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SOME NEW GENERALIZATIONS FOR $GA-$CONVEX FUNCTIONS

AHMET OCAK AKDEMIR♠, ▲M. EMIN ÖZDEMİR, MERVE AVCI♦, AND ♠ABDULLATIF YALÇIN

Abstract. In this paper, we establish an integral identity which gives several new equalities for special selections of n and based on this equality, we give some general integral inequalities for functions whose second derivatives of absolute values are $GA-$convex functions.

1. Introduction

We will start with the definiton of convexity that has utilization in all branches of mathematics and has several applications in mathematical analysis, optimization and statictics.

The function $f: I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on I, if the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The notion of convex functions has attract attention of several researchers have been studied on inequality theory. Remarkable studies have been improved for convex functions. One of them is Hermite-Hadamard inequality that gives us upper and lower bounds for the mean-value of a convex function which is given as:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}.$$

Anderson *et. al.* mentioned mean function in [4] as following:

Definition 1. *A function $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ is called a Mean function if*

(1) $M(x,y) = M(y,x)$,
(2) $M(x,x) = x$,
(3) $x < M(x,y) < y$, whenever $x < y$,
(4) $M(ax,ay) = aM(x,y)$ for all $a > 0$.

Based on the definition of mean function, let us recall special means (See [4])

1. Arithmetic Mean: $M(x,y) = A(x,y) = \frac{x+y}{2}$.
2. Geometric Mean: $M(x,y) = G(x,y) = \sqrt{xy}$.
3. Harmonic Mean: $M(x,y) = H(x,y) = 1/A\left(\frac{1}{x}, \frac{1}{y}\right)$.

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4. Logarithmic Mean: $M(x,y)=L(x,y)=(x-y)/(\log x-\log y)$ for $x\neq y$ nd $L(x,x)=x$.

5. Identric Mean: $M(x,y)=I(x,y)=(1/e)(x^x/y^y)^{1/(x-y)}$ for $x\neq y$ nd $I(x,x)=x$.

In [4], Anderson *et. al.* also gave a definition that include several different classes of convex functions as the following:

Definition 2. *Let $f:I\to(0,\infty)$ be continuous, where I is subinterval of $(0,\infty)$. Let M and N be any two Mean functions. We say f is MN-convex (concave) if*

$$f(M(x,y))\leq(\geq)N(f(x),f(y))$$

for all $x,y\in I$.

In [2], Niculescu mentioned the following considerable definitions:

The $AG-$convex functions (usually known as $\log-$convex functions) are those functions $f:I\to(0,\infty)$ for which

$$x,y\in I\text{ and }\lambda\in[0,1]\Longrightarrow f(\lambda x+(1-\lambda)y)\leq f(x)^{1-\lambda}f(y)^{\lambda}, \tag{1.1}$$

i.e., for which $\log f$ is convex.

The $GG-$convex functions (called in what follows multiplicatively convex functions) are those functions $f:I\to J$ (acting on subintervals of $(0,\infty)$) such that

$$x,y\in I\text{ and }\lambda\in[0,1]\Longrightarrow f\left(x^{1-\lambda}y^{\lambda}\right)\leq f(x)^{1-\lambda}f(y)^{\lambda}. \tag{1.2}$$

The class of all $GA-$convex functions is constituted by all functions $f:I\to\mathbb{R}$ (defined on subintervals of $(0,\infty)$) for which

$$x,y\in I\text{ and }\lambda\in[0,1]\Longrightarrow f\left(x^{1-\lambda}y^{\lambda}\right)\leq f(x)^{1-\lambda}f(y)^{\lambda}. \tag{1.3}$$

Besides, recall that the condition of $GA-$convexity is $x^2f''+xf'\geq0$ which implies all twice differentiable nondecreasing convex functions are also $GA-$convex.

In [1], authors proved the following lemma and established new inequalities of Hermite-Hadamard type.

Lemma 1. *Let $f:I\subset\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be differentiable mapping on I^o and $a,b\in I^o$ with $a<b$. If $f'\in L[a,b]$, then*

$$[bf(b)-af(a)]-\int_a^b f(x)\,dx=(\ln b-\ln a)\int_0^1 b^{2t}a^{2(1-t)}f'\left(b^ta^{1-t}\right)dt.$$

In [10], Latif gave the following integral identity and proved new inequalities of Hermite-Hadamard type.

Lemma 2. *Let $f:I\subset\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I^o and $a,b\in I^o$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\begin{aligned}&bf(b)-af(a)-\int_a^b f(x)\,dx\\&=\frac{\ln b-\ln a}{2}\left[\int_0^1 b^{1+t}a^{1-t}f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)dt+\int_0^1 b^{1-t}a^{1+t}f'\left(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\right)dt\right].\end{aligned}$$

For recent results, generalizations, improvements and counterparts see the papers [1]-[10] and references therein.

The main aim of this paper is to prove some new integral inequalities for $GA-$convex functions by using a new integral identity. Also some applications to special means are given.

2. A new lemma

We will give a new integral identity which is emboided in the following lemma to obtain our results.

Lemma 3. *Let $f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I^o and $a, b \in I^o$ with $a < b$. If $f'' \in L[a,b]$, then the following identity holds:*

(2.1)

$$\begin{aligned} H(a,b;n) &= \frac{a^{n+1}f'(a) - b^{n+1}f'(b)}{(n+1)(n+2)} - \frac{a^n f(a) - b^n f(b)}{(n+1)} - \int_a^b u^{n-1} f(u)\,du \\ &= \frac{\ln a - \ln x}{(n+1)(n+2)} \int_0^1 a^{(n+2)t} x^{(n+2)(1-t)} f''\left(a^t x^{1-t}\right) dt \\ &\quad + \frac{\ln x - \ln b}{(n+1)(n+2)} \int_0^1 x^{(n+2)t} b^{(n+2)(1-t)} f''\left(x^t b^{1-t}\right) dt \end{aligned}$$

for all $x \in [a,b]$ and $n \geq 0$.

Proof. Integrating by parts and by using the change of the variables, one can see that

$$\begin{aligned} &\int_0^1 a^{(n+2)t} x^{(n+2)(1-t)} f''\left(a^t x^{1-t}\right) dt \\ = &\frac{a^{n+1}f'(a) - x^{n+1}f'(x) - (n+2)a^n f(a) + (n+2)x^n f(x)}{\ln a - \ln x} \\ &- \frac{(n+1)(n+2)}{\ln a - \ln x} \int_a^x u^{n-1} f(u)\,du \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 x^{(n+2)t} b^{(n+2)(1-t)} f''\left(x^t b^{1-t}\right) dt \\ = &\frac{x^{n+1}f'(x) - b^{n+1}f'(b) - (n+2)x^n f(x) + (n+2)b^n f(b)}{\ln x - \ln b} \\ &- \frac{(n+1)(n+2)}{\ln x - \ln b} \int_x^b u^{n-1} f(u)\,du. \end{aligned}$$

Which completes the proof of 2.1. $\square$

Remark 1. *Several new equality can be derived from 2.1, by selecting of the special cases of n.*

3. NEW INEQUALITIES

Theorem 1. *$f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I^o, $a,b \in I^o$ with $a<b$ and $f'' \in L[a,b]$. If $|f''(x)|$ is $GA-$convex function on $[a,b]$, then the following inequality holds:*

$$
\begin{aligned}
& |H(a,b;n)| \\
\leq \;& \frac{x^{n+2}}{(n+1)(n+2)^3}\left[F_1(a,x)\right] + \frac{b^{n+2}}{(n+1)(n+2)^3}\left[F_2(b,x)\right]
\end{aligned}
$$

where

$$
\begin{aligned}
& F_1(a,x) \\
=\;& |f'(a)|\left(\frac{\left(\frac{a}{x}\right)^{n+2}\ln\left(\frac{a}{x}\right)^{n+2} - \left(\frac{a}{x}\right)^{n+2}+1}{\ln a - \ln x}\right) + |f'(x)|\left(\frac{\left(\frac{a}{x}\right)^{n+2} - \ln\left(\frac{a}{x}\right)^{n+2}-1}{\ln a - \ln x}\right) \\
& F_2(b,x) \\
=\;& |f'(x)|\left(\frac{\left(\frac{x}{b}\right)^{n+2}\ln\left(\frac{x}{b}\right)^{n+2} - \left(\frac{x}{b}\right)^{n+2}+1}{\ln b - \ln x}\right) + |f'(b)|\left(\frac{\left(\frac{x}{b}\right)^{n+2} - \ln\left(\frac{x}{b}\right)^{n+2}-1}{\ln b - \ln x}\right)
\end{aligned}
$$

for all $x \in [a,b]$ and $n \geq 0$.

Proof. From Lemma 3 and by using the $GA-$convexity of $|f''(x)|$, we have

$$
\begin{aligned}
& |H(a,b;n)| \\
\leq \;& \frac{\ln x - \ln a}{(n+1)(n+2)}\int_0^1 a^{(n+2)t}x^{(n+2)(1-t)}\left|f''\left(a^t x^{1-t}\right)\right| dt \\
& + \frac{\ln b - \ln x}{(n+1)(n+2)}\int_0^1 x^{(n+2)t}b^{(n+2)(1-t)}\left|f''\left(x^t b^{1-t}\right)\right| dt \\
\leq \;& \frac{\ln x - \ln a}{(n+1)(n+2)}\int_0^1 a^{(n+2)t}x^{(n+2)(1-t)}\left[t|f'(a)| + (1-t)|f'(x)|\right] dt \\
& + \frac{\ln b - \ln x}{(n+1)(n+2)}\int_0^1 x^{(n+2)t}b^{(n+2)(1-t)}\left[t|f'(x)| + (1-t)|f'(b)|\right] dt.
\end{aligned}
$$

By computing the above integrals and simplfying, we obtain

$$\begin{aligned}
& \left|H\left(a,b;n\right)\right| \\
\leq\ & \frac{x^{n+2}\left(\ln x-\ln a\right)}{\left(n+1\right)\left(n+2\right)}\left[\left|f'\left(a\right)\right|\left(\frac{\left(\frac{a}{x}\right)^{n+2}\ln\left(\frac{a}{x}\right)^{n+2}-\left(\frac{a}{x}\right)^{n+2}+1}{\left(\ln a^{n+2}-\ln x^{n+2}\right)^{2}}\right)\right. \\
& \left.+\left|f'\left(x\right)\right|\left(\frac{\left(\frac{a}{x}\right)^{n+2}-\ln\left(\frac{a}{x}\right)^{n+2}-1}{\left(\ln a^{n+2}-\ln x^{n+2}\right)^{2}}\right)\right] \\
& +\frac{b^{n+2}\left(\ln b-\ln x\right)}{\left(n+1\right)\left(n+2\right)}\left[\left|f'\left(x\right)\right|\left(\frac{\left(\frac{x}{b}\right)^{n+2}\ln\left(\frac{x}{b}\right)^{n+2}-\left(\frac{x}{b}\right)^{n+2}+1}{\left(\ln b^{n+2}-\ln x^{n+2}\right)^{2}}\right)\right. \\
& \left.+\left|f'\left(b\right)\right|\left(\frac{\left(\frac{x}{b}\right)^{n+2}-\ln\left(\frac{x}{b}\right)^{n+2}-1}{\left(\ln b^{n+2}-\ln x^{n+2}\right)^{2}}\right)\right].
\end{aligned}$$

Which completes the proof. □

Corollary 1. *Under the assumptions of Theorem 1, if we choose $n=0$, we have the following ineqality:*

$$\begin{aligned}
& \left|\frac{af'\left(a\right)-bf'\left(b\right)}{2}-f\left(a\right)+f\left(b\right)-\int_{a}^{b}\frac{f\left(u\right)}{u}du\right| \\
\leq\ & \frac{x^{2}}{8}\left[F_{3}(a,x)\right]+\frac{b^{2}}{8}\left[F_{4}(b,x)\right]
\end{aligned}$$

where

$$\begin{aligned}
F_{3}(a,x) &= \left|f'\left(a\right)\right|\left(\frac{\left(\frac{a}{x}\right)^{2}\ln\left(\frac{a}{x}\right)^{2}-\left(\frac{a}{x}\right)^{2}+1}{\ln a-\ln x}\right)+\left|f'\left(x\right)\right|\left(\frac{\left(\frac{a}{x}\right)^{2}-\ln\left(\frac{a}{x}\right)^{2}-1}{\ln a-\ln x}\right) \\
F_{4}(b,x) &= \left|f'\left(x\right)\right|\left(\frac{\left(\frac{x}{b}\right)^{2}\ln\left(\frac{x}{b}\right)^{2}-\left(\frac{x}{b}\right)^{2}+1}{\ln b-\ln x}\right)+\left|f'\left(b\right)\right|\left(\frac{\left(\frac{x}{b}\right)^{2}-\ln\left(\frac{x}{b}\right)^{2}-1}{\ln b-\ln x}\right)
\end{aligned}$$

for all $x\in\left[a,b\right]$.

Corollary 2. *Under the assumptions of Theorem 1, if we choose $n=1$, we have the following ineqality:*

$$\begin{aligned}
& \left|\frac{a^{2}f'\left(a\right)-b^{2}f'\left(b\right)}{6}-\frac{af\left(a\right)-bf\left(b\right)}{2}-\int_{a}^{b}f\left(u\right)du\right| \\
\leq\ & \frac{x^{3}}{54}\left[F_{5}(a,x)\right]+\frac{b^{3}}{54}\left[F_{6}(b,x)\right]
\end{aligned}$$

where

$$\begin{aligned}
F_{5}(a,x) &= \left|f'\left(a\right)\right|\left(\frac{\left(\frac{a}{x}\right)^{3}\ln\left(\frac{a}{x}\right)^{3}-\left(\frac{a}{x}\right)^{3}+1}{\ln a-\ln x}\right)+\left|f'\left(x\right)\right|\left(\frac{\left(\frac{a}{x}\right)^{3}-\ln\left(\frac{a}{x}\right)^{3}-1}{\ln a-\ln x}\right) \\
F_{6}(b,x) &= \left|f'\left(x\right)\right|\left(\frac{\left(\frac{x}{b}\right)^{3}\ln\left(\frac{x}{b}\right)^{3}-\left(\frac{x}{b}\right)^{3}+1}{\ln b-\ln x}\right)+\left|f'\left(b\right)\right|\left(\frac{\left(\frac{x}{b}\right)^{3}-\ln\left(\frac{x}{b}\right)^{3}-1}{\ln b-\ln x}\right)
\end{aligned}$$

for all $x\in\left[a,b\right]$.

Theorem 2. *$f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I^o, $a, b \in I^o$ with $a < b$ and $f'' \in L[a,b]$. If $|f''(x)|^q$ is $GA-$convex function on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&|H(a,b;n)| \\
\leq\ & \frac{(\ln x - \ln a)^{1-\frac{1}{q}}}{(n+1)(n+2)} L^{1-\frac{1}{q}}\left(x^{n+2}, a^{n+2}\right) \\
&\times \left(|f''(a)|^q \left[\frac{L\left(a^{n+2}, x^{n+2}\right) - a^{n+2}}{(n+2)} \right] \right. \\
&\left. + |f''(x)|^q \left[\frac{x^{n+2} - L\left(a^{n+2}, x^{n+2}\right)}{(n+2)} \right] \right)^{\frac{1}{q}} \\
&+ \frac{(\ln b - \ln x)^{1-\frac{1}{q}}}{(n+1)(n+2)} L^{1-\frac{1}{q}}\left(x^{n+2}, b^{n+2}\right) \\
&\times \left(|f''(x)|^q \left[\frac{L\left(x^{n+2}, b^{n+2}\right) - x^{n+2}}{(n+2)} \right] \right. \\
&\left. + |f''(b)|^q \left[\frac{b^{n+2} - L\left(x^{n+2}, b^{n+2}\right)}{(n+2)} \right] \right)^{\frac{1}{q}}.
\end{aligned}$$

for all $x \in [a,b]$, $n \geq 0$ and $q \geq 1$.

Proof. From Lemma 3, by using the $GA-$convexity of $|f'(x)|$ and by Hölder integral inequality, we have

$$\begin{aligned}
&|H(a,b;n)| \\
\leq\ & \frac{\ln x - \ln a}{(n+1)(n+2)} \int_0^1 a^{(n+2)t} x^{(n+2)(1-t)} \left| f''\left(a^t x^{1-t}\right) \right| dt \\
&+ \frac{\ln b - \ln x}{(n+1)(n+2)} \int_0^1 x^{(n+2)t} b^{(n+2)(1-t)} \left| f''\left(x^t b^{1-t}\right) \right| dt \\
\leq\ & \frac{\ln x - \ln a}{(n+1)(n+2)} \left(\int_0^1 a^{(n+2)t} x^{(n+2)(1-t)} dt \right)^{1-\frac{1}{q}} \\
&\times \left(\int_0^1 a^{(n+2)t} x^{(n+2)(1-t)} \left[t |f''(a)|^q + (1-t) |f''(x)|^q \right] dt \right)^{\frac{1}{q}} \\
&+ \frac{\ln b - \ln x}{(n+1)(n+2)} \left(\int_0^1 x^{(n+2)t} b^{(n+2)(1-t)} dt \right)^{1-\frac{1}{q}} \\
&\times \left(\int_0^1 x^{(n+2)t} b^{(n+2)(1-t)} \left[t |f''(x)|^q + (1-t) |f''(b)|^q \right] dt \right)^{\frac{1}{q}}.
\end{aligned}$$

By making use of the necessary computation, we get

$$
\begin{aligned}
& \left|H(a,b;n)\right| \\
\leq\ & \frac{(\ln x-\ln a)^{1-\frac{1}{q}}}{(n+1)(n+2)}L^{1-\frac{1}{q}}\left(x^{n+2},a^{n+2}\right) \\
& \times\left(x^{n+2}\left|f''(a)\right|^{q}\left[\frac{\left(\frac{a}{x}\right)^{n+2}\ln\left(\frac{a}{x}\right)^{n+2}-\left(\frac{a}{x}\right)^{n+2}+1}{(n+2)\left(\ln x^{n+2}-\ln a^{n+2}\right)}\right]\right. \\
& \left.+x^{n+2}\left|f''(x)\right|^{q}\left[\frac{\left(\frac{a}{x}\right)^{n+2}-\ln\left(\frac{a}{x}\right)^{n+2}-1}{(n+2)\left(\ln x^{n+2}-\ln a^{n+2}\right)}\right]\right)^{\frac{1}{q}} \\
& +\frac{(\ln b-\ln x)^{1-\frac{1}{q}}}{(n+1)(n+2)}L^{1-\frac{1}{q}}\left(x^{n+2},b^{n+2}\right) \\
& \times\left(b^{n+2}\left|f''(x)\right|^{q}\left[\frac{\left(\frac{x}{b}\right)^{n+2}\ln\left(\frac{x}{b}\right)^{n+2}-\left(\frac{x}{b}\right)^{n+2}+1}{(n+2)\left(\ln b^{n+2}-\ln x^{n+2}\right)}\right]\right. \\
& \left.+b^{n+2}\left|f''(b)\right|^{q}\left[\frac{\left(\frac{x}{b}\right)^{n+2}-\ln\left(\frac{x}{b}\right)^{n+2}-1}{(n+2)\left(\ln b^{n+2}-\ln x^{n+2}\right)}\right]\right)^{\frac{1}{q}}.
\end{aligned}
$$

Which completes the proof. □

Corollary 3. *Under the assumptions of Theorem 2, if we choose $n=0$, we have the following ineqality:*

$$
\begin{aligned}
& \left|\frac{af'(a)-bf'(b)}{2}-f(a)+f(b)-\int_a^b\frac{f(u)}{u}du\right| \\
\leq\ & \frac{(\ln x-\ln a)^{1-\frac{1}{q}}}{2}L^{1-\frac{1}{q}}\left(x^{2},a^{2}\right) \\
& \times\left(\left|f''(a)\right|^{q}\left[\frac{L\left(a^2,x^2\right)-a^2}{2}\right]+\left|f''(x)\right|^{q}\left[\frac{x^2-L\left(a^2,x^2\right)}{2}\right]\right)^{\frac{1}{q}} \\
& +\frac{(\ln b-\ln x)^{1-\frac{1}{q}}}{2}L^{1-\frac{1}{q}}\left(x^{2},b^{2}\right) \\
& \times\left(\left|f''(x)\right|^{q}\left[\frac{L\left(x^2,b^2\right)-x^2}{2}\right]+\left|f''(b)\right|^{q}\left[\frac{b^2-L\left(x^2,b^2\right)}{2}\right]\right)^{\frac{1}{q}}.
\end{aligned}
$$

for all $x\in[a,b]$.

Corollary 4. *Under the assumptions of Theorem 2, if we choose $n=1$, we have the following ineqality:*

$$\begin{aligned}
&\left|\frac{a^2f'(a)-b^2f'(b)}{6}-\frac{af(a)-bf(b)}{2}-\int_a^b f(u)\,du\right| \\
\leq\ & \frac{(\ln x-\ln a)^{1-\frac{1}{q}}}{6}L^{1-\frac{1}{q}}\left(x^3,a^3\right) \\
&\times\left(|f''(a)|^q\left[\frac{L\left(a^3,x^3\right)-a^3}{3}\right]+|f''(x)|^q\left[\frac{x^3-L\left(a^3,x^3\right)}{3}\right]\right)^{\frac{1}{q}} \\
&+\frac{(\ln b-\ln x)^{1-\frac{1}{q}}}{6}L^{1-\frac{1}{q}}\left(x^3,b^3\right) \\
&\times\left(|f''(x)|^q\left[\frac{L\left(x^3,b^3\right)-x^3}{3}\right]+|f''(b)|^q\left[\frac{b^3-L\left(x^3,b^3\right)}{3}\right]\right)^{\frac{1}{q}}.
\end{aligned}$$

for all $x\in[a,b]$.

Theorem 3. *$f:I\subset\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I^o, $a,b\in I^o$ with $a<b$ and $f'\in L[a,b]$. If $|f'(x)|^q$ is $GA-$convex function on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&|H(a,b;n)| \\
\leq\ & \frac{(\ln x-\ln a)^{\frac{1}{q}}}{(n+1)(n+2)}\left(\frac{q-1}{(n+2)q}\right)^{1-\frac{1}{q}}\left(a^{\frac{q(n+2)}{q-1}}-x^{\frac{q(n+2)}{q-1}}\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{|f''(a)|^q+|f''(x)|^q}{2}\right)^{\frac{1}{q}} \\
&+\frac{(\ln b-\ln x)^{\frac{1}{q}}}{(n+1)(n+2)}\left(\frac{q-1}{(n+2)q}\right)^{1-\frac{1}{q}}\left(x^{\frac{q(n+2)}{q-1}}-b^{\frac{q(n+2)}{q-1}}\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{|f''(x)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

for all $x\in[a,b]$ and $q>1$.

Proof. Since $|f'(x)|$ is $GA-$convex function on $[a,b]$, from Lemma 3 and by using Hölder integral inequality, we can write

$$\begin{aligned}
&|H(a,b;n)| \\
\leq\ & \frac{\ln x-\ln a}{(n+1)(n+2)}\int_0^1 a^{(n+2)t}x^{(n+2)(1-t)}\left|f''\left(a^t x^{1-t}\right)\right|dt \\
&+\frac{\ln b-\ln x}{(n+1)(n+2)}\int_0^1 x^{(n+2)t}b^{(n+2)(1-t)}\left|f''\left(x^t b^{1-t}\right)\right|dt \\
\leq\ & x^{n+2}\frac{\ln x-\ln a}{(n+1)(n+2)}\left(\int_0^1 \frac{a}{x}^{(n+2)\frac{qt}{q-1}}dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 t\left|f''(a)\right|^q+(1-t)\left|f''(x)\right|^q dt\right)^{\frac{1}{q}} \\
&+b^{(n+2)}\frac{\ln b-\ln x}{(n+1)(n+2)}\left(\int_0^1 \frac{x}{b}^{(n+2)\frac{qt}{q-1}}dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 t\left|f''(x)\right|^q+(1-t)\left|f''(b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

By a simple computation, we have

$$\begin{aligned}
&|H(a,b;n)| \\
\leq\ & \frac{(\ln x-\ln a)^{\frac{1}{q}}}{(n+1)(n+2)}\left(\frac{q-1}{(n+2)q}\right)^{1-\frac{1}{q}}\left(a^{\frac{q(n+2)}{q-1}}-x^{\frac{q(n+2)}{q-1}}\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{\left|f''(a)\right|^q+\left|f''(x)\right|^q}{2}\right)^{\frac{1}{q}} \\
&+\frac{(\ln b-\ln x)^{\frac{1}{q}}}{(n+1)(n+2)}\left(\frac{q-1}{(n+2)q}\right)^{1-\frac{1}{q}}\left(x^{\frac{q(n+2)}{q-1}}-b^{\frac{q(n+2)}{q-1}}\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{\left|f''(x)\right|^q+\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Which completes the proof. □

Corollary 5. *Under the assumptions of Theorem 3, if we choose $n=0$, we have the following ineqality:*

$$\begin{aligned}
&\left|\frac{af'(a)-bf'(b)}{2}-f(a)+f(b)-\int_a^b \frac{f(u)}{u}du\right| \\
\leq\ & \frac{(\ln x-\ln a)^{\frac{1}{q}}}{2}\left(\frac{q-1}{2q}\right)^{1-\frac{1}{q}}\left(a^{\frac{2q}{q-1}}-x^{\frac{2q}{q-1}}\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{|f''(a)|^q+|f''(x)|^q}{2}\right)^{\frac{1}{q}} \\
&+\frac{(\ln b-\ln x)^{\frac{1}{q}}}{2}\left(\frac{q-1}{2q}\right)^{1-\frac{1}{q}}\left(x^{\frac{2q}{q-1}}-b^{\frac{2q}{q-1}}\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{|f''(x)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

for all $x\in[a,b]$.

Corollary 6. *Under the assumptions of Theorem 3, if we choose $n=1$, we have the following ineqality:*

$$\begin{aligned}
&\left|\frac{a^2f'(a)-b^2f'(b)}{6}-\frac{af(a)-bf(b)}{2}-\int_a^b f(u)\,du\right| \\
\leq\ & \frac{(\ln x-\ln a)^{\frac{1}{q}}}{6}\left(\frac{q-1}{3q}\right)^{1-\frac{1}{q}}\left(a^{\frac{3q}{q-1}}-x^{\frac{3q}{q-1}}\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{|f''(a)|^q+|f''(x)|^q}{2}\right)^{\frac{1}{q}} \\
&+\frac{(\ln b-\ln x)^{\frac{1}{q}}}{6}\left(\frac{q-1}{3q}\right)^{1-\frac{1}{q}}\left(x^{\frac{3q}{q-1}}-b^{\frac{3q}{q-1}}\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{|f''(x)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

for all $x\in[a,b]$.

Theorem 4. $f:I\subset\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ *be a differentiable mapping on I^o, $a,b\in I^o$ with $a<b$ and $f'\in L[a,b]$. If $|f'(x)|^q$ is $GA-$convex function on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&|H(a,b;n)| \\
\leq\ & x^{n+2}\frac{(\ln x-\ln a)^{1-\frac{2}{q}}}{(n+1)(n+2)}(F_3(a,x))^{\frac{1}{q}} \\
&+b^{n+2}\frac{(\ln b-\ln x)^{1-\frac{2}{q}}}{(n+1)(n+2)}(F_4(x,b))^{\frac{1}{q}}.
\end{aligned}$$

where

$$
\begin{aligned}
& F_3(a,x) \\
= \ & \left|f''(a)\right|^q \left(\frac{\left(\frac{a}{x}\right)^{q(n+2)} \ln\left(\frac{a}{x}\right)^{q(n+2)} + 1}{q^2(n+2)^2}\right) + \left|f''(x)\right|^q \left(\frac{\left(\frac{a}{x}\right)^{q(n+2)} - \ln\left(\frac{a}{x}\right)^{q(n+2)} - 1}{q^2(n+2)^2}\right) \\
& F_4(x,b) \\
= \ & \left|f''(x)\right|^q \left(\frac{\left(\frac{x}{b}\right)^{q(n+2)} \ln\left(\frac{x}{b}\right)^{q(n+2)} + 1}{q^2(n+2)^2}\right) + \left|f''(b)\right|^q \left(\frac{\left(\frac{x}{b}\right)^{q(n+2)} - \ln\left(\frac{x}{b}\right)^{q(n+2)} - 1}{q^2(n+2)^2}\right)
\end{aligned}
$$

for all $x \in [a,b]$ *and* $q \geq 1$.

Proof. By a similar argument to the proof of previous theorem, since $|f'(x)|$ is $GA-$convex function on $[a,b]$, from Lemma 3 and by using another version of Hölder integral inequality, we have

$$
\begin{aligned}
& |H(a,b;n)| \\
\leq \ & \frac{\ln x - \ln a}{(n+1)(n+2)} \int_0^1 a^{(n+2)t} x^{(n+2)(1-t)} \left|f''\left(a^t x^{1-t}\right)\right| dt \\
& + \frac{\ln b - \ln x}{(n+1)(n+2)} \int_0^1 x^{(n+2)t} b^{(n+2)(1-t)} \left|f''\left(x^t b^{1-t}\right)\right| dt \\
\leq \ & x^{n+2} \frac{\ln x - \ln a}{(n+1)(n+2)} \left(\int_0^1 dt\right)^{1-\frac{1}{q}} \\
& \times \left(\int_0^1 \frac{a}{x}^{(n+2)qt} \left[t\left|f''(a)\right|^q + (1-t)\left|f''(x)\right|^q\right] dt\right)^{\frac{1}{q}} \\
& + b^{(n+2)} \frac{\ln b - \ln x}{(n+1)(n+2)} \left(\int_0^1 dt\right)^{1-\frac{1}{q}} \\
& \times \left(\int_0^1 \frac{x}{b}^{(n+2)\frac{qt}{q-1}} \left[t\left|f''(x)\right|^q + (1-t)\left|f''(b)\right|^q\right] dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

By computing the above integrals, we deduce

$$
\begin{aligned}
& |H(a,b;n)| \\
\leq\ & x^{n+2}\frac{\ln x-\ln a}{(n+1)(n+2)} \\
& \times\left(|f''(a)|^{q}\left(\frac{\left(\frac{a}{x}\right)^{q(n+2)}\ln\left(\frac{a}{x}\right)^{q(n+2)}+1}{\left(\ln\left(\frac{a}{x}\right)^{q(n+2)}\right)^{2}}\right)+|f''(x)|^{q}\left(\frac{\left(\frac{a}{x}\right)^{q(n+2)}-\ln\left(\frac{a}{x}\right)^{q(n+2)}-1}{\left(\ln\left(\frac{a}{x}\right)^{q(n+2)}\right)^{2}}\right)\right)^{\frac{1}{q}} \\
& +b^{(n+2)}\frac{\ln b-\ln x}{(n+1)(n+2)} \\
& \times\left(|f''(x)|^{q}\left(\frac{\left(\frac{x}{b}\right)^{q(n+2)}\ln\left(\frac{x}{b}\right)^{q(n+2)}+1}{\left(\ln\left(\frac{x}{b}\right)^{q(n+2)}\right)^{2}}\right)+|f''(b)|^{q}\left(\frac{\left(\frac{x}{b}\right)^{q(n+2)}-\ln\left(\frac{x}{b}\right)^{q(n+2)}-1}{\left(\ln\left(\frac{x}{b}\right)^{q(n+2)}\right)^{2}}\right)\right)^{\frac{1}{q}}.
\end{aligned}
$$

Which completes the proof. □

Corollary 7. *Under the assumptions of Theorem 4, if we choose $n=0$, we have the following ineqality:*

$$
\begin{aligned}
& \left|\frac{af'(a)-bf'(b)}{2}-f(a)+f(b)-\int_{a}^{b}\frac{f(u)}{u}du\right| \\
\leq\ & \frac{x^{2}(\ln x-\ln a)^{1-\frac{2}{q}}}{2} \\
& \times\left(|f''(a)|^{q}\left(\frac{\left(\frac{a}{x}\right)^{2q}\ln\left(\frac{a}{x}\right)^{2q}+1}{4q^{2}}\right)+|f''(x)|^{q}\left(\frac{\left(\frac{a}{x}\right)^{2q}-\ln\left(\frac{a}{x}\right)^{2q}-1}{4q^{2}}\right)\right) \\
& +\frac{b^{2}(\ln b-\ln x)^{1-\frac{2}{q}}}{2} \\
& \times\left(|f''(x)|^{q}\left(\frac{\left(\frac{x}{b}\right)^{2q}\ln\left(\frac{x}{b}\right)^{2q}+1}{4q^{2}}\right)+|f''(b)|^{q}\left(\frac{\left(\frac{x}{b}\right)^{2q}-\ln\left(\frac{x}{b}\right)^{2q}-1}{4q^{2}}\right)\right)^{\frac{1}{q}}
\end{aligned}
$$

for all $x\in[a,b]$.

Corollary 8. *Under the assumptions of Theorem 3, if we choose $n=1$, we have the following ineqality:*

$$\begin{aligned}
&\left|\frac{af'(a)-bf'(b)}{2}-f(a)+f(b)-\int_a^b\frac{f(u)}{u}du\right| \\
\leq\;& \frac{x^3(\ln x-\ln a)^{1-\frac{2}{q}}}{6} \\
&\times\left(|f''(a)|^q\left(\frac{\left(\frac{a}{x}\right)^{3q}\ln\left(\frac{a}{x}\right)^{3q}+1}{9q^2}\right)+|f''(x)|^q\left(\frac{\left(\frac{a}{x}\right)^{3q}-\ln\left(\frac{a}{x}\right)^{3q}-1}{9q^2}\right)\right) \\
&+\frac{b^3(\ln b-\ln x)^{1-\frac{2}{q}}}{6} \\
&\times\left(|f''(x)|^q\left(\frac{\left(\frac{x}{b}\right)^{3q}\ln\left(\frac{x}{b}\right)^{3q}+1}{9q^2}\right)+|f''(b)|^q\left(\frac{\left(\frac{x}{b}\right)^{3q}-\ln\left(\frac{x}{b}\right)^{3q}-1}{9q^2}\right)\right)^{\frac{1}{q}}
\end{aligned}$$

for all $x\in[a,b]$.

Remark 2. *Several applications can be given to special means of two real number, by choosing $f(x)=\frac{x^{s+1}}{s+1}$, $x\in\mathbb{R}_+$, $s>0$ ($|f''(x)|$ is $GA-$convex function).*

Remark 3. *In our results, if we set $a^{n+1}f'(a)=b^{n+1}f'(b)$, we can obtain several new inequalities.*

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♠Ağri İbrahim Çeçen University, Faculty of Science and Arts, Department of Mathematics, 04100, Ağri, Turkey
E-mail address: ahmetakdemir@agri.edu.tr

♣Atatürk University, K.K. Education Faculty, Department of Mathematics, 25240, Erzurum, Turkey
E-mail address: emos@atauni.edu.tr

♦Adiyaman University, Faculty of Science and Arts, Department of Mathematics, Adiyaman, Turkey
E-mail address: merveavci@ymail.com

E-mail address: latif.yalcin012@gmail.com