 by applying $f(x)=\frac{x^{s+1}}{s+1}$, $x\in\mathbb{R}_+$, $s>0$, where $|f'(x)|$ is $GG-$convex function. □

Proposition 2. *Suppose that $a,b\in\mathbb{R}_+$ and $s>0$. Then for all $q\geq 1$, one has the inequality*

$$\begin{aligned} &\left|(b-a)L_{s+1}^{s+1}(a,b)\right| \\ \leq\ & \frac{(x-a)}{L(a,x)}L^{1-\frac{1}{q}}\left(a^2,x^2\right)L^{\frac{1}{q}}\left(a^{sq+2},x^{sq+2}\right) \\ & +\frac{(b-x)}{L(x,b)}L^{1-\frac{1}{q}}\left(x^2,b^2\right)L^{\frac{1}{q}}\left(x^{sq+2},b^{sq+2}\right) \end{aligned}\tag{3.2}$$

for all $x\in[a,b]$.

Proof. The proof is immediate from Theorem 3 applied for $f(x)=\frac{x^{s+1}}{s+1}$, $x\in\mathbb{R}_+$, $s>0$ where $|f'(x)|^q$ is $GG-$convex function. □

Proposition 3. *Suppose that $a,b\in\mathbb{R}_+$ and $s>0$. Then for all $q>1$, we have*

$$\begin{aligned}&\left|(b-a)L_{s+1}^{s+1}(a,b)\right|\\ \leq\ &\frac{(x-a)}{L(a,x)}L^{1-\frac{1}{q}}\left(a^{\frac{2q}{q-1}},x^{\frac{2q}{q-1}}\right)L^{\frac{1}{q}}\left(a^{sq},x^{sq}\right)\\ &+\frac{(b-x)}{L(x,b)}L^{1-\frac{1}{q}}\left(x^{\frac{2q}{q-1}},b^{\frac{2q}{q-1}}\right)L^{\frac{1}{q}}\left(x^{sq},b^{sq}\right)\end{aligned}$$

Proof. The proof is immediate from Theorem 4 applied for $f(x)=\frac{x^{s+1}}{s+1}$, $x\in\mathbb{R}_+$, $s>0$ where $|f'(x)|^q$ is $GG-$convex function. □

Proposition 4. *Suppose that $a,b\in\mathbb{R}_+$ and $s>0$. Then for all $q\geq 1$, we have*

$$\begin{aligned}&\left|(b-a)L_{s+1}^{s+1}(a,b)\right|\\ \leq\ &\frac{(x-a)}{L(a,x)}L^{\frac{1}{q}}\left(a^{sq+2},x^{sq+2}\right)\\ &+\frac{(b-x)}{L(x,b)}L^{\frac{1}{q}}\left(x^{sq+2},bx^{sq+2}\right)\end{aligned}\tag{3.3}$$

Proof. It is easy to see that by applying $f(x)=\frac{x^{s+1}}{s+1}$ to Theorem 5, $x\in\mathbb{R}_+$, $s>0$, where $|f'(x)|^q$ is $GG-$convex function. □

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