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ON NEW INEQUALITIES OF HERMITE-HADAMARD-FEJER TYPE FOR GA-CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, firstly new Hermite-Hadamard type inequality for GA-convex function in fractional integral forms is given. Secondly, new Hermite-Hadamard-Fejer inequality for GA-convex function in fractional integral forms is established. Finally, an integral identity and some Hermite-Hadamard-Fejer type integral inequalities for GA-convex functions in fractional integral forms are obtained.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality [2].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In [1], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative,integrable and symmetric to $(a + b)/2$.

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2) see [3, 11, 12, 13].

Definition 1. [9, 10]. *A function $f : I \subseteq (0,\infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if*

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

In [8], Latif et al. established the following inequality which is the weighted generalization of Hermite-Hadamard inequality for GA-convex functions as follows:

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Theorem 2. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a GA-convex function and $a, b \in I$ with $a < b$. Let g:$[a,b] \to [0,\infty)$ be continuous positive mapping and geometrically symmetric to $\sqrt{ab}$. Then*

$$f\left(\sqrt{ab}\right)\int_a^b \frac{g(x)}{x}dx \le \int_a^b \frac{f(x)g(x)}{x}dx \le \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x}dx. \quad (1.3)$$

We will now give definitions of the right-hand side and left-hand side Hadamard fractional integrals which are used throughout this paper.

Definition 2. [7]. *Let $f \in L[a,b]$. The right-hand side and left-hand side Hadamard fractional integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $b > a \ge 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t},\ x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t},\ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [4, 5, 6, 14, 15].

In this paper, we gave new Hermite-Hadamard type inequality for GA-convex function in fractional integral forms. We established new Hermite-Hadamard-Fejer inequality for GA-convex function in fractional integral forms. We obtained an integral identity and some Hermite-Hadamard-Fejer type integral inequalities for GA-convex functions in fractional integral forms.

2. Main results

Throughout this section, let $\|g\|_\infty = \sup_{t\in[a,b]}|g(x)|$, for the continuous function $g : [a,b] \to \mathbb{R}$.

Lemma 1. *If $g : [a,b] \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is integrable and geometrically symmetric with respect to $\sqrt{ab}$ (i.e. $g\left(\frac{ab}{x}\right) = g(x)$ holds for all $x \in [a,b]$) with $a < b$, then*

$$J_{\sqrt{ab}-}^{\alpha}g(a) = J_{\sqrt{ab}+}^{\alpha}g(b) = \frac{1}{2}\left[J_{\sqrt{ab}-}^{\alpha}g(a) + J_{\sqrt{ab}+}^{\alpha}g(b)\right]$$

with $\alpha > 0$.

Proof. Since g is geometrically symmetric with respect to $\sqrt{ab}$, we have $g(ab/x) = g(x)$, for all $x \in [a,b]$. Hence, in the following integral if we setting $x = ab/t$ and $dx = -\left(ab/t^2\right)dt$ we have

$$\begin{aligned} J_{\sqrt{ab}-}^{\alpha}g(a) &= \frac{1}{\Gamma(\alpha)}\int_a^{\sqrt{ab}} \left(\ln\frac{x}{a}\right)^{\alpha-1} g(x)\frac{dx}{x} \\ &= \frac{1}{\Gamma(\alpha)}\int_{\sqrt{ab}}^b \left(\ln\frac{b}{t}\right)^{\alpha-1} g\left(\frac{ab}{t}\right)\frac{dt}{t} \\ &= \frac{1}{\Gamma(\alpha)}\int_{\sqrt{ab}}^b \left(\ln\frac{b}{t}\right)^{\alpha-1} g(t)\frac{dt}{t} = J_{\sqrt{ab}+}^{\alpha}g(b). \end{aligned}$$

This completes the proof. □

Theorem 3. *Let $f : [a,b] \subseteq \mathbb{R}_+ \to \mathbb{R}$ be GA-convex function with $a < b$ and $f \in L[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a) + J^{\alpha}_{\sqrt{ab}+}f(b)\right] \leq \frac{f(a)+f(b)}{2}. \quad (2.1)$$

Proof. Since f is a GA-convex function on $[a,b]$, we have for all $x, y \in [a,b]$

$$f(\sqrt{xy}) \leq \frac{f(x)+f(y)}{2}.$$

Choosing $x = a^{1-t}b^t$, $y = a^t b^{1-t}$, we get

$$f\left(\sqrt{ab}\right) \leq \frac{f\left(a^{1-t}b^t\right) + f\left(a^t b^{1-t}\right)}{2}. \quad (2.2)$$

Multiplying both sides of (2.2) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $\left[0,\frac{1}{2}\right]$, we obtain

$$\begin{aligned}
f\left(\sqrt{ab}\right) &\leq \frac{\alpha}{2^{1-\alpha}}\left[\int_0^{\frac{1}{2}} t^{\alpha-1}f\left(a^{1-t}b^t\right)dt + \int_0^{\frac{1}{2}} t^{\alpha-1}f\left(a^t b^{1-t}\right)dt\right] \\
&= \frac{\alpha}{2^{1-\alpha}}\left[\int_a^{\sqrt{ab}}\left(\frac{\ln\frac{x}{a}}{\ln\frac{b}{a}}\right)^{\alpha-1} f(x)\frac{dx}{x\ln\frac{b}{a}} + \int_{\sqrt{ab}}^{b}\left(\frac{\ln\frac{b}{x}}{\ln\frac{b}{a}}\right)^{\alpha-1} f(x)\frac{dx}{x\ln\frac{b}{a}}\right] \\
&= \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[\begin{array}{c} \frac{1}{\Gamma(\alpha)}\int_a^{\sqrt{ab}}\left(\ln\frac{x}{a}\right)^{\alpha-1} f(x)\frac{dx}{x} \\ +\frac{1}{\Gamma(\alpha)}\int_{\sqrt{ab}}^{b}\left(\ln\frac{b}{x}\right)^{\alpha-1} f(x)\frac{dx}{x} \end{array}\right] \\
&= \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a) + J^{\alpha}_{\sqrt{ab}+}f(b)\right]
\end{aligned}$$

and the first inequality is proved.

For the proof of the second inequality in (2.1), we first note that if f is a convex function, then for $t \in [0,1]$, it yields

$$f\left(a^{1-t}b^t\right) \leq (1-t)f(a) + tf(b)$$

and

$$f\left(a^t b^{1-t}\right) \leq tf(a) + (1-t)f(b).$$

By adding these inequalities, we have

$$f\left(a^{1-t}b^t\right) + f\left(a^t b^{1-t}\right) \leq f(a) + f(b) \quad (2.3)$$

Then multiplying both sides of (2.3) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $\left[0,\frac{1}{2}\right]$, we obtain

$$\int_0^{\frac{1}{2}} t^{\alpha-1}f\left(a^{1-t}b^t\right)dt + \int_0^{\frac{1}{2}} t^{\alpha-1}f\left(a^t b^{1-t}\right)dt \leq [f(a)+f(b)]\int_0^{\frac{1}{2}} t^{\alpha-1}dt$$

i.e.,

$$\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a) + J^{\alpha}_{\sqrt{ab}+}f(b)\right] \leq \frac{f(a)+f(b)}{2}$$

The proof is completed. □

Theorem 4. *Let $f : [a,b] \subseteq \mathbb{R}_{+} \rightarrow \mathbb{R}$ be GA-convex function with $a < b$ and $f \in L[a,b]$. If $g : [a,b] \rightarrow \mathbb{R}$ is nonnegative,integrable and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} & f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right] \leq \left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right] \\ \leq\ & \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right] \end{aligned} \tag{2.4}$$

with $\alpha > 0$.

Proof. Since f is a GA-convex function on $[a,b]$, we have for all $t \in [0,1]$

$$f\left(\sqrt{ab}\right) = f\left(\sqrt{a^{1-t}b^{t}a^{t}b^{1-t}}\right) \leq \frac{f\left(a^{1-t}b^{t}\right)+f\left(a^{t}b^{1-t}\right)}{2} \tag{2.5}$$

Multiplying both sides of (2.5) by $2t^{\alpha-1}g\left(a^{1-t}b^{t}\right)$ then integrating the resulting inequality with respect to t over $\left[0,\frac{1}{2}\right]$, we obtain

$$\begin{aligned} & 2f\left(\sqrt{ab}\right)\int_0^{\frac{1}{2}} t^{\alpha-1}g\left(a^{1-t}b^{t}\right)dt \\ \leq\ & \int_0^{\frac{1}{2}} t^{\alpha-1}\left[f\left(a^{1-t}b^{t}\right)+f\left(a^{t}b^{1-t}\right)\right]g\left(a^{1-t}b^{t}\right)dt \\ =\ & \int_0^{\frac{1}{2}} t^{\alpha-1}f\left(a^{1-t}b^{t}\right)g\left(a^{1-t}b^{t}\right)dt + \int_0^{\frac{1}{2}} t^{\alpha-1}f\left(a^{t}b^{1-t}\right)g\left(a^{1-t}b^{t}\right)dt. \end{aligned}$$

Setting $x = a^{1-t}b^{t}$, and $dx = a^{1-t}b^{t}\ln\left(\frac{b}{a}\right)dt$ gives

$$\begin{aligned} & \frac{2}{\left(\ln\frac{b}{a}\right)^{\alpha}}f\left(\sqrt{ab}\right)\int_a^{\sqrt{ab}}\left(\ln\frac{x}{a}\right)^{\alpha-1}g(x)\frac{dx}{x} \\ \leq\ & \frac{1}{\left(\ln\frac{b}{a}\right)^{\alpha}}\left\{\int_a^{\sqrt{ab}}\left(\ln\frac{x}{a}\right)^{\alpha-1}f(x)g(x)\frac{dx}{x}+\int_a^{\sqrt{ab}}\left(\ln\frac{x}{a}\right)^{\alpha-1}f\left(\frac{ab}{x}\right)g(x)\frac{dx}{x}\right\} \\ =\ & \frac{1}{\left(\ln\frac{b}{a}\right)^{\alpha}}\left\{\int_a^{\sqrt{ab}}\left(\ln\frac{x}{a}\right)^{\alpha-1}f(x)g(x)\frac{dx}{x}+\int_{\sqrt{ab}}^{b}\left(\ln\frac{b}{u}\right)^{\alpha-1}f(u)g\left(\frac{ab}{u}\right)\frac{du}{u}\right\} \\ =\ & \frac{1}{\left(\ln\frac{b}{a}\right)^{\alpha}}\left\{\int_a^{\sqrt{ab}}\left(\ln\frac{x}{a}\right)^{\alpha-1}f(x)g(x)\frac{dx}{x}+\int_{\sqrt{ab}}^{b}\left(\ln\frac{b}{u}\right)^{\alpha-1}f(u)g(u)\frac{du}{u}\right\}. \end{aligned}$$

Therefore, by Lemma 1 we have

$$\begin{aligned} \frac{\Gamma(\alpha)}{\left(\ln\frac{b}{a}\right)^{\alpha}}f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right] \leq\ & \frac{\Gamma(\alpha)}{\left(\ln\frac{b}{a}\right)^{\alpha}} \\ & \times\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right] \end{aligned}$$

and the first inequality is proved.

For the proof of the second inequality in (2.4) we first note that if f is a GA-convex function, then, for all $t \in [0,1]$, it yields

$$f\left(a^{1-t}b^{t}\right)+f\left(a^{t}b^{1-t}\right) \leq f(a)+f(b). \tag{2.6}$$

Then multiplying both sides of (2.6) by $t^{\alpha-1}g\left(a^{1-t}b^{t}\right)$ and integrating the resulting inequality with respect to t over $\left[0,\frac{1}{2}\right]$, we obtain

$$\begin{aligned} & \int_0^{\frac{1}{2}} t^{\alpha-1}f\left(a^{1-t}b^{t}\right)g\left(a^{1-t}b^{t}\right)dt + \int_0^{\frac{1}{2}} t^{\alpha-1}f\left(a^{t}b^{1-t}\right)g\left(a^{1-t}b^{t}\right)dt \\ \leq\ & [f(a)+f(b)]\int_0^{\frac{1}{2}} t^{\alpha-1}g\left(a^{1-t}b^{t}\right)dt \end{aligned}$$

i.e.

$$\begin{aligned}\frac{\Gamma(\alpha)}{\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right] &\leq \frac{\Gamma(\alpha)}{\left(\ln\frac{b}{a}\right)^{\alpha}}\left(\frac{f(a)+f(b)}{2}\right)\\ &\quad\times\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right]\end{aligned}$$

The proof is completed. □

Remark 1. *In Theorem 3,*

(1) if we take $\alpha=1$*, then inequality* (2.4) *becomes inequality* (1.3) *of Theorem 2,*

(2) if we take $g(x)=1$*, then inequality* (2.4) *becomes inequality* (2.1) *of Theorem 3.*

Lemma 2. *Let* $f:[a,b]\subseteq\mathbb{R}_{+}\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$ *and* $f'\in L[a,b]$*. If* $g:[a,b]\to\mathbb{R}$ *is integrable and geometrically symmetric with respect to* $\sqrt{ab}$ *then the following equality for fractional integrals holds:*

$$\begin{aligned}&f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right]-\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\\ &=\frac{1}{\Gamma(\alpha)}\left[\begin{array}{c}\int_a^{\sqrt{ab}}\left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\right)f'(t)\,dt\\ +\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right)f'(t)\,dt\end{array}\right]\end{aligned}\tag{2.7}$$

with $\alpha>0$.

Proof. It suffices to note that

$$\begin{aligned}I &= \int_a^{\sqrt{ab}}\left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\right)f'(t)\,dt+\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right)f'(t)\,dt\\ &= I_1+I_2.\end{aligned}$$

By integration by parts we get

$$\begin{aligned}I_1 &= \left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\right)f(t)\Big|_a^{\sqrt{ab}}-\int_a^{\sqrt{ab}}\left(\ln\frac{t}{a}\right)^{\alpha-1}g(t)f(t)\frac{dt}{t}\\ &= \left(\int_a^{\sqrt{ab}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right)f(\sqrt{ab})-\int_a^{\sqrt{ab}}\left(\ln\frac{b}{t}\right)^{\alpha-1}(fg)(t)\frac{dt}{t}\\ &= \Gamma(\alpha)\left[f(\sqrt{ab})J^{\alpha}_{\sqrt{ab}-}g(b)-J^{\alpha}_{\sqrt{ab}-}(fg)(b)\right]\end{aligned}$$

and similarly

$$\begin{aligned}I_2 &= \left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right)f(t)\Big|_{\sqrt{ab}}^b-\int_{\sqrt{ab}}^b\left(\ln\frac{b}{t}\right)^{\alpha-1}g(t)f(t)\frac{dt}{t}\\ &= \left(\int_{\sqrt{ab}}^b\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\right)f(\sqrt{ab})-\int_a^b\left(\ln\frac{t}{a}\right)^{\alpha-1}(fg)(t)\frac{dt}{t}\\ &= \Gamma(\alpha)\left[f\left(\sqrt{ab}\right)J^{\alpha}_{\sqrt{ab}+}g(b)-J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right].\end{aligned}$$

Thus, we can write

$$I=I_1+I_2=\Gamma(\alpha)\left\{\begin{array}{c}f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right]\\ -\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\end{array}\right\}.$$

Multiplying the both sides by $(\Gamma(\alpha))^{-1}$ we obtain (2.7) which completes the proof. □

Theorem 5. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|$ is GA-convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequality for fractional integrals holds*

$$\begin{aligned} & \left| f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a) + J^{\alpha}_{\sqrt{ab}+}g(b)\right] - \left[J^{\alpha}_{\sqrt{ab}-}(fg)(a) + J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right| \\ \le\ & \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[C_1(\alpha)\left|f'(a)\right| + C_2(\alpha)\left|f'(b)\right|\right], \end{aligned}$$

where

$$C_1(\alpha) = \int_0^{\frac{1}{2}} u^\alpha \left[(1-u)\left(a^{1-u}b^u\right) + u\left(a^u b^{1-u}\right)\right] du$$

and

$$C_2(\alpha) = \int_0^{\frac{1}{2}} u^\alpha \left[u\left(a^{1-u}b^u\right) + (1-u)\left(a^u b^{1-u}\right)\right] du$$

with $\alpha > 0$.

Proof. From Lemma 2 we have

$$\begin{aligned} & \left| f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a) + J^{\alpha}_{\sqrt{ab}+}g(b)\right] - \left[J^{\alpha}_{\sqrt{ab}-}(fg)(a) + J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right| \\ \le\ & \frac{1}{\Gamma(\alpha)}\left[\begin{array}{c} \int_a^{\sqrt{ab}}\left(\int_a^t \left(\ln\frac{s}{a}\right)^{\alpha-1}|g(s)|\frac{ds}{s}\right)|f'(t)|\,dt \\ +\int_{\sqrt{ab}}^b\left(\int_t^b \left(\ln\frac{b}{s}\right)^{\alpha-1}|g(s)|\frac{ds}{s}\right)|f'(t)|\,dt \end{array}\right] \\ \le\ & \frac{\|g\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{c} \int_a^{\sqrt{ab}}\left(\int_a^t \left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)|f'(t)|\,dt \\ +\int_{\sqrt{ab}}^b\left(\int_t^b \left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)|f'(t)|\,dt \end{array}\right]. \end{aligned}$$

Setting $t = a^{1-u}b^u$ and $dt = a^{1-u}b^u \ln\left(\frac{b}{a}\right) du$ gives

$$\begin{aligned} & \left| f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a) + J^{\alpha}_{\sqrt{ab}+}g(b)\right] - \left[J^{\alpha}_{\sqrt{ab}-}(fg)(a) + J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right| \\ \le\ & \frac{\|g\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{c} \int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du \\ +\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du \end{array}\right] \\ =\ & \frac{\|g\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{c} \int_0^{\frac{1}{2}}\left(\left.\frac{\left(\ln\frac{s}{a}\right)^\alpha}{\alpha}\right|_a^{a^{1-u}b^u}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du \\ +\int_{\frac{1}{2}}^1\left(\left.\frac{-\left(\ln\frac{b}{s}\right)^\alpha}{\alpha}\right|_{a^{1-u}b^u}^b\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du \end{array}\right] \\ =\ & \frac{\|g\|_\infty\left(\ln\frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)}\left[\begin{array}{c} \int_0^{\frac{1}{2}} u^\alpha\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)du \\ +\int_{\frac{1}{2}}^1 (1-u)^\alpha\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)du \end{array}\right] \end{aligned} \tag{2.8}$$

Since $|f'|$ is GA-convex on $[a,b]$, we know that for $u \in [0,1]$

$$\left|f'(a^{1-u}b^u)\right| \le (1-u)\left|f'(a)\right| + u\left|f'(b)\right|, \tag{2.9}$$

Hence, if we use (2.9) in (2.8), we obtain

$$\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right]-\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right|$$

$$\leq \frac{\|g\|_{\infty}\left(\ln\frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)}\left[\begin{array}{c}\int_0^{\frac{1}{2}}u^{\alpha}\left[(1-u)\left|f'(a)\right|+u\left|f'(b)\right|\right]\left(a^{1-u}b^{u}\right)du\\ +\int_{\frac{1}{2}}^{1}(1-u)^{\alpha}\left[(1-u)\left|f'(a)\right|+u\left|f'(b)\right|\right]\left(a^{1-u}b^{u}\right)du\end{array}\right]$$

$$= \frac{\|g\|_{\infty}\left(\ln\frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)}\left[\begin{array}{c}\int_0^{\frac{1}{2}}u^{\alpha}\left[(1-u)\left|f'(a)\right|+u\left|f'(b)\right|\right]\left(a^{1-u}b^{u}\right)du\\ +\int_0^{\frac{1}{2}}u^{\alpha}\left[u\left|f'(a)\right|+(1-u)\left|f'(b)\right|\right]\left(a^{u}b^{1-u}\right)du\end{array}\right]$$

$$= \frac{\|g\|_{\infty}\left(\ln\frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)}\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}u^{\alpha}\left[(1-u)\left(a^{1-u}b^{u}\right)+u\left(a^{u}b^{1-u}\right)\right]du\right)\left|f'(a)\right|\\ +\left(\int_0^{\frac{1}{2}}u^{\alpha}\left[u\left(a^{1-u}b^{u}\right)+(1-u)\left(a^{u}b^{1-u}\right)\right]du\right)\left|f'(b)\right|\end{array}\right]$$

This completes the proof. □

Corollary 1. *In Theorem 5;*

(1) If we take $\alpha=1$ we have the following Hermite-Hadamard-Fejer type inequality for GA-convex function which is related the left-hand side of (1.3)*:*

$$\left|f\left(\sqrt{ab}\right)\int_a^b\frac{g(x)}{x}dx-\int_a^b\frac{f(x)g(x)}{x}dx\right|$$
$$\leq \|g\|_{\infty}\ln^2\left(\frac{b}{a}\right)\left[C_1(1)\left|f'(a)\right|+C_2(1)\left|f'(b)\right|\right],$$

(2) If we take $g(x)=1$ we have following Hermite-Hadamard type inequality for GA-convex function in fractional integral forms which is related the left-hand side of (2.1)*:*

$$\left|f\left(\sqrt{ab}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a)+J^{\alpha}_{\sqrt{ab}+}f(b)\right]\right|$$
$$\leq \frac{\ln\left(\frac{b}{a}\right)}{2^{1-\alpha}}\left[C_1(\alpha)\left|f'(a)\right|+C_2(\alpha)\left|f'(b)\right|\right],$$

(3) If we take $\alpha=1$ and $g(x)=1$ we have the following Hermite-Hadamard type inequality for GA-convex function

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln\frac{b}{a}}\int_a^b\frac{f(x)}{x}dx\right|\leq\ln\left(\frac{b}{a}\right)\left[C_1(1)\left|f'(a)\right|+C_2(1)\left|f'(b)\right|\right].$$

Theorem 6. *Let $f:I\subseteq\mathbb{R}_+\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with $a<b$. If $|f'|^q, q\geq 1$, is GA-convex on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals hold:*

$$\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right]-\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right|$$

$$\leq \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)}(\alpha+1)^{1-\frac{1}{q}}\Gamma(\alpha+1)}$$
$$\times\left[\begin{array}{c}\left[C_3(\alpha)\left|f'(a)\right|^q+C_4(\alpha)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\\ +\left[C_5(\alpha)\left|f'(a)\right|^q+C_6(\alpha)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\end{array}\right],$$

where

$$C_3(\alpha)=\int_0^{\frac{1}{2}}u^{\alpha}(1-u)\left(a^{1-u}b^{u}\right)^q du,\quad C_4(\alpha)=\int_0^{\frac{1}{2}}u^{\alpha+1}\left(a^{1-u}b^{u}\right)^q du,$$

$$C_5(\alpha)=\int_0^{\frac{1}{2}} u^{\alpha+1}\left(a^u b^{1-u}\right)^q du, \quad C_6(\alpha)=\int_0^{\frac{1}{2}} u^{\alpha}\left(1-u\right)\left(a^u b^{1-u}\right)^q du.$$

and $\alpha>0$.

Proof. Using Lemma 2, we have

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g\left(a\right)+J^{\alpha}_{\sqrt{ab}+}g\left(b\right)\right]-\left[J^{\alpha}_{\sqrt{ab}-}\left(fg\right)\left(a\right)+J^{\alpha}_{\sqrt{ab}+}\left(fg\right)\left(b\right)\right]\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left[\begin{array}{c} \int_a^{\sqrt{ab}}\left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}\left|g(s)\right|\frac{ds}{s}\right)\left|f'\left(t\right)\right|dt \\ +\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\left|g(s)\right|\frac{ds}{s}\right)\left|f'\left(t\right)\right|dt \end{array}\right] \\
\leq\ & \frac{\left\|g\right\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{c} \int_a^{\sqrt{ab}}\left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(t\right)\right|dt \\ +\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(t\right)\right|dt \end{array}\right].
\end{aligned}$$

Setting $t=a^{1-u}b^u$ and $dt=a^{1-u}b^u\ln\left(\frac{b}{a}\right)du$ gives

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g\left(a\right)+J^{\alpha}_{\sqrt{ab}+}g\left(b\right)\right]-\left[J^{\alpha}_{\sqrt{ab}-}\left(fg\right)\left(a\right)+J^{\alpha}_{\sqrt{ab}+}\left(fg\right)\left(b\right)\right]\right| \\
\leq\ & \frac{\left\|g\right\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{c} \int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du \\ +\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du \end{array}\right] \\
=\ & \frac{\left\|g\right\|_\infty\ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\begin{array}{c} \int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)du \\ +\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)du \end{array}\right].
\end{aligned}$$

Using power mean inequality we have

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g\left(a\right)+J^{\alpha}_{\sqrt{ab}+}g\left(b\right)\right]-\left[J^{\alpha}_{\sqrt{ab}-}\left(fg\right)\left(a\right)+J^{\alpha}_{\sqrt{ab}+}\left(fg\right)\left(b\right)\right]\right| \\
\leq\ & \frac{\left\|g\right\|_\infty\ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\begin{array}{c} \left[\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)du\right]^{1-\frac{1}{q}} \\ \times\left[\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}} \\ +\left[\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)du\right]^{1-\frac{1}{q}} \\ \times\left[\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}} \end{array}\right] \\
=\ & \frac{\left\|g\right\|_\infty\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)}\left(\alpha+1\right)^{1-\frac{1}{q}}\Gamma(\alpha+1)} \\
&\times\left[\begin{array}{c} \left[\int_0^{\frac{1}{2}}u^\alpha\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}} \\ +\left[\int_{\frac{1}{2}}^1\left(1-u\right)^\alpha\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}} \end{array}\right]. \qquad (2.10)
\end{aligned}$$

Since $|f'|^q$ is GA-convex on $[a,b]$, we know that for $u\in[0,1]$

$$\left|f'(a^{1-u}b^u)\right|^q\leq\left(1-u\right)\left|f'\left(a\right)\right|^q+u\left|f'\left(b\right)\right|^q. \qquad (2.11)$$

Hence, if we use (2.11) in (2.10), we obtain

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g\left(a\right)+J^{\alpha}_{\sqrt{ab}+}g\left(b\right)\right]-\left[J^{\alpha}_{\sqrt{ab}-}\left(fg\right)\left(a\right)+J^{\alpha}_{\sqrt{ab}+}\left(fg\right)\left(b\right)\right]\right| \\
\leq\ & \frac{\left\|g\right\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)}\left(\alpha+1\right)^{1-\frac{1}{q}}\Gamma(\alpha+1)} \\
&\times\left[\begin{array}{c}\left[\int_0^{\frac{1}{2}}u^{\alpha}\left[(1-u)\left|f'\left(a\right)\right|^q+u\left|f'\left(b\right)\right|^q\right]\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}} \\ +\left[\int_{\frac{1}{2}}^1\left(1-u\right)^{\alpha}\left[(1-u)\left|f'\left(a\right)\right|^q+u\left|f'\left(b\right)\right|^q\right]\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}}\end{array}\right] \\
=\ & \frac{\left\|g\right\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)}\left(\alpha+1\right)^{1-\frac{1}{q}}\Gamma(\alpha+1)} \\
&\times\left[\begin{array}{c}\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}u^{\alpha}\left(1-u\right)\left(a^{1-u}b^u\right)^q du\right)\left|f'\left(a\right)\right|^q \\ +\left(\int_0^{\frac{1}{2}}u^{\alpha+1}\left(a^{1-u}b^u\right)^q du\right)\left|f'\left(b\right)\right|^q\end{array}\right]^{\frac{1}{q}} \\ +\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}u^{\alpha+1}\left(a^{u}b^{1-u}\right)^q du\right)\left|f'\left(a\right)\right|^q \\ \left(\int_0^{\frac{1}{2}}u^{\alpha}\left(1-u\right)\left(a^{u}b^{1-u}\right)^q du\right)\left|f'\left(b\right)\right|^q\end{array}\right]^{\frac{1}{q}}\end{array}\right]
\end{aligned}$$

This completes the proof. □

Corollary 2. *In Theorem 6;*

(1) If we take $\alpha=1$ we have the following Hermite-Hadamard-Fejer type inequality for GA-convex function which is related the left-hand side of (1.3)*:*

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)\int_a^b\frac{g\left(x\right)}{x}dx-\int_a^b\frac{f\left(x\right)g\left(x\right)}{x}dx\right| \\
\leq\ & \frac{\left\|g\right\|_{\infty}\ln^2\left(\frac{b}{a}\right)}{2^{3\left(1-\frac{1}{q}\right)}}\left[\begin{array}{c}\left[C_3(1)\left|f'\left(a\right)\right|^q+C_4(1)\left|f'\left(b\right)\right|^q\right]^{\frac{1}{q}} \\ +\left[C_5(1)\left|f'\left(a\right)\right|^q+C_6(1)\left|f'\left(b\right)\right|^q\right]^{\frac{1}{q}}\end{array}\right],
\end{aligned}$$

(2) If we take $g\left(x\right)=1$ we have following Hermite-Hadamard type inequality for GA-convex function in fractional integral forms which is related the left-hand side of (2.1)*:*

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)-\frac{\Gamma\left(\alpha+1\right)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f\left(a\right)+J^{\alpha}_{\sqrt{ab}+}f\left(b\right)\right]\right| \\
\leq\ & \frac{\ln\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)+(1-\alpha)}\left(\alpha+1\right)^{1-\frac{1}{q}}} \\
&\times\left[\begin{array}{c}\left[C_3(\alpha)\left|f'\left(a\right)\right|^q+C_4(\alpha)\left|f'\left(b\right)\right|^q\right]^{\frac{1}{q}} \\ +\left[C_5(\alpha)\left|f'\left(a\right)\right|^q+C_6(\alpha)\left|f'\left(b\right)\right|^q\right]^{\frac{1}{q}}\end{array}\right],
\end{aligned}$$

(3) If we take $\alpha=1$ and $g\left(x\right)=1$ we have the following Hermite-Hadamard type inequality for GA-convex function

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln\frac{b}{a}}\int_a^b\frac{f\left(x\right)}{x}dx\right| \\
\leq\ & \frac{\ln\left(\frac{b}{a}\right)}{2^{3\left(1-\frac{1}{q}\right)}}\left[\begin{array}{c}\left[C_3(1)\left|f'\left(a\right)\right|^q+C_4(1)\left|f'\left(b\right)\right|^q\right]^{\frac{1}{q}} \\ +\left[C_5(1)\left|f'\left(a\right)\right|^q+C_6(1)\left|f'\left(b\right)\right|^q\right]^{\frac{1}{q}}\end{array}\right].
\end{aligned}$$

Theorem 7. *Let $f:I\subseteq\mathbb{R}_+\rightarrow\mathbb{R}$ be a differentiable mapping on I° and $f'\in L\left[a,b\right]$ with $a<b$. If $\left|f'\right|^q,q>1,$ is GA-convex on $\left[a,b\right]$ and $g:\left[a,b\right]\rightarrow\mathbb{R}$ is continuous* *and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} & \left| f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right]-\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right| \\ \leq\ & \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{\alpha}}(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left[\begin{array}{c}\left(C_7\left|f'(a)\right|^q+C_8\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(C_9\left|f'(a)\right|^q+C_{10}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\end{array}\right], \end{aligned}$$

where

$$C_7=\int_0^{\frac{1}{2}}(1-u)\left(a^{1-u}b^u\right)^q du,\quad C_8=\int_0^{\frac{1}{2}}u\left(a^{1-u}b^u\right)^q du,$$

$$C_9=\int_{\frac{1}{2}}^{1}(1-u)\left(a^{1-u}b^u\right)^q du,\quad C_{10}=\int_{\frac{1}{2}}^{1}u\left(a^{1-u}b^u\right)^q du,$$

with $\alpha>0$ and $1/p+1/q=1$.

Proof. Using Lemma 2, setting $t=a^{1-u}b^u$ and $dt=a^{1-u}b^u\ln\left(\frac{b}{a}\right)du$, Hölder's inequality and (2.11) we have

$$\begin{aligned} & \left| f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right]-\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right| \\ \leq\ & \frac{1}{\Gamma(\alpha)}\left[\begin{array}{c}\int_a^{\sqrt{ab}}\left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}|g(s)|\frac{ds}{s}\right)|f'(t)|dt \\ +\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}|g(s)|\frac{ds}{s}\right)|f'(t)|dt\end{array}\right] \\ \leq\ & \frac{\|g\|_{\infty}}{\Gamma(\alpha)}\left[\begin{array}{c}\int_a^{\sqrt{ab}}\left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)|f'(t)|dt \\ +\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)|f'(t)|dt\end{array}\right] \\ =\ & \frac{\|g\|_{\infty}}{\Gamma(\alpha)}\left[\begin{array}{c}\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du \\ +\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du\end{array}\right] \\ =\ & \frac{\|g\|_{\infty}\ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\begin{array}{c}\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)du \\ +\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)du\end{array}\right] \\ \leq\ & \frac{\|g\|_{\infty}\ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)^p du\right)^{\frac{1}{p}} \\ \times\left(\int_0^{\frac{1}{2}}\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)^p du\right)^{\frac{1}{p}} \\ \times\left(\int_{\frac{1}{2}}^1\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}}\end{array}\right] \\ =\ & \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}}(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)} \\ & \times\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{1}{2}}^1\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}}\end{array}\right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}}(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)} \\
&\times\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left[(1-u)\left|f'(a)\right|^q+u\left|f'(b)\right|^q\right]\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{1}{2}}^1\left[(1-u)\left|f'(a)\right|^q+u\left|f'(b)\right|^q\right]\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}}\end{array}\right] \\
&= \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}}(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)} \\
&\times\left[\begin{array}{c}\left(\left[\int_0^{\frac{1}{2}}(1-u)\left(a^{1-u}b^u\right)^q du\right]\left|f'(a)\right|^q+\left[\int_0^{\frac{1}{2}}u\left(a^{1-u}b^u\right)^q du\right]\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(\left[\int_{\frac{1}{2}}^1(1-u)\left(a^{1-u}b^u\right)^q du\right]\left|f'(a)\right|^q+\left[\int_{\frac{1}{2}}^1 u\left(a^{1-u}b^u\right)^q du\right]\left|f'(b)\right|^q\right)^{\frac{1}{q}}\end{array}\right]
\end{aligned}$$

This completes the proof. □

Corollary 3. *In Theorem 7;*

(1) If we take $\alpha=1$ we have the following Hermite-Hadamard-Fejer type inequality for GA-convex function which is related the left-hand side of (1.3)*:*

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)\int_a^b\frac{g(x)}{x}dx-\int_a^b\frac{f(x)g(x)}{x}dx\right| \\
&\leq \frac{\|g\|_{\infty}\ln^2\left(\frac{b}{a}\right)}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[\begin{array}{c}\left(C_7\left|f'(a)\right|^q+C_8\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(C_9\left|f'(a)\right|^q+C_{10}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\end{array}\right],
\end{aligned}$$

(2) If we take $g(x)=1$ we have following Hermite-Hadamard type inequality for GA-convex function in fractional integral forms which is related the left-hand side of (2.1)*:*

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a)+J^{\alpha}_{\sqrt{ab}+}f(b)\right]\right| \\
&\leq \frac{\ln\left(\frac{b}{a}\right)}{2^{\left(\frac{\alpha p+1}{p}\right)+1-\alpha}(\alpha p+1)^{\frac{1}{p}}}\left[\begin{array}{c}\left(C_7\left|f'(a)\right|^q+C_8\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(C_9\left|f'(a)\right|^q+C_{10}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\end{array}\right],
\end{aligned}$$

(3) If we take $\alpha=1$ and $g(x)=1$ we have the following Hermite-Hadamard type inequality for GA-convex function

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln\frac{b}{a}}\int_a^b\frac{f(x)}{x}dx\right| \\
&\leq \frac{\ln\left(\frac{b}{a}\right)}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[\begin{array}{c}\left(C_7\left|f'(a)\right|^q+C_8\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(C_9\left|f'(a)\right|^q+C_{10}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\end{array}\right].
\end{aligned}$$

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