IMPROVING THE HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS

S. S. DRAGOMIR[1,2]

Abstract. In this paper we establish some inequalities that improve the celebrated Hermite-Hadamard inequality for convex functions Applications for special means are also provided.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers and $I \subseteq \mathbb{R}$ be an interval. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be *convex* in the classical sense if it satisfies the following inequality

$$f\left(tx + (1-t)t\right) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and $a, b \in I$ with $a < b$. Then the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds if f is convex, and is known in the literature as *Hermite-Hadamard inequality*, after the name of C. Hermite and J. Hadamard (see [19]). The inequalities in (1.1) hold in reversed direction if f is a concave function.

A vast literature related to (1.1) have been produced by a large number of mathematicians [15] since it is considered to be one of the most famous inequality for convex functions due to its usefulness and many applications in various branches of Pure and Applied Mathematics, such as Numerical Analysis [6], Information Theory [3], Operator Theory [12] and others.

Let $[a,b]$ be a compact interval of real numbers, $d_n := \left\{x_i | i = \overline{0,n}\right\} \subset [a,b]$, a division of the interval $[a,b]$, given by

$$d_n : a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b \ \ (n \geq 1)$$

and f a convex mapping on $[a,b]$. We consider the following sums [7], see also [15, p. 22]:

$$h_{d_n}(f) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i)$$

which we called *Hadamard's inferior sum*, and

$$H_{d_n}(f) := \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)$$

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 26D20.

Key words and phrases. Convex functions, Integral inequalities, Hermite-Hadamard type inequalities, Special means.

which was called *Hadamard's superior sum*.

In [7] we have proven amongst other that, if $f:[a,b]\to\mathbb{R}$ is convex, then we have the following refinement of the *Hermite-Hadamard inequality*:

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}h_{d_n}(f)\leq\frac{1}{b-a}\int_a^b f(t)\,dt \\
&\leq \frac{1}{b-a}H_{d_n}(f)\leq\frac{f(a)+f(b)}{2}
\end{aligned}
\tag{1.2}
$$

for any division d_n of the interval $[a,b]$.

If we write the inequality (1.2) for the division $d_2: a=x_0<x=x_1<x_2=b$ then we have

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq f\left(\frac{a+x}{2}\right)\frac{x-a}{b-a}+f\left(\frac{x+b}{2}\right)\frac{b-x}{b-a} \\
&\leq \frac{1}{b-a}\int_a^b f(t)\,dt \\
&\leq \frac{1}{2}\left[f(x)+\frac{f(a)(x-a)+f(b)(b-x)}{b-a}\right]\leq\frac{f(a)+f(b)}{2}
\end{aligned}
\tag{1.3}
$$

for any $x\in(a,b)$. The inequality (1.3) also holds for $x=a$ and $x=b$.

If f is concave, then the inequalities in (1.2) and (1.3) reverse.

If we take in (1.3) $x=\frac{a+b}{2}$ then we get

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\
&\leq \frac{1}{b-a}\int_a^b f(t)\,dt \\
&\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\leq\frac{f(a)+f(b)}{2}.
\end{aligned}
\tag{1.4}
$$

For recent results related to the Hermite-Hadamard inequality, see [1], [2], [4], [16], [17], [18] and [20]-[26].

Motivated by the above results, we establish in this paper some inequalities that improve the celebrated Hermite-Hadamard inequality for convex functions. Applications for special means are also provided.

2. Some Error Estimates

We have the following error bounds in approximating the integral of a convex function by the Hermite-Hadamard sums:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $d_n:=\left\{x_i|i=\overline{0,n}\right\}\subset[a,b]$ a division of the interval $[a,b]$ we have*

$$
0\leq\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{b-a}h_{d_n}(f)\leq\frac{1}{8}\cdot\frac{1}{b-a}B_{d_n}(f)
\tag{2.1}
$$

and

$$
0\leq\frac{1}{b-a}H_{d_n}(f)-\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{1}{8}\cdot\frac{1}{b-a}B_{d_n}(f),
\tag{2.2}
$$

where

$$B_{d_n}(f) := \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left[f'_-(x_{i+1}) - f'_+(x_i)\right].$$

The constant $\frac{1}{8}$ is best possible in both inequalities (2.1) and (2.2).

Proof. Let $f : [c,d] \to \mathbb{R}$ be a convex function on $[c,d]$. We use the inequality that has been established in [9]

$$0 \le \frac{1}{d-c}\int_c^d f(t)\,dt - f\left(\frac{c+d}{2}\right) \le \frac{1}{8}(d-c)\left[f'_-(d) - f'_+(c)\right] \tag{2.3}$$

and the inequality obtained in [10]

$$0 \le \frac{f(c)+f(d)}{2} - \frac{1}{d-c}\int_c^d f(t)\,dt \le \frac{1}{8}(d-c)\left[f'_-(d) - f'_+(c)\right]. \tag{2.4}$$

The constant $\frac{1}{8}$ is best possible in both (2.3) and (2.4).

Now, if we write the inequality (2.3) on the interval $[x_i, x_{i+1}]$, $i = \overline{0,n}$ then we have

$$0 \le \int_{x_i}^{x_{i+1}} f(t)\,dt - f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i) \le \frac{1}{8}(x_{i+1} - x_i)^2\left[f'_-(x_{i+1}) - f'_+(x_i)\right] \tag{2.5}$$

for any $i = \overline{0,n}$.

Moreover, if we sum the inequality (2.5) over i from 0 to $n-1$ we get the desired result (2.1).

The inequality (2.2) follows in a similar way from (2.4). We omit the details. □

If we denote the *norm* of the division $d_n := \left\{x_i | i = \overline{0,n}\right\}$ by

$$\upsilon(d_n) := \max_{i\in\{0,\ldots,n-1\}} (x_{i+1} - x_i)$$

and use the *Hölder's discrete inequality*, then we have

$$B_{d_n}(f) \le \begin{cases} \upsilon^2(d_n)\left[f'_-(b) - f'_+(a)\right], \\ \\ \left(\sum_{i=0}^{n-1}(x_{i+1}-x_i)^{2p}\right)^{1/p}\left(\sum_{i=0}^{n-1}\left[f'_-(x_{i+1}) - f'_+(x_i)\right]^q\right)^{1/q} \\ p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max_{i\in\{0,\ldots,n-1\}}\left[f'_-(x_{i+1}) - f'_+(x_i)\right]\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2. \end{cases} \tag{2.6}$$

Simpler bounds are provided below:

Corollary 1. *With the assumptions of Theorem 1 we have the inequalities:*

$$0 \le \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{b-a}h_{d_n}(f) \le \frac{1}{8}\cdot\frac{f'_-(b) - f'_+(a)}{b-a}\upsilon^2(d_n) \tag{2.7}$$

and

$$0 \le \frac{1}{b-a}H_{d_n}(f) - \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{8}\cdot\frac{f'_-(b) - f'_+(a)}{b-a}\upsilon^2(d_n). \tag{2.8}$$

The constant $\frac{1}{8}$ is best possible in both inequalities (2.7) and (2.8).

If we consider the division $d_2 : a = x_0 < x_1 = x < x_2 = b$, then from (2.1) and (2.2) we have

$$0 \le \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+x}{2}\right)\frac{x-a}{b-a} - f\left(\frac{x+b}{2}\right)\frac{b-x}{b-a} \tag{2.9}$$
$$\le \frac{1}{8}\cdot\frac{1}{b-a}\left[(x-a)^2\left[f'_-(x)-f'_+(a)\right]+(b-x)^2\left[f'_-(b)-f'_+(x)\right]\right]$$
$$\le \frac{1}{8}\cdot\frac{1}{b-a}\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^2\left[f'_-(b)-f'_+(a)\right], \\ \\ \left((x-a)^{2p}+(b-x)^{2p}\right)^{1/p} \\ \times\left(\left[f'_-(x)-f'_+(a)\right]^q+\left[f'_-(b)-f'_+(x)\right]^q\right)^{1/q} \\ p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max\left\{f'_-(x)-f'_+(a), f'_-(b)-f'_+(x)\right\} \\ \times\left[\frac{1}{2}(b-a)^2+2\left(x-\frac{a+b}{2}\right)^2\right],\end{cases}$$

and

$$0 \le \frac{1}{2}\left[f(x)+\frac{f(a)(x-a)+f(b)(b-x)}{b-a}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.10}$$
$$\le \frac{1}{8}\cdot\frac{1}{b-a}\left[(x-a)^2\left[f'_-(x)-f'_+(a)\right]+(b-x)^2\left[f'_-(b)-f'_+(x)\right]\right]$$
$$\le \frac{1}{8}\cdot\frac{1}{b-a}\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^2\left[f'_-(b)-f'_+(a)\right], \\ \\ \left((x-a)^{2p}+(b-x)^{2p}\right)^{1/p} \\ \times\left(\left[f'_-(x)-f'_+(a)\right]^q+\left[f'_-(b)-f'_+(x)\right]^q\right)^{1/q} \\ p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max\left\{f'_-(x)-f'_+(a), f'_-(b)-f'_+(x)\right\} \\ \times\left[\frac{1}{2}(b-a)^2+2\left(x-\frac{a+b}{2}\right)^2\right],\end{cases}$$

for any $x \in [a,b]$.

In particular, if we take $x = \frac{a+b}{2}$ in (2.9) and (2.10), then we get

$$0 \le \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \tag{2.11}$$
$$\le \frac{1}{32}(b-a)\left[f'_-(b)-f'_+(a)\right]$$

and

$$0 \le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.12}$$
$$\le \frac{1}{32}(b-a)\left[f'_-(b)-f'_+(a)\right].$$

3. Bounds for $h_{d_n}(f)$

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $\Phi : I \to \mathbb{R}$ is a convex function on I. Then Φ is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $\Phi'_-(x) \leq \Phi'_+(x) \leq \Phi'_-(y) \leq \Phi'_+(y)$ which shows that both Φ'_- and Φ'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $\Phi : I \to \mathbb{R}$, the subdifferential of Φ denoted by $\partial\Phi$ is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$\Phi(x) \geq \Phi(a) + (x - a)\varphi(a) \text{ for any } x, a \in I. \tag{3.1}$$

It is also well known that if Φ is convex on I, then $\partial\Phi$ is nonempty, Φ'_-, $\Phi'_+ \in \partial\Phi$ and if $\varphi \in \partial\Phi$, then

$$\Phi'_-(x) \leq \varphi(x) \leq \Phi'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If Φ is differentiable and convex on $\mathring{I}$, then $\partial\Phi = \{\Phi'\}$.

We have the following result:

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$ and $\varphi \in \partial f$ the subdifferential of f on $[a, b]$. Then for any $d_n := \left\{x_i | i = \overline{0, n}\right\} \subset [a, b]$, $(n \geq 2)$ a division of the interval $[a, b]$ we have*

$$\begin{aligned} 0 &\leq \frac{1}{b-a} h_{d_n}(f) - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{b-a} \sum_{i=0}^{n-1} \frac{x_{i+1}^2 - x_i^2}{2} \varphi\left(\frac{x_i + x_{i+1}}{2}\right) \\ &- \frac{a+b}{2} \frac{1}{b-a} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \varphi\left(\frac{x_i + x_{i+1}}{2}\right) \\ &\leq \begin{cases} \frac{1}{2} \frac{f'_-\left(\frac{b+x_{n-1}}{2}\right) - f'_+\left(\frac{a+x_1}{2}\right)}{b-a} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left| \frac{x_i + x_{i+1}}{2} - \frac{b + x_{n-1} + x_1 + a}{4} \right| \\ \\ \frac{1}{2} \frac{\frac{b+x_{n-1}}{2} - \frac{a+x_1}{2}}{b-a} \\ \times \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left| \varphi\left(\frac{x_i + x_{i+1}}{2}\right) - \frac{f'_-\left(\frac{b+x_{n-1}}{2}\right) + f'_+\left(\frac{a+x_1}{2}\right)}{2} \right| \end{cases} \\ &\leq \frac{1}{4} \left(f'_-\left(\frac{b + x_{n-1}}{2}\right) - f'_+\left(\frac{a + x_1}{2}\right)\right) \left(\frac{b + x_{n-1}}{2} - \frac{a + x_1}{2}\right). \end{aligned} \tag{3.2}$$

Proof. We use the following reverse of Jensen's discrete inequality that holds for the convex function $\Phi : [m, M] \to \mathbb{R}$, the section of the subdifferential $\varphi \in \partial\Phi$, the points $y_i \in [m, M]$, $i \in \{1, ..., k\}$ and the probability sequence $p_i \geq 0$, $i \in \{1, ..., k\}$

with $\sum_{i=1}^{k} p_i = 1$ (see [14], [8] and [13])

$$
\begin{aligned}
(3.3)\qquad 0 &\leq \sum_{i=1}^{k} p_i \Phi(y_i) - \Phi\left(\sum_{i=1}^{k} p_i y_i\right) \\
&\leq \sum_{i=1}^{k} p_i y_i \varphi(y_i) - \sum_{i=1}^{k} p_i \varphi(y_i) \sum_{i=1}^{k} p_i y_i \\
&\leq \begin{cases} \frac{1}{2}\left(\Phi'_-(M) - \Phi'_+(m)\right) \sum_{i=1}^{k} p_i \left| y_i - \frac{m+M}{2} \right| \\ \\ \frac{1}{2}(M-m) \sum_{i=1}^{k} p_i \left| \varphi(y_i) - \frac{\Phi'_-(M)+\Phi'_+ m}{2} \right| \end{cases} \\
&\leq \frac{1}{4}\left(\Phi'_-(M) - \Phi'_+(m)\right)(M-m).
\end{aligned}
$$

Now, if we write the inequality (3.3) for $\Phi = f$, $y_i = \frac{x_i + x_{i+1}}{2}$, $p_i = \frac{x_{i+1} - x_i}{b-a}$, $i \in \{0, ..., n-1\}$,

$$
M = \max_{i \in \{0,\dots,n-1\}} \left\{ \frac{x_i + x_{i+1}}{2} \right\} = \frac{b + x_{n-1}}{2}
$$

and

$$
m = \min_{i \in \{0,\dots,n-1\}} \left\{ \frac{x_i + x_{i+1}}{2} \right\} = \frac{a + x_1}{2},
$$

then we get

$$
\begin{aligned}
(3.4)\quad 0 &\leq \frac{1}{b-a} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \Phi\left(\frac{x_i + x_{i+1}}{2}\right) - \Phi\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{b-a} \sum_{i=0}^{n-1} \frac{x_{i+1}^2 - x_i^2}{2} \varphi\left(\frac{x_i + x_{i+1}}{2}\right) \\
&- \frac{a+b}{2} \frac{1}{b-a} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \varphi\left(\frac{x_i + x_{i+1}}{2}\right) \\
&\leq \begin{cases} \frac{1}{2} \frac{f'_-\left(\frac{b+x_{n-1}}{2}\right) - f'_+\left(\frac{a+x_1}{2}\right)}{b-a} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left| \frac{x_i + x_{i+1}}{2} - \frac{b + x_{n-1} + x_1 + a}{4} \right| \\ \\ \frac{1}{2} \frac{\frac{b+x_{n-1}}{2} - \frac{a+x_1}{2}}{b-a} \\ \times \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left| \varphi\left(\frac{x_i + x_{i+1}}{2}\right) - \frac{f'_-\left(\frac{b+x_{n-1}}{2}\right) + f'_+\left(\frac{a+x_1}{2}\right)}{2} \right| \end{cases} \\
&\leq \frac{1}{4}\left(f'_-\left(\frac{b+x_{n-1}}{2}\right) - f'_+\left(\frac{a+x_1}{2}\right)\right)\left(\frac{b+x_{n-1}}{2} - \frac{a+x_1}{2}\right),
\end{aligned}
$$

which proves (3.2). □

If we consider the division $d_2 : a = x_0 < x_1 = x < x_2 = b$, then from (3.2) we have

$$
\begin{aligned}
(3.5)\qquad & 0 \le f\left(\frac{a+x}{2}\right)\frac{x-a}{b-a} + f\left(\frac{x+b}{2}\right)\frac{b-x}{b-a} - f\left(\frac{a+b}{2}\right) \\
& \le \frac{1}{b-a}\left[\frac{x^2-a^2}{2}\varphi\left(\frac{a+x}{2}\right) + \frac{b^2-x^2}{2}\varphi\left(\frac{x+b}{2}\right)\right] \\
& - \frac{a+b}{2}\frac{1}{b-a}\left[(x-a)\varphi\left(\frac{a+x}{2}\right) + (b-x)\varphi\left(\frac{x+b}{2}\right)\right] \\
& \le \begin{cases} \frac{1}{8}\left[f'_-\left(\frac{b+x}{2}\right) - f'_+\left(\frac{a+x}{2}\right)\right](b-a) \\ \\ \frac{1}{4}\left[(x-a)\left|\varphi\left(\frac{a+x}{2}\right) - \frac{f'_-\left(\frac{b+x}{2}\right)+f'_+\left(\frac{a+x}{2}\right)}{2}\right|\right. \\ \left. +(b-x)\left|\varphi\left(\frac{x+b}{2}\right) - \frac{f'_-\left(\frac{b+x}{2}\right)+f'_+\left(\frac{a+x}{2}\right)}{2}\right|\right] \end{cases} \\
& \le \frac{1}{8}\left[f'_-\left(\frac{b+x}{2}\right) - f'_+\left(\frac{a+x}{2}\right)\right](b-a),
\end{aligned}
$$

for any $x \in [a,b]$ and $\varphi \in \partial\Phi$.

If in (3.5) we take $x = \frac{a+b}{2}$, then we get

$$
\begin{aligned}
(3.6)\qquad & 0 \le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - f\left(\frac{a+b}{2}\right) \\
& \le \left[\frac{3a+b}{8}\varphi\left(\frac{3a+b}{4}\right) + \frac{a+3b}{8}\varphi\left(\frac{a+3b}{4}\right)\right] \\
& - \frac{a+b}{4}\left[\varphi\left(\frac{3a+b}{4}\right) + \varphi\left(\frac{a+3b}{4}\right)\right] \\
& \le \begin{cases} \frac{1}{8}\left[f'_-\left(\frac{3b+a}{4}\right) - f'_+\left(\frac{a+3b}{4}\right)\right](b-a) \\ \\ \frac{1}{8}(b-a)\left[\left|\varphi\left(\frac{3a+b}{4}\right) - \frac{f'_-\left(\frac{3b+a}{4}\right)+f'_+\left(\frac{3a+b}{4}\right)}{2}\right|\right. \\ \left. +\left|\varphi\left(\frac{a+3b}{2}\right) - \frac{f'_-\left(\frac{3b+a}{4}\right)+f'_+\left(\frac{3a+b}{4}\right)}{2}\right|\right] \end{cases} \\
& \le \frac{1}{8}\left[f'_-\left(\frac{3b+a}{4}\right) - f'_+\left(\frac{a+3b}{4}\right)\right](b-a).
\end{aligned}
$$

The following results also holds:

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then for any $d_n := \left\{x_i | i = \overline{0,n}\right\} \subset [a,b]$, $(n \ge 2)$ a division of the interval $[a,b]$ we have*

$$
\begin{aligned}
(3.7)\qquad & 0 \le \frac{1}{b-a}h_{d_n}(f) - f\left(\frac{a+b}{2}\right) \\
& \le \frac{1}{4}\frac{(x_{n-1}-a)(b-x_1)}{\frac{b+x_{n-1}}{2} - \frac{a+x_1}{2}} \sup_{t\in\left(\frac{a+x_1}{2},\frac{b+x_{n-1}}{2}\right)} \Psi_f\left(t; \frac{a+x_1}{2}, \frac{b+x_{n-1}}{2}\right) \\
& \le \frac{1}{4}\left(f'_-\left(\frac{b+x_{n-1}}{2}\right) - f'_+\left(\frac{a+x_1}{2}\right)\right)\left(\frac{b+x_{n-1}}{2} - \frac{a+x_1}{2}\right)
\end{aligned}
$$

and

$$0 \le \frac{1}{b-a} h_{d_n}(f) - f\left(\frac{a+b}{2}\right) \tag{3.8}$$
$$\le \frac{1}{2}\left(\frac{b+x_{n-1}}{2} - \frac{a+x_1}{2}\right)\Psi_f\left(\frac{a+x_1+b+x_{n-1}}{4}; \frac{a+x_1}{2}, \frac{b+x_{n-1}}{2}\right)$$
$$\le \frac{1}{4}\left(f'_-\left(\frac{b+x_{n-1}}{2}\right) - f'_+\left(\frac{a+x_1}{2}\right)\right)\left(\frac{b+x_{n-1}}{2} - \frac{a+x_1}{2}\right),$$

where $\Psi_f(\cdot; a, b) : (a, b) \to \mathbb{R}$ *is defined by*

$$\Psi_f(t; a, b) := \frac{f(b) - f(t)}{b-t} - \frac{f(t) - f(a)}{t-a}.$$

Proof. For a convex function $\Phi : [m, M] \to \mathbb{R}$, consider the attached function $\Psi_\Phi(\cdot; m, M) : (m, M) \to \mathbb{R}$ defined by

$$\Psi_\Phi(t; m, M) = \frac{\Phi(M) - \Phi(t)}{M-t} - \frac{\Phi(t) - \Phi(m)}{t-m}.$$

In [13] we obtained the following reverses of Jensen's inequality

$$0 \le \sum_{i=1}^k p_i \Phi(y_i) - \Phi\left(\sum_{i=1}^k p_i y_i\right) \tag{3.9}$$
$$\le \frac{\left(M - \sum_{i=1}^k p_i y_i\right)\left(\sum_{i=1}^k p_i y_i - m\right)}{M-m} \sup_{t\in(m,M)} \Psi_\Phi(t; m, M)$$
$$\le \frac{\Phi'_-(M) - \Phi'_+(m)}{M-m}\left(M - \sum_{i=1}^k p_i y_i\right)\left(\sum_{i=1}^k p_i y_i - m\right)$$
$$\le \frac{1}{4}\left(\Phi'_-(M) - \Phi'_+(m)\right)(M-m)$$

and

$$0 \le \sum_{i=1}^k p_i \Phi(y_i) - \Phi\left(\sum_{i=1}^k p_i y_i\right) \le \frac{1}{2}(M-m)\Psi_\Phi\left(\sum_{i=1}^k p_i y_i; m, M\right) \tag{3.10}$$
$$\le \frac{1}{4}\left(\Phi'_-(M) - \Phi'_+(m)\right)(M-m),$$

where $y_i \in [m, M]$ and $p_i \ge 0$, $i \in \{1, ..., k\}$ with $\sum_{i=1}^k p_i = 1$.

If we write the inequality (3.9) for $\Phi = f$, $y_i = \frac{x_i + x_{i+1}}{2}$, $p_i = \frac{x_{i+1} - x_i}{b-a}$, $i \in \{0, ..., n-1\}$,

$$M = \max_{i\in\{0,\dots,n-1\}} \left\{\frac{x_i + x_{i+1}}{2}\right\} = \frac{b + x_{n-1}}{2}$$

and

$$m = \min_{i\in\{0,\dots,n-1\}} \left\{\frac{x_i + x_{i+1}}{2}\right\} = \frac{a + x_1}{2},$$

then we get

$$0 \leq \frac{1}{b-a}\sum_{i=0}^{n-1}(x_{i+1}-x_i) f\left(\frac{x_i+x_{i+1}}{2}\right) - f\left(\frac{a+b}{2}\right) \tag{3.11}$$
$$\leq \frac{\left(\frac{b+x_{n-1}}{2}-\frac{a+b}{2}\right)\left(\frac{a+b}{2}-\frac{a+x_1}{2}\right)}{\frac{b+x_{n-1}}{2}-\frac{a+x_1}{2}}$$
$$\times \sup_{t\in\left(\frac{a+x_1}{2},\frac{b+x_{n-1}}{2}\right)} \Psi_f\left(t;\frac{a+x_1}{2},\frac{b+x_{n-1}}{2}\right)$$
$$\leq \frac{f'_-\left(\frac{b+x_{n-1}}{2}\right)-f'_+\left(\frac{a+x_1}{2}\right)}{\frac{b+x_{n-1}}{2}-\frac{a+x_1}{2}}$$
$$\times\left(\frac{b+x_{n-1}}{2}-\frac{a+b}{2}\right)\left(\frac{a+b}{2}-\frac{a+x_1}{2}\right)$$
$$=\frac{1}{4}\left(f'_-\left(\frac{b+x_{n-1}}{2}\right)-f'_+\left(\frac{a+x_1}{2}\right)\right)\left(\frac{b+x_{n-1}}{2}-\frac{a+x_1}{2}\right),$$

which proves (3.7).

The inequality (3.8) follows by (3.10). □

If we consider the division $d_2 : a = x_0 < x_1 = x < x_2 = b$, then from (3.7) and (3.8) we have

$$0 \leq f\left(\frac{a+x}{2}\right)\frac{x-a}{b-a} + f\left(\frac{x+b}{2}\right)\frac{b-x}{b-a} - f\left(\frac{a+b}{2}\right) \tag{3.12}$$
$$\leq \frac{1}{2}\frac{(x-a)(b-x)}{b-a}\sup_{t\in\left(\frac{a+x}{2},\frac{b+x}{2}\right)}\Psi_f\left(t;\frac{a+x}{2},\frac{x+b}{2}\right)$$
$$\leq \frac{1}{8}\left(f'_-\left(\frac{b+x}{2}\right)-f'_+\left(\frac{a+x}{2}\right)\right)(b-a)$$

and

$$0 \leq f\left(\frac{a+x}{2}\right)\frac{x-a}{b-a} + f\left(\frac{x+b}{2}\right)\frac{b-x}{b-a} - f\left(\frac{a+b}{2}\right) \tag{3.13}$$
$$\leq \frac{1}{4}(b-a)\Psi_f\left(\frac{a+2x+b}{4};\frac{a+x}{2},\frac{b+x}{2}\right)$$
$$\leq \frac{1}{8}\left(f'_-\left(\frac{b+x}{2}\right)-f'_+\left(\frac{a+x}{2}\right)\right)(b-a),$$

for any $x \in [a,b]$.

We also have:

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then for any $d_n := \left\{x_i | i = \overline{0,n}\right\} \subset [a,b]$, $(n \geq 2)$ a division of the interval $[a,b]$ we have*

$$\begin{aligned}(3.14)\qquad 0 &\leq \frac{1}{b-a} h_{d_n}(f) - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{\frac{b+x_{n-1}}{2} - \frac{a+x_1}{2}} \max\left\{x_{n-1}-a, b-x_1\right\} \\ &\times \left(\frac{f\left(\frac{a+x_1}{2}\right) + f\left(\frac{b+x_{n-1}}{2}\right)}{2} - f\left(\frac{b+x_{n-1}+x_1+a}{4}\right)\right).\end{aligned}$$

Proof. In [13] we obtained the following reverse of Jensen's inequality as well

$$\begin{aligned}(3.15)\qquad 0 &\leq \sum_{i=1}^{k} p_i \Phi(y_i) - \Phi\left(\sum_{i=1}^{k} p_i y_i\right) \\ &\leq 2 \max\left\{\frac{M - \sum_{i=1}^{k} p_i y_i}{M-m}, \frac{\sum_{i=1}^{k} p_i y_i - m}{M-m}\right\} \\ &\times \left(\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right)\end{aligned}$$

where $y_i \in [m, M]$ and $p_i \geq 0$, $i \in \{1, ..., k\}$ with $\sum_{i=1}^{k} p_i = 1$.

If we write the inequality (3.15) for $\Phi = f$, $y_i = \frac{x_i + x_{i+1}}{2}$, $p_i = \frac{x_{i+1} - x_i}{b-a}$, $i \in \{0, ..., n-1\}$,

$$M = \max_{i \in \{0,\dots,n-1\}} \left\{\frac{x_i + x_{i+1}}{2}\right\} = \frac{b + x_{n-1}}{2}$$

and

$$m = \min_{i \in \{0,\dots,n-1\}} \left\{\frac{x_i + x_{i+1}}{2}\right\} = \frac{a + x_1}{2},$$

then we get

$$\begin{aligned}(3.16)\qquad 0 &\leq \frac{1}{b-a} \sum_{i=0}^{n-1} (x_{i+1} - x_i) f\left(\frac{x_i + x_{i+1}}{2}\right) - f\left(\frac{a+b}{2}\right) \\ &\leq 2 \max\left\{\frac{\frac{b+x_{n-1}}{2} - \frac{a+b}{2}}{\frac{b+x_{n-1}}{2} - \frac{a+x_1}{2}}, \frac{\frac{a+b}{2} - \frac{a+x_1}{2}}{\frac{b+x_{n-1}}{2} - \frac{a+x_1}{2}}\right\} \\ &\times \left(\frac{f\left(\frac{a+x_1}{2}\right) + f\left(\frac{b+x_{n-1}}{2}\right)}{2} - f\left(\frac{b+x_{n-1}+a+x}{4}\right)\right),\end{aligned}$$

which proves the inequality (3.14). □

If we consider the division $d_2 : a = x_0 < x_1 = x < x_2 = b$, then from (3.14) we have

$$0 \leq f\left(\frac{a+x}{2}\right)\frac{x-a}{b-a} + f\left(\frac{x+b}{2}\right)\frac{b-x}{b-a} - f\left(\frac{a+b}{2}\right) \tag{3.17}$$
$$\leq \left(1 + \frac{2}{b-a}\left|x - \frac{a+b}{2}\right|\right)$$
$$\times \left(\frac{f\left(\frac{a+x}{2}\right) + f\left(\frac{b+x}{2}\right)}{2} - f\left(\frac{b+2x+a}{4}\right)\right),$$

for any $x \in [a,b]$.

4. Bounds for $H_{d_n}(f)$

For a division $d_n := \left\{x_i | i = \overline{0,n}\right\} \subset [a,b]$ of the interval $[a,b]$, we consider the sums

$$\delta_{d_n} := \frac{1}{2(b-a)}\sum_{i=0}^{n-1}(x_{i+1} - x_i)\left[(b - x_i)(x_i - a) + (b - x_{i+1})(x_{i+1} - a)\right]$$

and

$$\Delta_{d_n} := \frac{1}{b-a}\sum_{i=0}^{n-1}(x_{i+1} - x_i)\left(b - \frac{x_i + x_{i+1}}{2}\right)\left(\frac{x_i + x_{i+1}}{2} - a\right).$$

Since the function $g : [a,b] \to \mathbb{R}$, $g(t) = (b-t)(t-a)$ is concave, then

$$\frac{g(x_i) + g(x_{i+1})}{2} \leq g\left(\frac{x_i + x_{i+1}}{2}\right)$$

for any $i \in \{0, ..., n-1\}$, which implies that

$$\delta_{d_n} \leq \Delta_{d_n} \text{ for any division } d_n. \tag{4.1}$$

Also by Jensen's discrete inequality for the concave function g we have

$$\begin{aligned}
\Delta_{d_n} &= \sum_{i=0}^{n-1}\frac{x_{i+1} - x_i}{b-a}\left(b - \frac{x_i + x_{i+1}}{2}\right)\left(\frac{x_i + x_{i+1}}{2} - a\right) \\
&\leq \left(b - \sum_{i=0}^{n-1}\frac{x_{i+1} - x_i}{b-a}\frac{x_i + x_{i+1}}{2}\right)\left(\sum_{i=0}^{n-1}\frac{x_{i+1} - x_i}{b-a}\frac{x_i + x_{i+1}}{2} - a\right) \\
&= \left(b - \frac{1}{2}\sum_{i=0}^{n-1}\frac{x_{i+1}^2 - x_i^2}{b-a}\right)\left(\frac{1}{2}\sum_{i=0}^{n-1}\frac{x_{i+1}^2 - x_i^2}{b-a} - a\right) \\
&= \left(b - \frac{a+b}{2}\right)\left(\frac{a+b}{2} - a\right) = \frac{1}{4}(b-a)^2,
\end{aligned}$$

therefore

$$\Delta_{d_n} \leq \frac{1}{4}(b-a)^2 \text{ for any division } d_n. \tag{4.2}$$

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $d_n:=\left\{x_i|i=\overline{0,n}\right\}\subset[a,b]$, $(n\geq 2)$ a division of the interval $[a,b]$ we have*

$$\begin{aligned}(4.3)\qquad 0 &\leq \inf_{t\in(a,b)}\Psi_f(t;a,b)\,\delta_{d_n}\leq\frac{f(a)+f(b)}{2}(b-a)-H_{d_n}(f)\\ &\leq \sup_{t\in(a,b)}\Psi_f(t;a,b)\,\delta_{d_n}\leq \sup_{t\in(a,b)}\Psi_f(t;a,b)\,\Delta_{d_n},\end{aligned}$$

where $\Psi_f(\cdot;a,b):(a,b)\to\mathbb{R}$ is defined by

$$\Psi_f(t;a,b):=\frac{f(b)-f(t)}{b-t}-\frac{f(t)-f(a)}{t-a}.$$

Proof. Observe that for the convex function $f:[a,b]\to\mathbb{R}$ we have

$$\begin{aligned}(4.4)\qquad 0 &\leq \frac{(b-t)f(a)+(t-a)f(b)}{b-a}-f(t)\\ &=\frac{(t-a)(f(b)-f(t))-(b-t)(f(t)-f(a))}{b-a}\\ &=\frac{(t-a)(b-t)}{b-a}\left(\frac{f(b)-f(t)}{b-t}-\frac{f(t)-f(a)}{t-a}\right)\\ &=\frac{(t-a)(b-t)}{b-a}\Psi_f(t;a,b)\end{aligned}$$

for any $t\in[a,b]$.

Since, see also [13], we have

$$0\leq\Psi_f(t;a,b)\leq f'_-(b)-f'_+(a)$$

for any $t\in(a,b)$, then from (4.4) we have

$$\begin{aligned}(4.5)\qquad 0 &\leq \frac{(t-a)(b-t)}{b-a}\inf_{t\in(a,b)}\Psi_f(t;a,b)\leq\frac{(b-t)f(a)+(t-a)f(b)}{b-a}-f(t)\\ &\leq\frac{(t-a)(b-t)}{b-a}\sup_{t\in(a,b)}\Psi_f(t;a,b)\leq\frac{f'_-(b)-f'_+(a)}{b-a}(t-a)(b-t)\end{aligned}$$

for any $t\in(a,b)$.

Now, if we write the inequality (4.5) for x_i and x_{i+1}, $i\in\{0,...,n-1\}$, add the obtained inequalities and divide by 2 then we get

$$\begin{aligned}(4.6)\quad 0 &\leq \frac{1}{2}\left[\frac{(x_i-a)(b-x_i)}{b-a}+\frac{(x_{i+1}-a)(b-x_{i+1})}{b-a}\right]\inf_{t\in(a,b)}\Psi_f(t;a,b)\\ &\leq\frac{1}{2}\left[\frac{(b-x_i)f(a)+(x_i-a)f(b)}{b-a}+\frac{(b-x_{i+1})f(a)+(x_{i+1}-a)f(b)}{b-a}\right]\\ &\quad-\frac{f(x_i)+f(x_{i+1})}{2}\\ &\leq\frac{1}{2}\left[\frac{(x_i-a)(b-x_i)}{b-a}+\frac{(x_{i+1}-a)(b-x_{i+1})}{b-a}\right]\sup_{t\in(a,b)}\Psi_f(t;a,b)\end{aligned}$$

for any $i\in\{0,...,n-1\}$.

Now, if we multiply (4.6) by $x_{i+1} - x_i > 0$ and sum over i from 0 to $n-1$, then we get

$$0 \le \frac{1}{2} \inf_{t\in(a,b)} \Psi_f(t;a,b) \times \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left[\frac{(x_i - a)(b - x_i)}{b-a} + \frac{(x_{i+1} - a)(b - x_{i+1})}{b-a} \right] \le \frac{1}{2(b-a)} \sum_{i=0}^{n-1} (x_{i+1} - x_i) [(b - x_i) f(a) + (x_i - a) f(b) + (b - x_{i+1}) f(a) + (x_{i+1} - a) f(b)] - \sum_{i=0}^{n-1} (x_{i+1} - x_i) \frac{f(x_i) + f(x_{i+1})}{2} \le \frac{1}{2} \sup_{t\in(a,b)} \Psi_f(t;a,b) \times \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left[\frac{(x_i - a)(b - x_i)}{b-a} + \frac{(x_{i+1} - a)(b - x_{i+1})}{b-a} \right]. \tag{4.7}$$

Now, observe that

$$\begin{aligned} &\frac{1}{2(b-a)} \sum_{i=0}^{n-1} (x_{i+1} - x_i) [(b - x_i) f(a) + (x_i - a) f(b) \\ &+ (b - x_{i+1}) f(a) + (x_{i+1} - a) f(b)] \\ &= \frac{1}{2(b-a)} \sum_{i=0}^{n-1} (x_{i+1} - x_i) [(2b - x_i - x_{i+1}) f(a) + (x_i + x_{i+1} - 2a) f(b)] \\ &= \frac{1}{b-a} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left[\left(b - \frac{x_i + x_{i+1}}{2} \right) f(a) + \left(\frac{x_i + x_{i+1}}{2} - a \right) f(b) \right] \\ &= \frac{1}{b-a} \\ &\times \left[\left(b(b-a) - \sum_{i=0}^{n-1} \frac{(x_{i+1}^2 - x_i^2)}{2} \right) f(a) + \left(\sum_{i=0}^{n-1} \frac{(x_{i+1}^2 - x_i^2)}{2} - a(b-a) \right) f(b) \right] \\ &= \frac{1}{b-a} \left[\left(b(b-a) - \frac{1}{2}\left(b^2 - a^2\right) \right) f(a) + \left(\frac{1}{2}\left(b^2 - a^2\right) - a(b-a) \right) f(b) \right] \\ &= (b-a) \frac{f(a) + g(b)}{2} \end{aligned}$$

and by (4.7) we obtain the desired result (4.3). □

Remark 1. *Since* $\sup_{t\in(a,b)} \Psi_f(t;a,b) \le f'_-(b) - f'_+(a)$, *then we have from (4.3) that*

$$0 \le \frac{f(a) + f(b)}{2}(b-a) - H_{d_n}(f) \le \left[f'_-(b) - f'_+(a) \right] \delta_{d_n} \le \left[f'_-(b) - f'_+(a) \right] \Delta_{d_n}.$$

We also have

$$0 \le \frac{f(a) + f(b)}{2}(b-a) - H_{d_n}(f) \le \frac{1}{4}(b-a)^2 \sup_{t\in(a,b)} \Psi_f(t;a,b).$$

If we consider the division $d_2 : a = x_0 < x_1 = x < x_2 = b$, then

$$\delta_{d_2} = \frac{1}{2(b-a)}\left[(x-a)^2(b-x) + (b-x)^2(x-a)\right] = \frac{1}{2}(x-a)(b-x)$$

and

$$\begin{aligned}
&\Delta_{d_2} \\
&= \frac{1}{b-a}\left[(x-a)\left(b-\frac{a+x}{2}\right)\left(\frac{a+x}{2}-a\right) + (b-x)\left(b-\frac{x+b}{2}\right)\left(\frac{x+b}{2}-a\right)\right] \\
&= \frac{1}{2(b-a)}\left[(x-a)^2\left(b-\frac{a+x}{2}\right) + (b-x)^2\left(\frac{x+b}{2}-a\right)\right],
\end{aligned}$$

for $x \in [a,b]$.

If we use the inequality (4.3), then we get

$$\begin{aligned}
(4.8)\qquad 0 &\leq \frac{1}{2}\frac{(x-a)(b-x)}{b-a}\inf_{t\in(a,b)}\Psi_f(t;a,b) \\
&\leq \frac{f(a)+f(b)}{2} - \frac{1}{2}\left[f(x) + \frac{f(a)(x-a)+f(b)(b-x)}{b-a}\right] \\
&\leq \frac{1}{2}\frac{(x-a)(b-x)}{b-a}\sup_{t\in(a,b)}\Psi_f(t;a,b) \leq \frac{1}{2}(x-a)(b-x)\frac{f'_-(b)-f'_+(a)}{b-a}
\end{aligned}$$

for any $x \in [a,b]$.

If we take in (4.8) $x = \frac{a+b}{2}$, then we get

$$\begin{aligned}
(4.9)\qquad \frac{1}{4}(b-a)\inf_{t\in(a,b)}\Psi_f(t;a,b) &\leq \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{4}(b-a)\sup_{t\in(a,b)}\Psi_f(t;a,b) \\
&\leq \frac{1}{4}(b-a)\left[f'_-(b)-f'_+(a)\right].
\end{aligned}$$

We also have:

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then for any $d_n := \left\{x_i | i = \overline{0,n}\right\} \subset [a,b]$, $(n \geq 2)$ a division of the interval $[a,b]$ we have*

$$\begin{aligned}
(4.10)\qquad 0 &\leq \left(b-a-\eta_{d_n}\right)\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right] \\
&\leq \frac{f(a)+f(b)}{2}(b-a) - H_{d_n}(f) \\
&\leq \left(b-a+\eta_{d_n}\right)\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right],
\end{aligned}$$

where

$$0 \leq \eta_{d_n} := \frac{1}{b-a}\sum_{i=0}^{n-1}\left[\left|x_i - \frac{a+b}{2}\right| + \left|x_{i+1} - \frac{a+b}{2}\right|\right](x_{i+1}-x_i) \leq b-a.$$

Proof. We recall the following result obtained by the author in [11] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$0 \leq n \min_{i \in \{1,\dots,n\}} \{p_i\} \left[\frac{1}{n} \sum_{i=1}^{n} \Phi(x_i) - \Phi\left(\frac{1}{n} \sum_{i=1}^{n} x_i \right) \right] \tag{4.11}$$
$$\leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right)$$
$$\leq n \max_{i \in \{1,\dots,n\}} \{p_i\} \left[\frac{1}{n} \sum_{i=1}^{n} \Phi(x_i) - \Phi\left(\frac{1}{n} \sum_{i=1}^{n} x_i \right) \right],$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i \in \{1,\dots,n\}}$ are vectors and $\{p_i\}_{i \in \{1,\dots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^{n} p_i > 0$.

For $n = 2$ we deduce from (4.11) that

$$0 \leq 2 \min\{\lambda, 1-\lambda\} \left[\frac{\Phi(x) + \Phi(y)}{2} - \Phi\left(\frac{x+y}{2} \right) \right] \tag{4.12}$$
$$\leq \lambda \Phi(x) + (1-\lambda) \Phi(y) - \Phi(\lambda x + (1-\lambda) y)$$
$$\leq 2 \max\{\lambda, 1-\lambda\} \left[\frac{\Phi(x) + \Phi(y)}{2} - \Phi\left(\frac{x+y}{2} \right) \right]$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

If we replace in (3.10) $\Phi = f$, $C = [x,y] = [a,b]$ and $\lambda = \frac{b-t}{b-a}$, then we get:

$$0 \leq \frac{2}{b-a} \min\{b-t, t-a\} \left[\frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2} \right) \right] \tag{4.13}$$
$$\leq \frac{b-t}{b-a} f(a) + \frac{t-a}{b-a} f(b) - f(t)$$
$$\leq \frac{2}{b-a} \max\{b-t, t-a\} \left[\frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2} \right) \right],$$

for any $t \in [a,b]$.

Since

$$\min\{\alpha, \beta\} = \frac{1}{2}(\beta + \alpha) - \frac{1}{2}|\beta - \alpha|$$

and

$$\max\{\alpha, \beta\} = \frac{1}{2}(\beta + \alpha) + \frac{1}{2}|\beta - \alpha|$$

then

$$\min\{b-t, t-a\} = \frac{1}{2}(b-a) - \left| t - \frac{a+b}{2} \right|$$

and

$$\max\{b-t, t-a\} = \frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right|$$

for $t \in [a,b]$.

Therefore, by (4.13) we have

$$0 \leq \left[1-\frac{2}{b-a}\left|t-\frac{a+b}{2}\right|\right]\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right] \leq \frac{b-t}{b-a}f(a)+\frac{t-a}{b-a}f(b)-f(t) \leq \left[1+\frac{2}{b-a}\left|t-\frac{a+b}{2}\right|\right]\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right], \tag{4.14}$$

for $t \in [a,b]$.

Now, if we write the inequality (4.14) for x_i and x_{i+1}, $i \in \{0,...,n-1\}$, add the obtained inequalities and divide by 2 then we get

$$\begin{aligned} 0 &\leq \left[1-\frac{1}{b-a}\left[\left|x_i-\frac{a+b}{2}\right|+\left|x_{i+1}-\frac{a+b}{2}\right|\right]\right] \\ &\times \left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right] \\ &\leq \frac{1}{2}\left[\frac{(b-x_i)f(a)+(x_i-a)f(b)}{b-a}+\frac{(b-x_{i+1})f(a)+(x_{i+1}-a)f(b)}{b-a}\right] \\ &- \frac{f(x_i)+f(x_{i+1})}{2} \\ &\leq \left[1+\frac{1}{b-a}\left[\left|x_i-\frac{a+b}{2}\right|+\left|x_{i+1}-\frac{a+b}{2}\right|\right]\right] \\ &\times \left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right] \end{aligned} \tag{4.15}$$

for any $i \in \{0,...,n-1\}$.

Now, if we multiply (4.15) by $x_{i+1}-x_i>0$ and sum over i from 0 to $n-1$, then we get the desired result (4.10). □

If we consider the division $d_2 : a = x_0 < x_1 = x < x_2 = b$, then

$$\eta_{d_2} = \frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|, \; x \in [a,b]$$

and by (4.10) we get

$$\begin{aligned} 0 &\leq \left(\frac{1}{2}(b-a)-\left|x-\frac{a+b}{2}\right|\right)\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right] \\ &\leq \frac{f(a)+f(b)}{2}-\frac{1}{2}\left[f(x)+\frac{f(a)(x-a)+f(b)(b-x)}{b-a}\right] \\ &\leq \left(\frac{3}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right)\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right] \end{aligned} \tag{4.16}$$

for any $x \in [a,b]$.

5. Applications for Special Means

For $x \neq y$ and $p \in \mathbb{R} \setminus \{-1,0\}$, we define the *p-logarithmic mean* (*generalized logarithmic mean*) $L_p(x,y)$ by

$$L_p(x,y) := \left[\frac{y^{p+1}-x^{p+1}}{(p+1)(y-x)}\right]^{1/p}.$$

In fact the singularities at $p = -1, 0$ are removable and L_p can be defined for $p = -1, 0$ so as to make $L_p(x, y)$ a continuous function of p. In the limit as $p \to 0$ we obtain the *identric mean* $I(x, y)$, given by

$$I(x,y) := \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{1/(y-x)},$$

and in the case $p \to -1$ the *logarithmic mean* $L(x, y)$, given by

$$L(x,y) := \frac{y-x}{\ln y - \ln x}.$$

In each case we define the mean as x when $y = x$, which occurs as the limiting value of $L_p(x, y)$ for $y \to x$.

In addition we have the *arithmetic, geometric and harmonic means*, defined respectively by

$$A(x,y) := \frac{x+y}{2}, \qquad G(x,y) := \sqrt{xy} \quad \text{and} \quad H(x,y) := \frac{2xy}{x+y}.$$

The first two arise from $L_p(x, y)$ in the respective cases $p = 1$, $p = -2$. Remarkably there is no value of p for which $L_p = H$ (see [5, p. 347]). However H is connected with the generalized logarithmic-mean by

$$H(x,y) = \left[A\left(x^{-1}, y^{-1}\right)\right]^{-1}. \tag{5.1}$$

If we use the inequalities (2.1) and (2.2) for the convex functions $f(t) = t^p$, $p \in (-\infty, 0) \cup (1, \infty)$, $f(t) = \frac{1}{t}$ and $f(t) = -\ln t$ for $t \in [a, b] \subset (0, \infty)$ then we get

$$\begin{aligned} 0 &\le L_p^p(a,b) - \frac{1}{b-a}\sum_{i=0}^{n-1} A^p\left(x_i, x_{i+1}\right)\left(x_{i+1}-x_i\right) \\ &\le \frac{1}{8}\frac{p}{b-a}\sum_{i=0}^{n-1}\left(x_{i+1}-x_i\right)^2\left(x_{i+1}^{p-1}-x_i^{p-1}\right) \le \frac{1}{8}p\frac{b^{p-1}-a^{p-1}}{b-a}\upsilon^2\left(d_n\right), \end{aligned} \tag{5.2}$$

$$\begin{aligned} 0 &\le \frac{1}{b-a}\sum_{i=0}^{n-1} A\left(x_i^p, x_{i+1}^p\right)\left(x_{i+1}-x_i\right) - L_p^p(a,b) \\ &\le \frac{1}{8}\frac{p}{b-a}\sum_{i=0}^{n-1}\left(x_{i+1}-x_i\right)^2\left(x_{i+1}^{p-1}-x_i^{p-1}\right) \le \frac{1}{8}p\frac{b^{p-1}-a^{p-1}}{b-a}\upsilon^2\left(d_n\right), \end{aligned} \tag{5.3}$$

$$\begin{aligned} 0 &\le L^{-1}(a,b) - \frac{1}{b-a}\sum_{i=0}^{n-1} A^{-1}\left(x_i, x_{i+1}\right)\left(x_{i+1}-x_i\right) \\ &\le \frac{1}{8}\cdot\frac{1}{b-a}\sum_{i=0}^{n-1}\left(x_{i+1}-x_i\right)^3\frac{x_{i+1}+x_i}{x_{i+1}^2 x_i^2} \le \frac{1}{8}\cdot\frac{b+a}{b^2a^2}\upsilon^2\left(d_n\right), \end{aligned} \tag{5.4}$$

$$\begin{aligned} 0 &\le \frac{1}{b-a}\sum_{i=0}^{n-1} H^{-1}\left(x_i, x_{i+1}\right)\left(x_{i+1}-x_i\right) - L^{-1}(a,b) \\ &\le \frac{1}{8}\cdot\frac{1}{b-a}\sum_{i=0}^{n-1}\left(x_{i+1}-x_i\right)^3\frac{x_{i+1}+x_i}{x_{i+1}^2 x_i^2} \le \frac{1}{8}\cdot\frac{b+a}{b^2a^2}\upsilon^2\left(d_n\right), \end{aligned} \tag{5.5}$$

$$(5.6)\qquad 1\leq\frac{\prod_{i=0}^{n-1}\left[A\left(x_i,x_{i+1}\right)\right]^{\frac{x_{i+1}-x_i}{b-a}}}{I\left(a,b\right)}\leq\exp\left(\frac{1}{8}\cdot\frac{1}{b-a}\sum_{i=0}^{n-1}\frac{\left(x_{i+1}-x_i\right)^3}{x_{i+1}x_i}\right)\leq\exp\left(\frac{1}{8ba}\upsilon^2\left(d_n\right)\right)$$

and

$$(5.7)\qquad 1\leq\frac{I\left(a,b\right)}{\prod_{i=0}^{n-1}\left[G\left(x_i,x_{i+1}\right)\right]^{\frac{x_{i+1}-x_i}{b-a}}}\leq\exp\left(\frac{1}{8}\cdot\frac{1}{b-a}\sum_{i=0}^{n-1}\frac{\left(x_{i+1}-x_i\right)^3}{x_{i+1}x_i}\right)\leq\exp\left(\frac{1}{8ba}\upsilon^2\left(d_n\right)\right)$$

for any division $d_n:=\left\{x_i|i=\overline{0,n}\right\}\subset[a,b]$, $(n\geq 2)$.

If we use the first part of the inequality (3.2) for the convex functions $f(t)=t^p$, $p\in(-\infty,0)\cup(1,\infty)$, $f(t)=\frac{1}{t}$ and $f(t)=-\ln t$ for $t\in[a,b]\subset(0,\infty)$ then we get

$$(5.8)\qquad 0\leq\frac{1}{b-a}\sum_{i=0}^{n-1}A^p\left(x_i,x_{i+1}\right)\left(x_{i+1}-x_i\right)-A^p\left(a,b\right)\leq\frac{1}{b-a}p\sum_{i=0}^{n-1}A^{p-1}\left(x_i,x_{i+1}\right)\left[A\left(x_i,x_{i+1}\right)-A\left(a,b\right)\right]\left(x_{i+1}-x_i\right),$$

$$(5.9)\qquad 0\leq\frac{1}{b-a}\sum_{i=0}^{n-1}A^{-1}\left(x_i,x_{i+1}\right)\left(x_{i+1}-x_i\right)-A^{-1}\left(a,b\right)\leq\frac{1}{b-a}\left[\sum_{i=0}^{n-1}\left(x_{i+1}-x_i\right)A^{-1}\left(x_i,x_{i+1}\right)\left[A\left(a,b\right)A^{-2}\left(x_i,x_{i+1}\right)-1\right]\right]$$

and

$$(5.10)\qquad 1\leq\frac{A\left(a,b\right)}{\prod_{i=0}^{n-1}\left[A\left(x_i,x_{i+1}\right)\right]^{\frac{x_{i+1}-x_i}{b-a}}}\leq\exp\left(\frac{1}{b-a}\sum_{i=0}^{n-1}\left(x_{i+1}-x_i\right)\left[A\left(a,b\right)A^{-1}\left(x_i,x_{i+1}\right)-1\right]\right).$$

If we use the inequality (4.10) for the same functions as above, we have

$$(5.11)\qquad 0\leq\left(1-\frac{\eta_{d_n}}{b-a}\right)\left[A\left(a^p,b^p\right)-A^p\left(a,b\right)\right]\leq A\left(a^p,b^p\right)-\frac{1}{b-a}\sum_{i=0}^{n-1}A\left(x_i^p,x_{i+1}^p\right)\left(x_{i+1}-x_i\right)\leq\left(1+\frac{\eta_{d_n}}{b-a}\right)\left[A\left(a^p,b^p\right)-A^p\left(a,b\right)\right],$$

$$
\begin{aligned}
(5.12) \qquad 0 &\leq \left(1-\frac{\eta_{d_n}}{b-a}\right)\left[H^{-1}(a,b)-A^{-1}(a,b)\right] \\
&\leq H^{-1}(a,b)-\frac{1}{b-a}\sum_{i=0}^{n-1}H^{-1}(x_i,x_{i+1})(x_{i+1}-x_i) \\
&\leq \left(1+\frac{\eta_{d_n}}{b-a}\right)\left[H^{-1}(a,b)-A^{-1}(a,b)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
(5.13) \qquad 1 &\leq \left(\frac{A(a,b)}{G(a,b)}\right)^{\left(1-\frac{\eta_{d_n}}{b-a}\right)} \leq \frac{\prod_{i=0}^{n-1}\left[G(x_i,x_{i+1})\right]^{\frac{x_{i+1}-x_i}{b-a}}}{G(a,b)} \\
&\leq \left(\frac{A(a,b)}{G(a,b)}\right)^{\left(1+\frac{\eta_{d_n}}{b-a}\right)},
\end{aligned}
$$

where

$$
\eta_{d_n} := \frac{1}{b-a}\sum_{i=0}^{n-1}\left[\left|x_i-\frac{a+b}{2}\right|+\left|x_{i+1}-\frac{a+b}{2}\right|\right](x_{i+1}-x_i).
$$

References

[1] M. Alomari and M. Darus, The Hadamard's inequality for s-convex function. *Int. J. Math. Anal.* (Ruse) **2** (2008), no. 13-16, 639–646.

[2] M. Alomari and M. Darus, Hadamard-type inequalities for s-convex functions. *Int. Math. Forum* **3** (2008), no. 37-40, 1965–1975.

[3] N. S. Barnett, P. Cerone and S. S. Dragomir, Some new inequalities for Hermite-Hadamard divergence in information theory. in *Stochastic Analysis and Applications.* Vol. **3**, 7–19, Nova Sci. Publ., Hauppauge, NY, 2003.

[4] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities. *Comput. Math. Appl.* **58** (2009), no. 9, 1869–1877.

[5] P. S. Bullen, D. S. Mitrinović and P. M. Vasić, *Means and their Inequalities*, Reidel, Dordrecht (1988).

[6] P. Cerone and S. S. Dragomir, *Mathematical Inequalities. A Perspective.* CRC Press, Boca Raton, FL, 2011. x+391 pp. ISBN: 978-1-4398-4896-8.

[7] S. S. Dragomir, Some remarks on Hadamard's inequality for convex functions, *Extracta Math.* (Spain), **9**(2) (1994), 88-94.

[8] S. S. Dragomir, On a reverse of Jensen's inequality for isotonic linear functionals, *J. Inequal. Pure Appl. Math.* **2**(3) (2001), Article 36.

[9] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No. 2, Article 31.

[10] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No. 3, Article 35.

[11] S. S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), 471-476.

[12] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.* Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1

[13] S. S. Dragomir, Some reverses of the Jensen inequality with applications, *Bull. Aust. Math. Soc.* **87** (2013), 177–194

[14] S. S. Dragomir and N. M. Ionescu, Some converse of Jensen's inequality and applications, *Rev. Anal. Numér. Théor. Approx.* **23**(1) (1994), 71–78.

[15] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000. [Online http://rgmia.org/monographs/hermite_hadamard.html].

[16] U. S. Kirmaci, M. Klaričić Bakula, M. E Özdemir and J. Pečarić, Hadamard-type inequalities for s-convex functions. *Appl. Math. Comput.* **193** (2007), no. 1, 26–35.
[17] M. A. Latif, On some inequalities for h-convex functions. *Int. J. Math. Anal.* (Ruse) **4** (2010), no. 29-32, 1473–1482.
[18] M. A. Noor, K. I. Noor and M. U. Awan, Some inequalities for geometrically-arithmetically h-convex functions, *Creat. Math. Inform.* **23** (2014), No. 1, 91 - 98.
[19] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications* (Mathematics in Science and Engineering), Boston/San Diego/New York/London/Sydney/Tokyo/Toronto, 187, Academic Press Inc., 1992.
[20] M. Radulescu, S. Radulescu and P. Alexandrescu, On the Godunova-Levin-Schur class of functions. *Math. Inequal. Appl.* **12** (2009), no. 4, 853–862.
[21] M. Z. Sarikaya, A. Saglam, and H. Yildirim, On some Hadamard-type inequalities for h-convex functions. *J. Math. Inequal.* **2** (2008), no. 3, 335–341.
[22] E. Set, M. E. Özdemir and M. Z. Sarıkaya, New inequalities of Ostrowski's type for s-convex functions in the second sense with applications. *Facta Univ. Ser. Math. Inform.* **27** (2012), no. 1, 67–82.
[23] M. Z. Sarikaya, E. Set and M. E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions. *Acta Math. Univ. Comenian.* (N.S.) **79** (2010), no. 2, 265–272.
[24] M. Tunç, Ostrowski-type inequalities via h-convex functions with applications to special means. *J. Inequal. Appl.* **2013**, 2013:326.
[25] S. Varošanec, On h-convexity. *J. Math. Anal. Appl.* **326** (2007), no. 1, 303–311.
[26] X.-M. Zhang, Y.-M. Chu and X.-H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its application, *Journal of Inequalities and Applications*, Volume **2010**, Article ID 507560, 11 pages.

[1]Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa