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HERMITE-HADAMARD-FEJER TYPE INEQUALITIES FOR HARMONICALLY CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, firstly, Hermite–Hadamard-Fejér type inequality for harmonically convex functions in fractional integral forms have been established. Secondly, an integral identity and some Hermite-Hadamard-Fejer type integral inequalities for harmonically convex functions in fractional integral forms have been obtained. The some results presented here would provide extensions of those given in earlier works.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality [5].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In [4], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative,integrable and symmetric to $(a + b)/2$.

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2) see [1, 6, 7, 15, 17].

We recall the following inequality and special functions which are known as Beta and hypergeometric function respectively

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \ x, y > 0,$$

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \frac{1}{\beta(b, c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \\ c &> b > 0, |z| < 1 \text{ (see [12])}. \end{aligned}$$

Lemma 1. [14, 19]. *For $0 < \alpha \leq 1$ and $0 \leq a < b$ we have*

$$|a^\alpha - b^\alpha| \leq (b-a)^\alpha.$$

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Following definitions and mathematical preliminaries of fractional calculus theory are used further in this paper.

Definition 1. [12]. *Let $f\in L[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of oder $\alpha>0$ with $a\geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\left(x-t\right)^{\alpha-1}f(t)dt,\ \ x>a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}\left(t-x\right)^{\alpha-1}f(t)dt,\ \ x<b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha)=\int\limits_{0}^{\infty}e^{-t}t^{\alpha-1}dt$ and $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [3, 8, 9, 16, 18, 19].

In [11], İşcan gave definition of harmonically convex functions and established following Hermite-Hadamard type inequality for harmonically convex functions as follows:

Definition 2. *Let $I\subset\mathbb{R}\backslash\{0\}$ be a real interval. A function $f:I\rightarrow\mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)\,y}\right)\leq tf\left(y\right)+(1-t)\,f\left(x\right) \tag{1.3}$$

for all $x,y\in I$ and $t\in[0,1]$. If the inequality in (1.3) is reversed, then f is said to be harmonically concave.

Theorem 2. [11]. *Let $f:I\subset\mathbb{R}\backslash\{0\}\rightarrow\mathbb{R}$ be a harmonically convex function and $a,b\in I$ with $a<b$. If $f\in L[a,b]$ then the following inequalities hold:*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_{a}^{b}\frac{f\left(x\right)}{x^{2}}dx\leq\frac{f\left(a\right)+f\left(b\right)}{2}. \tag{1.4}$$

In [10], İşcan and Wu represented Hermite-Hadamard's inequalities for harmonically convex functions in fractional integral forms as follows:

Theorem 3. *Let $f:I\subset(0,\infty)\rightarrow\mathbb{R}$ be a function such that $f\in L[a,b]$, where $a,b\in I$ with $a<b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned}f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma\left(\alpha+1\right)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c}J_{1/a-}^{\alpha}\left(f\circ h\right)\left(1/b\right)\\+J_{1/b+}^{\alpha}\left(f\circ h\right)\left(1/a\right)\end{array}\right\}\\ &\leq \frac{f\left(a\right)+f\left(b\right)}{2}\end{aligned} \tag{1.5}$$

with $\alpha>0$ and $h(x)=1/x$.

In [13] Latif et. al. gave the following definition:

Definition 3. *A function $g:[a,b]\subseteq\mathbb{R}\backslash\{0\}\rightarrow\mathbb{R}$ is said to be harmonically symmetric with respect to $2ab/a+b$ if*

$$g(x)=g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right)$$

holds for all $x\in[a,b]$.

In [2] Chan and Wu represented Hermite-Hadamard-Fejer inequality for harmonically convex functions as follows:

Theorem 4. *Let $f:I\subseteq\mathbb{R}\backslash\{0\}\rightarrow\mathbb{R}$ be a harmonically convex function and $a,b\in I$ with $a<b$. If $f\in L[a,b]$ and $g:[a,b]\subseteq\mathbb{R}\backslash\{0\}\rightarrow\mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then*

$$\begin{aligned}f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{g(x)}{x^2}dx &\leq \int_a^b\frac{f(x)g(x)}{x^2}dx\\ &\leq \frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx. \qquad (1.6)\end{aligned}$$

In this paper, we firstly represented Hermite-Hadamard-Fejér inequality for harmonically convex function in fractional integral forms which is the weighted generalization of Hermite-Hadamard inequality for harmonically convex functions (1.5). Secondly, we obtained some new inequalities connected with the right-hand side of Hermite-Hadamard-Fejér type integral inequality for harmonically convex function in fractional integrals.

2. Main results

Throughout this section, we take $\|g\|_\infty=\sup_{t\in[a,b]}|g(x)|$, for the continuous function $g:[a,b]\rightarrow\mathbb{R}$.

Lemma 2. *If $g:[a,b]\subseteq\mathbb{R}\backslash\{0\}\rightarrow\mathbb{R}$ is integrable and harmonically symmetric with respect to $2ab/a+b$ with $a<b$, then*

$$J_{1/b+}^{\alpha}(g\circ h)(1/a)=J_{1/a-}^{\alpha}(g\circ h)(1/b)=\frac{1}{2}\left[\begin{array}{c}J_{1/b+}^{\alpha}(g\circ h)(1/a)\\+J_{1/a-}^{\alpha}(g\circ h)(1/b)\end{array}\right]$$

with $\alpha>0$ and $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

Proof. Since g is harmonically symmetric with respect to $2ab/a+b$, using Definition 3 we have $g\left(\frac{1}{x}\right)=g(\frac{1}{(\frac{1}{a})+(\frac{1}{b})-x})$, for all $x\in\left[\frac{1}{b},\frac{1}{a}\right]$. Hence, in the following integral setting $t=(\frac{1}{a})+(\frac{1}{b})-x$ and $dt=-dx$ gives

$$\begin{aligned}J_{1/b+}^{\alpha}(g\circ h)(1/a) &= \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}g\left(\frac{1}{t}\right)dt\\ &= \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1}g\left(\frac{1}{(1/a)+(1/b)-x}\right)dx\\ &= \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1}g\left(\frac{1}{x}\right)dx=J_{1/a-}^{\alpha}(g\circ h)(1/b).\end{aligned}$$

This completes the proof. □

Theorem 5. *Let $f:[a,b]\rightarrow\mathbb{R}$ be harmonically convex function with $a<b$ and $f\in L[a,b]$. If $g:[a,b]\rightarrow\mathbb{R}$ is nonnegative, integrable and harmonically symmetric*

with respect to $2ab/a+b$, *then the following inequalities for fractional integrals hold:*

$$
\begin{aligned}
& f\left(\frac{2ab}{a+b}\right)\left[J_{1/b+}^{\alpha}\left(g\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(g\circ h\right)(1/b)\right] \\
\leq\; & \left[J_{1/b+}^{\alpha}\left(fg\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(fg\circ h\right)(1/b)\right] \\
\leq\; & \frac{f(a)+f(b)}{2}\left[J_{1/b+}^{\alpha}\left(g\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(g\circ h\right)(1/b)\right] \qquad (2.1)
\end{aligned}
$$

with $\alpha>0$ *and* $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

Proof. Since f is a harmonically convex function on $[a,b]$, we have for all $t\in[0,1]$

$$
\begin{aligned}
f\left(\frac{2ab}{a+b}\right) &= f\left(\frac{2\left(\frac{ab}{ta+(1-t)b}\right)\left(\frac{ab}{tb+(1-t)a}\right)}{\left(\frac{ab}{ta+(1-t)b}\right)+\left(\frac{ab}{tb+(1-t)a}\right)}\right) \\
&\leq \frac{f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)}{2}. \qquad (2.2)
\end{aligned}
$$

Multiplying both sides of (2.2) by $2t^{\alpha-1}g\left(\frac{ab}{tb+(1-t)a}\right)$ then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$
\begin{aligned}
& 2f\left(\frac{2ab}{a+b}\right)\int_0^1 t^{\alpha-1}g\left(\frac{ab}{tb+(1-t)a}\right)dt \\
\leq\; & \int_0^1 t^{\alpha-1}\left[f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)\right]g\left(\frac{ab}{tb+(1-t)a}\right)dt \\
=\; & \int_0^1 t^{\alpha-1}f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt \\
& +\int_0^1 t^{\alpha-1}f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt.
\end{aligned}
$$

Since g is harmonically symmetric with respect to $2ab/a+b$, using Definition 3 we have $g\left(\frac{1}{x}\right)=g(\frac{1}{(\frac{1}{a})+(\frac{1}{b})-x})$, for all $x\in\left[\frac{1}{b},\frac{1}{a}\right]$. Setting $x=\frac{tb+(1-t)a}{ab}$, and $dx=\left(\frac{b-a}{ab}\right)dt$ gives

$$
\begin{aligned}
& 2\left(\frac{ab}{b-a}\right)^{\alpha}f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1}g\left(\frac{1}{x}\right)dx \\
\leq\; & \left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c}\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1}f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)g\left(\frac{1}{x}\right)dx \\ +\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1}f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx\end{array}\right\} \\
=\; & \left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c}\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-x\right)^{\alpha-1}f\left(\frac{1}{x}\right)g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)dx \\ +\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1}f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx\end{array}\right\} \\
=\; & \left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c}\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-x\right)^{\alpha-1}f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx \\ +\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1}f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx\end{array}\right\}
\end{aligned}
$$

Therefore, by Lemma 2 we have

$$
\begin{aligned}
& \left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)f\left(\frac{2ab}{a+b}\right)\left[J_{1/b+}^{\alpha}\left(g\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(g\circ h\right)(1/b)\right] \\
\leq\; & \left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)\left[J_{1/b+}^{\alpha}\left(fg\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(fg\circ h\right)(1/b)\right]
\end{aligned}
$$

and the first inequality is proved.

For the proof of the second inequality in (2.1) we first note that if f is a harmonically convex function, then, for all $t\in[0,1]$, it yields

$$f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)\leq f(a)+f(b). \tag{2.3}$$

Then multiplying both sides of (2.3) by $t^{\alpha-1}g\left(\frac{ab}{tb+(1-t)a}\right)$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned} &\int_0^1 t^{\alpha-1}f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt \\ &+\int_0^1 t^{\alpha-1}f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt \\ \leq\ & [f(a)+f(b)]\int_0^1 t^{\alpha-1}g\left(\frac{ab}{tb+(1-t)a}\right)dt \end{aligned}$$

i.e.

$$\begin{aligned} &\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)\left[J^{\alpha}_{1/b+}\left(fg\circ h\right)(1/a)+J^{\alpha}_{1/a-}\left(fg\circ h\right)(1/b)\right] \\ \leq\ &\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)\left(\frac{f(a)+f(b)}{2}\right)\left[J^{\alpha}_{1/b+}\left(g\circ h\right)(1/a)+J^{\alpha}_{1/a-}\left(g\circ h\right)(1/b)\right] \end{aligned}$$

The proof is completed. □

Remark 1. *In Theorem 5,*

(i) if we take $\alpha=1$, then inequality (2.1) becomes inequality (1.6) of Theorem 4.

(ii) if we take $g(x)=1$, then inequality (2.1) becomes inequality (1.5) of Theorem 3.

(iii) if we take $\alpha=1$ and $g(x)=1$, then inequality (2.1) becomes inequality (1.4) of Theorem 2.

Lemma 3. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ and $a<b$. If $g:[a,b]\to\mathbb{R}$ is integrable and harmonically symmetric with respect to $2ab/a+b$, then the following equality for fractional integrals hold:*

$$\begin{aligned} &\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}\left(g\circ h\right)(1/a)+J^{\alpha}_{1/a-}\left(g\circ h\right)(1/b)\right] \\ &-\left[J^{\alpha}_{1/b+}\left(fg\circ h\right)(1/a)+J^{\alpha}_{1/a-}\left(fg\circ h\right)(1/b)\right] \\ =\ &\frac{1}{\Gamma(\alpha)}\int_{\frac{1}{b}}^{\frac{1}{a}}\left[\begin{array}{c}\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\alpha-1}\left(g\circ h\right)(s)ds \\ -\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\alpha-1}\left(g\circ h\right)(s)ds\end{array}\right]\left(f\circ h\right)'(t)dt \end{aligned} \tag{2.4}$$

with $\alpha>0$ and $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

Proof. It suffices to note that

$$\begin{aligned}
I &= \int_{\frac{1}{b}}^{\frac{1}{a}} \begin{bmatrix} \int_{\frac{1}{b}}^{t} \left(\frac{1}{a}-s\right)^{\alpha-1} (g\circ h)(s)ds \\ -\int_{t}^{\frac{1}{a}} \left(s-\frac{1}{b}\right)^{\alpha-1} (g\circ h)(s)ds \end{bmatrix} (f\circ h)'(t)dt \\
&= \int_{\frac{1}{b}}^{\frac{1}{a}} \left[\int_{\frac{1}{b}}^{t} \left(\frac{1}{a}-s\right)^{\alpha-1} (g\circ h)(s)ds\right] (f\circ h)'(t)dt \\
&\quad -\int_{\frac{1}{b}}^{\frac{1}{a}} \left[\int_{t}^{\frac{1}{a}} \left(s-\frac{1}{b}\right)^{\alpha-1} (g\circ h)(s)ds\right] (f\circ h)'(t)dt \\
&= I_1 - I_2.
\end{aligned}$$

By integration by parts and Lemma 2 we get

$$\begin{aligned}
I_1 &= \left(\int_{\frac{1}{b}}^{t} \left(\frac{1}{a}-s\right)^{\alpha-1} (g\circ h)(s)ds\right) (f\circ h)(t)\Big|_{\frac{1}{b}}^{\frac{1}{a}} \\
&\quad -\int_{\frac{1}{b}}^{\frac{1}{a}} \left(\frac{1}{a}-t\right)^{\alpha-1} (g\circ h)(t)(f\circ h)(t)dt \\
&= \left(\int_{\frac{1}{b}}^{\frac{1}{a}} \left(\frac{1}{a}-s\right)^{\alpha-1} (g\circ h)(s)ds\right) f(a) \\
&\quad -\int_{\frac{1}{b}}^{\frac{1}{a}} \left(\frac{1}{a}-t\right)^{\alpha-1} (g\circ h)(t)(f\circ h)(t)dt \\
&= \Gamma(\alpha)\left[f(a)J_{1/b+}^{\alpha}(g\circ h)(1/a) - J_{1/b+}^{\alpha}(fg\circ h)(1/a)\right] \\
&= \Gamma(\alpha)\begin{bmatrix} \frac{f(a)}{2}\left[J_{1/b+}^{\alpha}(g\circ h)(1/a) + J_{1/a-}^{\alpha}(g\circ h)(1/b)\right] \\ -J_{1/b+}^{\alpha}(fg\circ h)(1/a) \end{bmatrix}
\end{aligned}$$

and similarly

$$\begin{aligned}
I_2 &= \left(\int_{t}^{\frac{1}{a}} \left(s-\frac{1}{b}\right)^{\alpha-1} (g\circ h)(s)ds\right) (f\circ h)(t)\Big|_{\frac{1}{b}}^{\frac{1}{a}} \\
&\quad -\int_{\frac{1}{b}}^{\frac{1}{a}} \left(t-\frac{1}{b}\right)^{\alpha-1} (g\circ h)(t)(f\circ h)(t)dt \\
&= -\left(\int_{\frac{1}{b}}^{\frac{1}{a}} \left(s-\frac{1}{b}\right)^{\alpha-1} (g\circ h)(s)ds\right) f(b) \\
&\quad +\int_{\frac{1}{b}}^{\frac{1}{a}} \left(t-\frac{1}{b}\right)^{\alpha-1} (g\circ h)(t)(f\circ h)(t)dt \\
&= \Gamma(\alpha)\left[-f(b)J_{1/a-}^{\alpha}(g\circ h)(1/b) + J_{1/a-}^{\alpha}(fg\circ h)(1/b)\right] \\
&= \Gamma(\alpha)\begin{bmatrix} -\frac{f(b)}{2}\left[J_{1/b+}^{\alpha}(g\circ h)(1/a) + J_{1/a-}^{\alpha}(g\circ h)(1/b)\right] \\ +J_{1/a-}^{\alpha}(fg\circ h)(1/b) \end{bmatrix}.
\end{aligned}$$

Thus, we can write

$$I = I_1 - I_2 = \Gamma(\alpha)\left\{ \begin{array}{c} \left(\frac{f(a)+f(b)}{2}\right)\left[J_{1/b+}^{\alpha}(g\circ h)(1/a) + J_{1/a-}^{\alpha}(g\circ h)(1/b)\right] \\ -\left[J_{1/b+}^{\alpha}(fg\circ h)(1/a) + J_{1/a-}^{\alpha}(fg\circ h)(1/b)\right] \end{array} \right\}.$$

Multiplying the both sides by $(\Gamma(\alpha))^{-1}$ we obtain (2.4) which completes the proof. □

Remark 2. *In Lemma 3, if we take $g(x) = 1$, then equality (2.4) becomes equality in* [10, Lemma 3].

Theorem 6. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ and $a < b$. If $|f'|$ is harmonically convex on $[a,b]$, $g : [a,b] \to \mathbb{R}$ is continuous and harmonically symmetric with respect to $2ab/a+b$, then the following inequality for fractional integrals hold:*

$$\left| \begin{array}{c} \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}\left(g\circ h\right)(1/a)+J^{\alpha}_{1/a-}\left(g\circ h\right)(1/b)\right] \\ -\left[J^{\alpha}_{1/b+}\left(fg\circ h\right)(1/a)+J^{\alpha}_{1/a-}\left(fg\circ h\right)(1/b)\right] \end{array} \right|$$
$$\leq \frac{\|g\|_\infty ab\,(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[C_1(\alpha)\left|f'(a)\right|+C_2(\alpha)\left|f'(b)\right|\right] \tag{2.5}$$

where

$$C_1(\alpha) = \left[\begin{array}{c} \frac{b^{-2}}{\alpha+2}\ {}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{(\alpha+1)(\alpha+2)}\ {}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right) \\ +\frac{2(a+b)^{-2}}{(\alpha+1)(\alpha+2)}\ {}_2F_1\left(2,\alpha+1;\alpha+3;\frac{b-a}{b+a}\right) \end{array} \right]$$

$$C_2(\alpha) = \left[\begin{array}{c} \frac{b^{-2}}{(\alpha+1)(\alpha+2)}\ {}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{\alpha+2}\ {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right) \\ +\left(\frac{a+b}{2}\right)^{-2}\frac{1}{\alpha+1}\ {}_2F_1\left(2,\alpha+1;\alpha+2;\frac{b-a}{b+a}\right) \\ -\frac{2(a+b)^{-2}}{(\alpha+1)(\alpha+2)}\ {}_2F_1\left(2,\alpha+1;\alpha+3;\frac{b-a}{b+a}\right) \end{array} \right]$$

with $0 < \alpha \leq 1$ and $h(x) = 1/x$, $x \in \left[\frac{1}{b},\frac{1}{a}\right]$.

Proof. From Lemma 3 we have

$$\left| \begin{array}{c} \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}\left(g\circ h\right)(1/a)+J^{\alpha}_{1/a-}\left(g\circ h\right)(1/b)\right] \\ -\left[J^{\alpha}_{1/b+}\left(fg\circ h\right)(1/a)+J^{\alpha}_{1/a-}\left(fg\circ h\right)(1/b)\right] \end{array} \right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{b}}^{\frac{1}{a}}\left| \begin{array}{c} \int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\alpha-1}(g\circ h)(s)ds \\ -\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds \end{array} \right|\left|(f\circ h)'(t)\right|dt \tag{2.6}$$

Since g is harmonically symmetric with respect to $2ab/a+b$, using Definition 3 we have $g\left(\frac{1}{x}\right) = g\left(\frac{1}{(\frac{1}{a})+(\frac{1}{b})-x}\right)$, for all $x \in \left[\frac{1}{b},\frac{1}{a}\right]$.

$$\left|\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\alpha-1}(g\circ h)(s)ds-\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds\right|$$
$$= \left|\int_{\frac{1}{a}+\frac{1}{b}-t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds+\int_{\frac{1}{a}}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds\right|$$
$$= \left|\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds\right|$$
$$\leq \begin{cases} \int_{t}^{\frac{1}{a}+\frac{1}{b}-t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)\right|ds, & t\in\left[\frac{1}{b},\frac{a+b}{2ab}\right] \\ \int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)\right|ds, & t\in\left[\frac{a+b}{2ab},\frac{1}{a}\right] \end{cases} \tag{2.7}$$

If we use (2.7) in (2.6), we have

$$\begin{aligned}
&\left|\begin{array}{c} \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right]\end{array}\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left[\begin{array}{c} \int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_t^{\frac{1}{a}+\frac{1}{b}-t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)\right|ds\right)\left|(f\circ h)'(t)\right|dt \\ +\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)\right|ds\right)\left|(f\circ h)'(t)\right|dt\end{array}\right] \\
\leq\ & \frac{\|g\|_{\infty}}{\Gamma(\alpha)}\left[\begin{array}{c} \int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_t^{\frac{1}{a}+\frac{1}{b}-t}\left(s-\frac{1}{b}\right)^{\alpha-1}ds\right)\left|(f\circ h)'(t)\right|dt \\ +\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}ds\right)\left|(f\circ h)'(t)\right|dt\end{array}\right] \\
=\ & \frac{\|g\|_{\infty}}{\Gamma(\alpha)}\left[\begin{array}{c} \int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_t^{\frac{1}{a}+\frac{1}{b}-t}\left(s-\frac{1}{b}\right)^{\alpha-1}ds\right)\frac{1}{t^2}\left|f'(\frac{1}{t})\right|dt \\ +\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}ds\right)\frac{1}{t^2}\left|f'(\frac{1}{t})\right|dt\end{array}\right]
\end{aligned}$$

Setting $t=\frac{ub+(1-u)a}{ab}$, and $dt=\left(\frac{b-a}{ab}\right)du$ gives

$$\begin{aligned}
&\left|\begin{array}{c} \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right]\end{array}\right| \\
\leq\ & \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{c} \int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du \\ +\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du\end{array}\right] \quad (2.8)
\end{aligned}$$

Since $|f'|$ is harmonically convex on $[a,b]$, we have

$$\left|f'(\frac{ab}{ub+(1-u)a})\right|\leq u\left|f'(a)\right|+(1-u)\left|f'(b)\right| \quad (2.9)$$

If we use (2.9) in (2.8), we have

$$\begin{aligned}
&\left|\begin{array}{c} \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right]\end{array}\right| \\
\leq\ & \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} \\
& \times\left[\begin{array}{c} \left[\begin{array}{c}\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}udu \\ +\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}udu\end{array}\right]\left|f'(a)\right| \\ +\left[\begin{array}{c}\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}(1-u)du \\ +\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}(1-u)du\end{array}\right]\left|f'(b)\right|\end{array}\right] \quad (2.10)
\end{aligned}$$

Calculating following integrals by Lemma 1, we have

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2} udu + \int_{\frac{1}{2}}^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2} udu \\
= & \int_0^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2} udu + 2\int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2} udu \\
\leq & \int_0^1 \frac{u^{\alpha+1}}{(ub+(1-u)a)^2} du - \int_0^1 \frac{u(1-u)^\alpha}{(ub+(1-u)a)^2} du + 2\int_0^{\frac{1}{2}} \frac{(1-2u)^\alpha}{(ub+(1-u)a)^2} udu \\
= & \int_0^1 \frac{(1-u)^{\alpha+1}}{(ua+(1-u)b)^2} du - \int_0^1 \frac{(1-u)u^\alpha}{(ua+(1-u)b)^2} du + \frac{1}{2}\int_0^1 \frac{u(1-u)^\alpha}{\left(\frac{u}{2}b+\left(1-\frac{u}{2}\right)a\right)^2} du \\
= & \int_0^1 (1-u)^{\alpha+1} b^{-2} \left(1-u\left(1-\frac{a}{b}\right)\right)^{-2} du \\
& -\int_0^1 (1-u) u^\alpha b^{-2} \left(1-u\left(1-\frac{a}{b}\right)\right)^{-2} du \\
& +\frac{1}{2}\int_0^1 (1-v) v^\alpha \left(\frac{a+b}{2}\right)^{-2} \left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2} dv \\
= & \left[\begin{array}{c}
\frac{b^{-2}}{\alpha+2}\ {}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right) \\
-\frac{b^{-2}}{(\alpha+1)(\alpha+2)}\ {}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right) \\
+\frac{2(a+b)^{-2}}{(\alpha+1)(\alpha+2)}\ {}_2F_1\left(2,\alpha+1;\alpha+3;\frac{b-a}{b+a}\right)
\end{array}\right] \\
= & C_1(\alpha)
\end{aligned} \tag{2.11}$$

and similarly we get

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2}(1-u)\,du + \int_{\frac{1}{2}}^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2}(1-u)\,du \\
\leq & \int_0^1 \frac{u(1-u)^\alpha}{(ua+(1-u)b)^2} du - \int_0^1 \frac{u^{\alpha+1}}{(ua+(1-u)b)^2} du \\
& + \int_0^1 \frac{(1-u)^\alpha}{\left(\frac{u}{2}b+\left(1-\frac{u}{2}\right)a\right)^2} du - \frac{1}{2}\int_0^1 \frac{u(1-u)^\alpha}{\left(\frac{u}{2}b+\left(1-\frac{u}{2}\right)a\right)^2} du \\
= & \int_0^1 \frac{u(1-u)^\alpha}{(ua+(1-u)b)^2} du - \int_0^1 \frac{u^{\alpha+1}}{(ua+(1-u)b)^2} du \\
& + \left(\frac{a+b}{2}\right)^{-2} \int_0^1 v^\alpha \left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2} dv \\
& -\frac{1}{2}\left(\frac{a+b}{2}\right)^{-2} \int_0^1 (1-v) v^\alpha \left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2} dv \\
= & \left[\begin{array}{c}
\frac{b^{-2}}{(\alpha+1)(\alpha+2)}\ {}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right) \\
-\frac{b^{-2}}{\alpha+2}\ {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right) \\
+\left(\frac{a+b}{2}\right)^{-2} \frac{1}{\alpha+1}\ {}_2F_1\left(2,\alpha+1;\alpha+2;\frac{b-a}{b+a}\right) \\
-\frac{2(a+b)^{-2}}{(\alpha+1)(\alpha+2)}\ {}_2F_1\left(2,\alpha+1;\alpha+3;\frac{b-a}{b+a}\right)
\end{array}\right] \\
= & C_2(\alpha)
\end{aligned} \tag{2.12}$$

If we use (2.11) and (2.12) in (2.10), we have (2.5). This completes the proof. □

Corollary 1. *In Theorem 6;*

(1) If we take $\alpha = 1$ we have the following Hermite-Hadamard-Fejer inequality for harmonically convex functions which is related the right-hand side of (1.6)*:*

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx \right| \\ \leq & \frac{\|g\|_\infty (b-a)^2}{2}\left[C_1(1)\left|f'(a)\right| + C_2(1)\left|f'(b)\right|\right], \end{aligned}$$

(2) If we take $g(x) = 1$ we have following Hermite-Hadamard type inequality for harmonically convex function in fractional integral forms which is related the right-hand side of (1.5)*:*

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^\alpha \left\{ \begin{array}{c} J^\alpha_{1/a-}(f\circ h)(1/b) \\ +J^\alpha_{1/b+}(f\circ h)(1/a) \end{array} \right\} \right| \\ \leq & \frac{ab(b-a)}{2}\left[C_1(\alpha)\left|f'(a)\right| + C_2(\alpha)\left|f'(b)\right|\right], \end{aligned}$$

(3) If we take $\alpha = 1$ and $g(x) = 1$ we have the following Hermite-Hadamard type inequality for harmonically convex function which is related the right-hand side of (1.4)*:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \leq \frac{ab(b-a)}{2}\left[C_1(1)\left|f'(a)\right| + C_2(1)\left|f'(b)\right|\right].$$

Theorem 7. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$ and $a < b$. If $|f'|^q, q \geq 1$, is harmonically convex on $[a,b]$, $g : [a,b] \to \mathbb{R}$ is continuous and harmonically symmetric with respect to $2ab/a+b$, then the following inequality for fractional integrals hold:*

$$\begin{aligned} & \left| \begin{array}{c} \frac{f(a)+f(b)}{2}\left[J^\alpha_{1/b+}(g\circ h)(1/a) + J^\alpha_{1/a-}(g\circ h)(1/b)\right] \\ -\left[J^\alpha_{1/b+}(fg\circ h)(1/a) + J^\alpha_{1/a-}(fg\circ h)(1/b)\right] \end{array} \right| \\ \leq & \frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^\alpha \\ & \times \left[\begin{array}{c} C_3^{1-\frac{1}{q}}(\alpha)\left[\left(\begin{array}{c} C_4(\alpha)\left|f'(a)\right|^q \\ +C_5(\alpha)\left|f'(b)\right|^q \end{array} \right)\right]^{\frac{1}{q}} \\ +C_6^{1-\frac{1}{q}}(\alpha)\left[\left(\begin{array}{c} C_7(\alpha)\left|f'(a)\right|^q \\ +C_8(\alpha)\left|f'(b)\right|^q \end{array} \right)\right]^{\frac{1}{q}} \end{array} \right] \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} C_3(\alpha) &= \frac{2(a+b)^{-2}}{(\alpha+1)}\ {}_2F_1\left(2, \alpha+1; \alpha+3; \tfrac{b-a}{b+a}\right), \\ C_4(\alpha) &= \frac{(a+b)^{-2}}{(\alpha+1)(\alpha+2)}\ {}_2F_1\left(2, \alpha+1; \alpha+3; \tfrac{b-a}{b+a}\right), \\ C_5(\alpha) &= C_3(\alpha) - C_4(\alpha), \\ C_6(\alpha) &= \left[\begin{array}{c} \frac{b^{-2}}{(\alpha+1)}\ {}_2F_1\left(2, 1; \alpha+2; 1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{(\alpha+1)}\ {}_2F_1\left(2, \alpha+1; \alpha+2; 1-\frac{a}{b}\right) + C_3(\alpha) \end{array} \right] \\ C_7(\alpha) &= \left[\begin{array}{c} \frac{b^{-2}}{(\alpha+2)}\ {}_2F_1\left(2, 1; \alpha+3; 1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{(\alpha+1)(\alpha+2)}\ {}_2F_1\left(2, \alpha+1; \alpha+3; 1-\frac{a}{b}\right) + C_4(\alpha) \end{array} \right] \\ C_8(\alpha) &= C_6(\alpha) - C_7(\alpha), \end{aligned}$$

with $0 < \alpha \leq 1$ and $h(x) = 1/x$, $x \in \left[\frac{1}{b}, \frac{1}{a}\right]$.

Proof. Using (2.8), power mean inequality and the harmonically convexity of $|f'|^q$, it follows that

$$\begin{aligned}
&\left|\begin{array}{c}\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right]\\ -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right]\end{array}\right| \\
\leq\ & \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{c}\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du\\ +\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du\end{array}\right] \\
\leq\ & \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\\ \times\left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|^q du\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\\ \times\left(\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|^q du\right)^{\frac{1}{q}}\end{array}\right] \\
\leq\ & \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} \\
&\times\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\\ \times\left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left[u|f'(a)|^q+(1-u)|f'(b)|^q\right]du\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\\ \times\left(\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left[u|f'(a)|^q+(1-u)|f'(b)|^q\right]du\right)^{\frac{1}{q}}\end{array}\right] \\
=\ & \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} \\
&\times\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\\ \times\left(\begin{array}{c}\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}udu\,|f'(a)|^q\\ +\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}(1-u)\,du\,|f'(b)|^q\end{array}\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\\ \times\left(\begin{array}{c}\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}udu\,|f'(a)|^q\\ +\int_{\frac{1}{2}}^{1}\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}(1-u)\,du\,|f'(b)|^q\end{array}\right)^{\frac{1}{q}}\end{array}\right] \qquad (2.14)
\end{aligned}$$

Calculating following integrals by Lemma 1, we have

$$\begin{aligned}
\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du &\leq \int_0^{\frac{1}{2}}\frac{(1-2u)^{\alpha}}{(ub+(1-u)a)^2}du \\
&= \frac{1}{2}\int_0^1\frac{(1-u)^{\alpha}}{\left(\frac{u}{2}b+(1-\frac{u}{2})a\right)^2}du \\
&= 2(a+b)^{-2}\int_0^1 v^{\alpha}\left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2}dv \\
&= \frac{2(a+b)^{-2}}{(\alpha+1)}\ {}_2F_1\left(2,\alpha+1;\alpha+2;\tfrac{b-a}{b+a}\right) \\
&= C_3(\alpha) \qquad (2.15)
\end{aligned}$$

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}udu &\leq \int_0^{\frac{1}{2}} \frac{(1-2u)^{\alpha}}{(ub+(1-u)a)^2}udu \\ &= \frac{1}{4}\int_0^1 \frac{u(1-u)^{\alpha}}{\left(\frac{u}{2}b+(1-\frac{u}{2})a\right)^2}du \\ &= (a+b)^{-2}\int_0^1 (1-v)v^{\alpha}\left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2}du \\ &= \frac{(a+b)^{-2}}{(\alpha+1)(\alpha+2)}\ {}_2F_1\left(2,\alpha+1;\alpha+3;\tfrac{b-a}{b+a}\right) \\ &= C_4(\alpha) \qquad (2.16)\end{aligned}$$

$$\int_0^{\frac{1}{2}} \frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}(1-u)\,du \leq C_3(\alpha)-C_4(\alpha)=C_5(\alpha) \qquad (2.17)$$

$$\begin{aligned}\int_{\frac{1}{2}}^1 \frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du &= \int_0^1 \frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du+\int_0^{\frac{1}{2}} \frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du \\ &\leq \left[\begin{array}{c}\frac{b^{-2}}{(\alpha+1)}\ {}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{(\alpha+1)}\ {}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right)\ +C_3(\alpha)\end{array}\right] \\ &= C_6(\alpha) \qquad (2.18)\end{aligned}$$

$$\begin{aligned}\int_{\frac{1}{2}}^1 \frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}udu &= \int_0^1 \frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}udu+\int_0^{\frac{1}{2}} \frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}udu \\ &\leq \left[\begin{array}{c}\frac{b^{-2}}{(\alpha+2)}\ {}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{(\alpha+1)(\alpha+2)}\ {}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right)\ +C_4(\alpha)\end{array}\right] \\ &= C_7(\alpha) \qquad (2.19)\end{aligned}$$

$$\int_{\frac{1}{2}}^1 \frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}(1-u)\,du \leq C_6(\alpha)-C_7(\alpha)=C_8(\alpha) \qquad (2.20)$$

If we use $(2.15-2.20)$ in (2.14), we have (2.13).This completes the proof. □

Corollary 2. *In Theorem 7;*

(1) If we take $\alpha=1$ *we have the following Hermite-Hadamard-Fejer inequality for harmonically convex functions which is related the right-hand side of* (1.6)*:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx-\int_a^b \frac{f(x)g(x)}{x^2}dx\right| \\ \leq\ & \frac{\|g\|_{\infty}(b-a)^2}{2}\left[\begin{array}{c} C_3^{1-\frac{1}{q}}(1)\left[\left(\begin{array}{c} C_4(1)\left|f'(a)\right|^q \\ +C_5(1)\left|f'(b)\right|^q\end{array}\right)\right]^{\frac{1}{q}} \\ +C_6^{1-\frac{1}{q}}(1)\left[\left(\begin{array}{c} C_7(1)\left|f'(a)\right|^q \\ +C_8(1)\left|f'(b)\right|^q\end{array}\right)\right]^{\frac{1}{q}}\end{array}\right],\end{aligned}$$

(2) If we take $g(x)=1$ *we have following Hermite-Hadamard type inequality for harmonically convex function in fractional integral forms which is related the*

right-hand side of (1.5)*:*

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a} \right)^{\alpha} \left\{ \begin{array}{c} J_{1/a-}^{\alpha} (f \circ h)(1/b) \\ +J_{1/b+}^{\alpha} (f \circ h)(1/a) \end{array} \right\} \right| \\ \leq & \frac{ab(b-a)}{2} \left[\begin{array}{c} C_3^{1-\frac{1}{q}}(\alpha) \left[\left(\begin{array}{c} C_4(\alpha) |f'(a)|^q \\ +C_5(\alpha) |f'(b)|^q \end{array} \right) \right]^{\frac{1}{q}} \\ +C_6^{1-\frac{1}{q}}(\alpha) \left[\left(\begin{array}{c} C_7(\alpha) |f'(a)|^q \\ +C_8(\alpha) |f'(b)|^q \end{array} \right) \right]^{\frac{1}{q}} \end{array} \right], \end{aligned}$$

(3) If we take $\alpha = 1$ *and* $g(x) = 1$ *we have the following Hermite-Hadamard type inequality for harmonically convex function which is related the right-hand side of* (1.4)*:*

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ \leq & \frac{ab(b-a)}{2} \left[\begin{array}{c} C_3^{1-\frac{1}{q}}(1) \left[\left(\begin{array}{c} C_4(1) |f'(a)|^q \\ +C_5(1) |f'(b)|^q \end{array} \right) \right]^{\frac{1}{q}} \\ +C_6^{1-\frac{1}{q}}(1) \left[\left(\begin{array}{c} C_7(1) |f'(a)|^q \\ +C_8(1) |f'(b)|^q \end{array} \right) \right]^{\frac{1}{q}} \end{array} \right]. \end{aligned}$$

We can state another inequality for $q > 1$ as follows:

Theorem 8. *Let* $f : I \subset (0, \infty) \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a, b]$*, where* $a, b \in I$ *and* $a < b$*. If* $|f'|^q, q > 1$*, is harmonically convex on* $[a, b]$*,* $g : [a, b] \to \mathbb{R}$ *is continuous and harmonically symmetric with respect to* $2ab/a + b$ *, then the following inequality for fractional integrals hold:*

$$\begin{aligned} & \left| \begin{array}{c} \frac{f(a)+f(b)}{2} \left[J_{1/b+}^{\alpha} (g \circ h)(1/a) + J_{1/a-}^{\alpha} (g \circ h)(1/b) \right] \\ - \left[J_{1/b+}^{\alpha} (fg \circ h)(1/a) + J_{1/a-}^{\alpha} (fg \circ h)(1/b) \right] \end{array} \right| \\ \leq & \frac{\|g\|_{\infty} ab(b-a)}{\Gamma(\alpha+1)} \left(\frac{b-a}{ab} \right)^{\alpha} \\ & \times \left[\begin{array}{c} C_9^{\frac{1}{p}}(\alpha) \left[\frac{|f'(a)|^q + 3|f'(b)|^q}{8} \right]^{\frac{1}{q}} \\ +C_{10}^{\frac{1}{p}}(\alpha) \left[\frac{3|f'(a)|^q + |f'(b)|^q}{8} \right]^{\frac{1}{q}} \end{array} \right] \end{aligned} \tag{2.21}$$

where

$$\begin{aligned} C_9(\alpha) &= \left(\frac{a+b}{2} \right)^{-2p} \frac{1}{2(\alpha p+1)} \ {}_2F_1\left(2p, \alpha p+1; \alpha p+2; \tfrac{b-a}{b+a}\right), \\ C_{10}(\alpha) &= b^{-2p} \frac{1}{2(\alpha p+1)} \ {}_2F_1\left(2p, 1; \alpha p+2; \tfrac{1}{2}\left(1 - \tfrac{a}{b}\right)\right), \end{aligned}$$

with $0 < \alpha \leq 1$*,* $h(x) = 1/x$*,* $x \in \left[\frac{1}{b}, \frac{1}{a}\right]$ *and* $1/p + 1/q = 1$*.*

Proof. Using (2.8), Hölder's inequality and the harmonically convexity of $|f'|^q$, it follows that

$$\begin{aligned}
&\left| \begin{array}{c} \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right] \end{array}\right| \\
\leq\ & \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{c} \int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du \\ +\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du \end{array}\right] \\
\leq\ & \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{c} \left(\int_0^{\frac{1}{2}}\frac{[(1-u)^{\alpha}-u^{\alpha}]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}} \\ \times\left(\int_0^{\frac{1}{2}}\left|f'(\frac{ab}{ub+(1-u)a})\right|^q du\right)^{\frac{1}{q}} \end{array}\right. \\
&\left.\begin{array}{c} +\left(\int_{\frac{1}{2}}^1\frac{[u^{\alpha}-(1-u)^{\alpha}]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}} \\ \times\left(\int_{\frac{1}{2}}^1\left|f'(\frac{ab}{ub+(1-u)a})\right|^q du\right)^{\frac{1}{q}} \end{array}\right] \\
\leq\ & \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{c} \left(\int_0^{\frac{1}{2}}\frac{[(1-u)^{\alpha}-u^{\alpha}]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}} \\ \times\left(\int_0^{\frac{1}{2}}u\left|f'(a)\right|^q+(1-u)\left|f'(b)\right|^q du\right)^{\frac{1}{q}} \end{array}\right. \\
&\left.\begin{array}{c} +\left(\int_{\frac{1}{2}}^1\frac{[u^{\alpha}-(1-u)^{\alpha}]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}} \\ \times\left(\int_{\frac{1}{2}}^1u\left|f'(a)\right|^q+(1-u)\left|f'(b)\right|^q du\right)^{\frac{1}{q}} \end{array}\right] \\
=\ & \frac{\|g\|_{\infty}ab(b-a)}{2\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} \\
&\times\left[\left(\int_0^{\frac{1}{2}}\frac{[(1-u)^{\alpha}-u^{\alpha}]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right]^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\frac{[u^{\alpha}-(1-u)^{\alpha}]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left[\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right]^{\frac{1}{q}}\right] \qquad (2.22)
\end{aligned}$$

Calculating following integrals by Lemma 1, we have

$$\begin{aligned}
\int_0^{\frac{1}{2}}\frac{[(1-u)^{\alpha}-u^{\alpha}]^p}{(ub+(1-u)a)^{2p}}du &\leq \int_0^{\frac{1}{2}}\frac{(1-2u)^{\alpha p}}{(ub+(1-u)a)^{2p}}du \\
&= \frac{1}{2}\int_0^1\frac{(1-u)^{\alpha p}}{\left(\frac{u}{2}b+(1-\frac{u}{2})a\right)^{2p}}du \\
&= \frac{1}{2}\int_0^1 v^{\alpha p}\left(\frac{a+b}{2}\right)^{-2p}\left[1-v\left(\frac{b-a}{b+a}\right)\right]^{-2p}dv \\
&= \left(\frac{a+b}{2}\right)^{-2p}\frac{1}{2(\alpha p+1)}\ {}_2F_1\left(2p,\alpha p+1;\alpha p+2;\tfrac{b-a}{b+a}\right) \\
&= C_9(\alpha) \qquad (2.23)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_{\frac{1}{2}}^{1}\frac{[u^{\alpha}-(1-u)^{\alpha}]^{p}}{(ub+(1-u)a)^{2p}}du &\leq \int_{\frac{1}{2}}^{1}\frac{(2u-1)^{\alpha p}}{(ub+(1-u)a)^{2p}}du \\ &= \int_{0}^{\frac{1}{2}}\frac{(1-2u)^{\alpha p}}{(ua+(1-u)b)^{2p}}du \\ &= \frac{1}{2}\int_{0}^{1}\frac{(1-v)^{\alpha p}}{\left(\frac{v}{2}a+(1-\frac{v}{2})b\right)^{2p}}dv \\ &= \frac{1}{2}\int_{0}^{1}(1-v)^{\alpha p}\,b^{-2p}\left(1-\frac{v}{2}\left(1-\frac{a}{b}\right)\right)^{-2p}dv \\ &= b^{-2p}\frac{1}{2(\alpha p+1)}\ {}_2F_1\left(2p,1;\alpha p+2;\tfrac{1}{2}\left(1-\tfrac{a}{b}\right)\right) \\ &= C_{10}(\alpha) \qquad (2.24)\end{aligned}$$

If we use (2.23) and (2.24) in (2.22), we have (2.21).This completes the proof. □

Corollary 3. *In Theorem 8;*

(1) If we take $\alpha=1$ we have the following Hermite-Hadamard-Fejer inequality for harmonically convex functions which is related the right-hand side of (1.6)*:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{f(x)g(x)}{x^2}dx\right| \\ &\leq \frac{\|g\|_{\infty}(b-a)^2}{2}\left[\begin{array}{c}C_9^{\frac{1}{p}}(1)\left[\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right]^{\frac{1}{q}} \\ +C_{10}^{\frac{1}{p}}(1)\left[\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right]^{\frac{1}{q}}\end{array}\right]\end{aligned}$$

(2) If we take $g(x)=1$ we have following Hermite-Hadamard type inequality for harmonically convex function in fractional integral forms which is related the right-hand side of (1.5)*:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c}J_{1/a-}^{\alpha}(f\circ h)(1/b) \\ +J_{1/b+}^{\alpha}(f\circ h)(1/a)\end{array}\right\}\right| \\ &\leq \frac{ab(b-a)}{2}\left[\begin{array}{c}C_9^{\frac{1}{p}}(\alpha)\left[\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right]^{\frac{1}{q}} \\ +C_{10}^{\frac{1}{p}}(\alpha)\left[\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right]^{\frac{1}{q}}\end{array}\right]\end{aligned}$$

(3) If we take $\alpha=1$ and $g(x)=1$ we have the following Hermite-Hadamard type inequality for harmonically convex function which is related the right-hand side of (1.4)*:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\ &\leq \frac{ab(b-a)}{2}\left[\begin{array}{c}C_9^{\frac{1}{p}}(1)\left[\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right]^{\frac{1}{q}} \\ +C_{10}^{\frac{1}{p}}(1)\left[\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right]^{\frac{1}{q}}\end{array}\right]\end{aligned}$$

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