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# INEQUALITIES OF FEJÉR TYPE FOR $\eta$-CONVEX FUNCTIONS ON LINEAR SPACES

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Abstract. Some inequalities of Fejér type for $\eta$-convex functions defined on convex subsets in real or complex linear spaces are given. Applications for Hermite-Hadamard, trapezoid and mid-point type inequalities are provided as well.

## 1. Introduction

Let $I$ be an interval in real line $\mathbb{R}$. Consider $\eta : A \times A \to B$ for appropriate $A, B \subseteq \mathbb{R}$.

**Definition 1** ([38]). *A function $f : I \to \mathbb{R}$ is called convex with respect to $\eta$ (briefly $\eta$-convex), if*

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y)),$$

*for all $x, y \in I$ and $t \in [0, 1]$.*

In the above definition if we set $\eta(x, y) = x - y$, then we recapture the classic definition of a convex function.

The following characterization of $\eta$-convexity holds [38]:

**Theorem 1.** *A function $f : I \to \mathbb{R}$ is $\eta$-convex if and only if for any $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$,*

$$\det \begin{pmatrix} 1 & x_1 & \eta(f(x_1), f(x_3)) \\ 1 & x_2 & f(x_2) - f(x_3) \\ 1 & x_3 & 0 \end{pmatrix} \geq 0 \tag{1.1}$$

*and*

$$f(x_1) \leq f(x_3) + \eta(f(x_1), f(x_3)). \tag{1.2}$$

The following result is of importance [38]:

**Theorem 2.** *Suppose that $f : I \to \mathbb{R}$ is a $\eta$-convex function and $\eta$ is bounded from above on $f(I) \times f(I)$. Then $f$ satisfies a Lipschitz condition on any closed interval $[a, b]$ contained in the interior $I^\circ$ of $I$. Hence, $f$ is absolutely continuous on $[a, b]$ and continuous on $I^\circ$.*

We have the following Hermite-Hadamard type inequality [38]:

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)(b-a) - \frac{1}{2}\int_a^b \eta\left(f(a+b-x), f(x)\right) dx \\ &\leq \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} + \frac{\eta(f(a), f(b)) + \eta(f(b), f(a))}{4}, \end{aligned} \tag{1.3}$$

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provided that $f:[a,b]\to\mathbb{R}$ is a $\eta$-convex function, $\eta$ is bounded from above on $f([a,b])\times f([a,b])$.

A function $g:[a,b]\to\mathbb{R}$ is said to be symmetric with respect to $\frac{a+b}{2}$ on $[a,b]$ if

$$g(x)=g(a+b-x),\text{ for any } a\le x\le b.$$

We also have the following Fejér type inequalities [38]:

**Theorem 3.** *Suppose that $f:[a,b]\to\mathbb{R}$ is a $\eta$-convex function such that $\eta$ is bounded from above on $f([a,b])\times f([a,b])$ and $g:[a,b]\to\mathbb{R}^{+}$ is integrable and symmetric with respect to $\frac{a+b}{2}$. Then*

$$\int_a^b f(x)g(x)dx\le\frac{f(a)+f(b)}{2}\int_a^b g(x)dx+\frac{\eta(f(a),f(b))+\eta(f(b),f(a))}{2(b-a)}\int_a^b(b-x)g(x)dx \tag{1.4}$$

*and*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx-\frac{1}{2}\int_a^b\eta\left(f(a+b-x),f(x)\right)g(x)dx\le\int_a^b f(x)g(x)dx. \tag{1.5}$$

For other results see [38].

For various Hermite-Hadamard type inequalities, see [1]-[37] and [39]-[54].

Motivated by the above results, we obtain in this paper some inequalities of Fejér type for $\eta$-convex functions defined on convex subsets in real or complex linear spaces. Applications for Hermite-Hadamard, trapezoid and mid-point type inequalities are provided as well.

## 2. Some Preliminary Facts

Let $C$ be a convex set in the linear space $X$. Consider a function $\eta:A\times A\to B$ for appropriate $A,B\subseteq\mathbb{R}$. We can introduce the following concept for functions defined on a convex subset $C$ in a linear space $X$:

**Definition 2.** *A function $f:C\to\mathbb{R}$ is called $\eta$-convex if*

$$f(tx+(1-t)y)\le f(y)+t\eta(f(x),f(y)), \tag{2.1}$$

*for all $x,y\in C$ and $t\in[0,1]$.*

We observe that by taking $x=y$ in (2.1) we get $t\eta(f(x),f(x))\ge0$ for any $x\in C$ and $t\in[0,1]$, which implies that

$$\eta(f(x),f(x))\ge0 \tag{2.2}$$

for any $x\in C$. Also, if we take $t=1$ in (2.1) we get

$$f(x)-f(y)\le\eta(f(x),f(y)) \tag{2.3}$$

for any $x,y\in C$. The second condition obviously implies the first.

So, if we want to define $\eta$-convex functions $f$ on a convex subset $C$ of a linear space with images in an interval $I$ of real numbers, we should assume that

$$\eta(a,b)\ge a-b\text{ for any } a,b\in I. \tag{2.4}$$

If in Definition 2 we take $\eta(a,b) = a - b$ then we recover the concept of the usual convexity.

We observe that if $f : C \to I$ is a convex function and $\eta : I \times I \to \mathbb{R}$ is an arbitrary bifunction that satisfies the condition (2.4) on $I$, then for any $x, y \in C$ and $t \in [0,1]$ we have

$$f(tx + (1-t)y) \leq f(y) + t\,[f(x) - f(y)] \leq f(y) + t\eta(f(x), f(y)),$$

showing that $f$ is $\eta$-convex.

There exists $\eta$-convex functions for some bifunctions $\eta$ that are not convex. Following [38] we have the following simple examples:

**Example 1.** *a. Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} -x, & x \geq 0; \\ x, & x < 0. \end{cases}$$

*and define a bifunction $\eta$ as $\eta(x,y) = -x - y$, for all $x, y \in \mathbb{R}^- = (-\infty, 0]$.*

*It is not hard to check that $f$ is a $\eta$-convex function but not a convex one.*

*b. Define the function $f : \mathbb{R} \to \mathbb{R}^+$ by*

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ 1, & x > 1. \end{cases}$$

*and define*

$$\eta(x,y) = \begin{cases} x + y, & x \leq y; \\ 2(x+y), & x > y. \end{cases}$$

*for all $x, y \in \mathbb{R}^+ = [0, +\infty)$. Then $f$ is $\eta$-convex but is not convex.*

For $x, y \in C$ define $g_{x,y} : [0,1] \to \mathbb{R}$ by $g_{x,y}(t) = f(tx + (1-t)y)$.

**Lemma 1.** *The function $f : C \to \mathbb{R}$ is $\eta$-convex on $C$ if and only if the function $g_{x,y}$ is $\eta$-convex on $[0,1]$ for any $x, y \in C$.*

*Proof.* Assume that $x, y \in C$ with $x \neq y$ and let $t_1, t_2 \in [0,1]$ and $\lambda \in [0,1]$. Then

$$\begin{aligned} &g_{x,y}(\lambda t_1 + (1-\lambda) t_2) \\ &= f((\lambda t_1 + (1-\lambda) t_2) x + (1 - \lambda t_1 - (1-\lambda) t_2) y) \\ &= f((\lambda t_1 + (1-\lambda) t_2) x + (\lambda + 1 - \lambda - \lambda t_1 - (1-\lambda) t_2) y) \\ &= f[\lambda (t_1 x + (1-t_1) y) + (1-\lambda)(t_2 x + (1-t_2) y)] \end{aligned} \tag{2.5}$$

By the $\eta$-convexity of $f$ we have

$$\begin{aligned} &f[\lambda (t_1 x + (1-t_1) y) + (1-\lambda)(t_2 x + (1-t_2) y)] \\ &\leq f(t_2 x + (1-t_2) y) + \lambda \eta (f(t_1 x + (1-t_1) y), f(t_2 x + (1-t_2) y)) \\ &= g_{x,y}(t_2) + \lambda \eta (g_{x,y}(t_1), g_{x,y}(t_2)). \end{aligned} \tag{2.6}$$

Then by (2.5) and (2.6) we get

$$g_{x,y}(\lambda t_1 + (1-\lambda) t_2) \leq g_{x,y}(t_2) + \lambda \eta (g_{x,y}(t_1), g_{x,y}(t_2)) \tag{2.7}$$

that shows that $g_{x,y}$ is $\eta$-convex on $[0,1]$.

In the case when $x = y$ the inequality (2.7) is trivially satisfied.

Now let $x, y \in C$ with $x \neq y$ and assume that $g_{x,y}$ is $\eta$-convex on $[0,1]$. Then by (2.7) for $\lambda = t$, $t_1 = 1$ and $t_2 = 0$, we have

$$\begin{aligned} f(tx+(1-t)y) &= g_{x,y}(t) = g_{x,y}(t \cdot 1 + (1-t)\cdot 0) \\ &\leq g_{x,y}(0) + t\eta\left(g_{x,y}(1), g_{x,y}(0)\right) = f(y) + t\eta\left(f(x), f(y)\right). \end{aligned}$$

Since this inequality also holds when $x = y$, we conclude that $f$ is $\eta$-convex on $C$. □

**Remark 1.** *If the function $f : C \to \mathbb{R}$ is $\eta$-convex on $C$ and $\eta$ is bounded from above on $f(C) \times f(C)$, then for any $x, y \in C$ with $x \neq y$ the function $g_{x,y}$ is Lipschitzian on $[0,1]$. Hence, $g_{x,y}$ is absolutely continuous on $[0,1]$.*

We have the following lemma:

**Lemma 2.** *Let $f : C \to \mathbb{R}$ be a $\eta$-convex function on $C$. Then for any $x, y \in C$ and $t \in [0,1]$ we have the inequalities*

$$\begin{aligned} &\frac{1}{2}\left[f(tx+(1-t)y) + f((1-t)x+ty)\right] \\ &\leq \min\left\{f(y) + \frac{1}{2}\eta(f(x), f(y)), f(x) + \frac{1}{2}\eta(f(y), f(x))\right\} \\ &\leq \frac{1}{2}\left[f(x)+f(y)\right] + \frac{1}{4}\left[\eta(f(x), f(y)) + \eta(f(y), f(x))\right], \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\frac{1}{2}\left[f(tx+(1-t)y) + f((1-t)x+ty)\right] \\ &\leq \frac{1}{2}\left[f(x)+f(y)\right] + \frac{1}{2}\left[\eta(f(x), f(y)) + \eta(f(y), f(x))\right] t \end{aligned} \tag{2.9}$$

*and*

$$\begin{aligned} &f(tx+(1-t)y) \\ &\leq \frac{1}{2}\left[f(x)+f(y)\right] + \frac{1}{2}\left[t\eta(f(x), f(y)) + (1-t)\eta(f(y), f(x))\right]. \end{aligned} \tag{2.10}$$

*Proof.* If we replace in (2.1) $t$ with $1-t$, then we get

$$f((1-t)x+ty) \leq f(y) + (1-t)\eta(f(x), f(y)) \tag{2.11}$$

for all $x, y \in C$ and $t \in [0,1]$.

If we add (2.1) with (2.11) and divide by 2 we get

$$\frac{1}{2}\left[f(tx+(1-t)y) + f((1-t)x+ty)\right] \leq f(y) + \frac{1}{2}\eta(f(x), f(y))$$

for all $x, y \in C$ and $t \in [0,1]$.

If in this inequality we replace $x$ with $y$ we also have

$$\frac{1}{2}\left[f(tx+(1-t)y) + f((1-t)x+ty)\right] \leq f(x) + \frac{1}{2}\eta(f(y), f(x)) \tag{2.12}$$

for all $x, y \in C$ and $t \in [0,1]$.

Making use of (2.11) and (2.12) we get (2.8).

Now, if we replace $x$ with $y$ in (2.1), then we get

$$f(ty+(1-t)x) \leq f(x) + t\eta(f(y), f(x)), \tag{2.13}$$

for all $x, y \in C$ and $t \in [0,1]$.

If we add (2.1) with (2.13) we get

$$f(tx+(1-t)y)+f(ty+(1-t)x)$$
$$\leq f(y)+f(x)+t\left[\eta(f(x),f(y))+\eta(f(y),f(x))\right]$$

which is equivalent to (2.9).

Finally, if we change $x$ with $y$ in (2.11) we get

$$f((1-t)\,y+tx)\leq f(x)+(1-t)\,\eta(f(y),f(x)) \tag{2.14}$$

for all $x,y\in C$ and $t\in[0,1]$.

If we add (2.1) with (2.14) we get

$$2f(tx+(1-t)y)\leq f(y)+f(x)+t\eta(f(x),f(y))+(1-t)\,\eta(f(y),f(x))$$

for all $x,y\in C$ and $t\in[0,1]$ and the inequality (2.10) is proved. □

**Corollary 1.** *Let $f:C\to\mathbb{R}$ be a $\eta$-convex function on $C$. Then for any $x,y\in C$ we have*

$$f\left(\frac{x+y}{2}\right)\leq\min\left\{f\left(y\right)+\frac{1}{2}\eta(f(x),f(y)),f\left(x\right)+\frac{1}{2}\eta(f(y),f(x))\right\} \tag{2.15}$$
$$\leq\frac{1}{2}\left[f\left(x\right)+f\left(y\right)\right]+\frac{1}{4}\left[\eta(f(x),f(y))+\eta(f(y),f(x))\right].$$

If follows from Lemma 2 by taking $t=\frac{1}{2}$.

**Corollary 2.** *Let $f:C\to\mathbb{R}$ be a $\eta$-convex function on $C$. Then for any $a,b\in C$ and $t\in[0,1]$ we have*

$$f\left(\frac{a+b}{2}\right) \tag{2.16}$$
$$\leq\min\left\{f\left(ta+(1-t)\,b\right)+\frac{1}{2}\eta(f((1-t)\,a+tb),f(ta+(1-t)\,b)),\right.$$
$$\left.f\left((1-t)\,a+tb\right)+\frac{1}{2}\eta(f(ta+(1-t)\,b),f((1-t)\,a+tb))\right\}$$
$$\leq\frac{1}{2}\left[f\left((1-t)\,a+tb\right)+f\left(ta+(1-t)\,b\right)\right]$$
$$+\frac{1}{4}\eta(f((1-t)\,a+tb),f(ta+(1-t)\,b))$$
$$+\frac{1}{4}\eta(f(ta+(1-t)\,b),f((1-t)\,a+tb)).$$

It follows by Corollary 2 by taking $x=ta+(1-t)\,b$ and $y=(1-t)\,a+tb$, $t\in[0,1]$.

## 3. Fejér Type Inequalities

In what follows we assume that $w:C\to[0,\infty)$ is such that for any $x,y\in C$, $x\neq y$ the function $w_{x,y}:[0,1]\to[0,\infty)$, $w_{x,y}\left(t\right)=w\left((1-t)\,x+ty\right)$ is Lebesgue integrable on $[0,1]$. We call this type of functions *Lebesgue integrable on $C$* for simplicity.

We say that $w$ is *symmetric on the segment* $[x,y]:=\{(1-s)\,x+sy,s\in[0,1]\}$ if

$$w_{x,y}\left(t\right)=w\left((1-t)\,x+ty\right)=w\left(tx+(1-t)\,y\right)=w_{x,y}\left(1-t\right)=w_{y,x}\left(t\right) \tag{3.1}$$

for any $t \in [0,1]$.

If $C = I$ an interval in $\mathbb{R}$ and $[a,b] \subset I$ then $w : I \to [0,\infty)$ is *symmetric on* $[a,b]$ iff $w(a+b-x) = w(x)$ for any $x \in [a,b]$.

**Theorem 4.** *Let $f : C \to \mathbb{R}$ be a $\eta$-convex function on $C$ with $\eta$ bounded from above on $f(C) \times f(C)$. If $w : C \to [0,\infty)$ is Lebesgue integrable on $C$, then for any $x, y \in C$, $x \neq y$ we have the weighted inequalities*

$$\begin{aligned}
&\frac{1}{2}\int_0^1 [f(tx+(1-t)y) + f((1-t)x+ty)]\, w((1-t)x+ty)\,dt \\
&\le \min\left\{ f(y) + \frac{1}{2}\eta(f(x),f(y)), f(x) + \frac{1}{2}\eta(f(y),f(x)) \right\} \\
&\times \int_0^1 w((1-t)x+ty)\,dt \\
&\le \frac{1}{2}[f(x)+f(y)] \int_0^1 w((1-t)x+ty)\,dt \\
&+ \frac{1}{4}[\eta(f(x),f(y)) + \eta(f(y),f(x))] \int_0^1 w((1-t)x+ty)\,dt,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\frac{1}{2}\int_0^1 [f(tx+(1-t)y) + f((1-t)x+ty)]\, w((1-t)x+ty)\,dt \\
&\le \frac{1}{2}[f(x)+f(y)] \int_0^1 w((1-t)x+ty)\,dt \\
&+ \frac{1}{2}[\eta(f(x),f(y)) + \eta(f(y),f(x))] \int_0^1 t w((1-t)x+ty)\,dt
\end{aligned} \tag{3.3}$$

*and*

$$\begin{aligned}
&\int_0^1 f(tx+(1-t)y) w((1-t)x+ty)\,dt \\
&\le \frac{1}{2}[f(x)+f(y)] \int_0^1 w((1-t)x+ty)\,dt \\
&+ \frac{1}{2}\eta(f(x),f(y)) \int_0^1 t w((1-t)x+ty)\,dt \\
&+ \frac{1}{2}\eta(f(y),f(x)) \int_0^1 (1-t) w((1-t)x+ty)\,dt.
\end{aligned} \tag{3.4}$$

*Proof.* The involved integrals above exist since $\eta$ is bounded from above on $f(C) \times f(C)$ and therefore on any interval $[x,y] \subset C$. Also, according to Lemma $g_{x,y} : [0,1] \to \mathbb{R}$, $g_{x,y}(t) = f(tx+(1-t)y)$ is $\eta$-convex on any interval $[x,y] \subset C$ and by Theorem 2 it is integrable on $[0,1]$ for any $x, y \in C$, $x \neq y$.

The inequalities (3.2)-(3.4) follow by Lemma 2 on multiplying the inequalities with $w((1-t)x+ty) \ge 0$, $t \in [0,1]$ and then integrating over $t$ on $[0,1]$. □

**Corollary 3.** *With the assumptions of Theorem 4 and if for some $x, y \in C$, $x \neq y$ the function $w$ is symmetric on the segment $[x, y]$, then we have the inequalities*

$$\begin{aligned} &\int_0^1 [f((1-t)x+ty)]\, w((1-t)x+ty)\, dt \\ &\le \min\left\{ f(y)+\frac{1}{2}\eta(f(x),f(y)), f(x)+\frac{1}{2}\eta(f(y),f(x)) \right\} \\ &\times \int_0^1 w((1-t)x+ty)\, dt \\ &\le \frac{1}{2}[f(x)+f(y)] \int_0^1 w((1-t)x+ty)\, dt \\ &+ \frac{1}{4}[\eta(f(x),f(y))+\eta(f(y),f(x))] \int_0^1 w((1-t)x+ty)\, dt \end{aligned} \tag{3.5}$$

*and*

$$\begin{aligned} &\int_0^1 [f((1-t)x+ty)]\, w((1-t)x+ty)\, dt \\ &\le \frac{1}{2}[f(x)+f(y)] \int_0^1 w((1-t)x+ty)\, dt \\ &+ \frac{1}{2}[\eta(f(x),f(y))+\eta(f(y),f(x))] \int_0^1 tw((1-t)x+ty)\, dt. \end{aligned} \tag{3.6}$$

*Proof.* By the symmetry of $w$ on the segment $[x, y]$ we have

$$\begin{aligned} &\frac{1}{2}\int_0^1 [f(tx+(1-t)y)+f((1-t)x+ty)]\, w((1-t)x+ty)\, dt \\ &= \frac{1}{2}\int_0^1 f(tx+(1-t)y) w((1-t)x+ty)\, dt \\ &+ \frac{1}{2}\int_0^1 f((1-t)x+ty) w((1-t)x+ty)\, dt \\ &= \frac{1}{2}\int_0^1 f(tx+(1-t)y) w(tx+(1-t)y)\, dt \\ &+ \frac{1}{2}\int_0^1 f((1-t)x+ty) w((1-t)x+ty)\, dt \\ &:= K \end{aligned} \tag{3.7}$$

Changing the variable $s = 1-t$, $t \in [0, 1]$, we have

$$\begin{aligned} &\int_0^1 f(tx+(1-t)y) w(tx+(1-t)y)\, dt \\ &= \int_0^1 f((1-s)x+sy) w((1-s)x+sy)\, ds \end{aligned}$$

and by (3.7) we get

$$K = \int_0^1 f((1-t)x+ty) w((1-t)x+ty)\, dt$$

and by (2.9) and (2.10) we get (3.5) and (3.6).

We remark that, by the symmetry of $w$ on the segment $[x,y]$ we have

$$\int_0^1 (1-t)\, w\left((1-t)\,x+ty\right) dt = \int_0^1 (1-t)\, w\left(tx+(1-t)\,y\right) dt.$$

Changing the variable $s=1-t$, $t\in[0,1]$, we have

$$\int_0^1 (1-t)\, w\left(tx+(1-t)\,y\right) dt = \int_0^1 sw\left((1-s)\,x+sy\right) ds$$

and by (3.4) we also get (3.6). □

**Remark 2.** *If $f:[a,b]\to\mathbb{R}$ is a $\eta$-convex function, $\eta$ is bounded from above on $f([a,b])\times f([a,b])$ and $g:[a,b]\to\mathbb{R}^{+}$ is integrable, then by Theorem 4 we have*

$$\begin{aligned}
&\frac{1}{2}\int_a^b \left[f(x)+f(a+b-x)\right] g(x)\, dx \\
&\leq \min\left\{f(b)+\frac{1}{2}\eta(f(a),f(b)),\, f(a)+\frac{1}{2}\eta(f(b),f(a))\right\}\int_a^b g(x)\, dx \\
&\leq \frac{1}{2}\left[f(a)+f(b)\right]\int_a^b g(x)\, dx \\
&+\frac{1}{4}\left[\eta(f(a),f(b))+\eta(f(b),f(a))\right]\int_a^b g(x)\, dx,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\frac{1}{2}\int_a^b \left[f(x)+f(a+b-x)\right] g(x)\, dx \\
&\leq \frac{1}{2}\left[f(a)+f(b)\right]\int_a^b g(x)\, dx \\
&+\frac{1}{2}\left[\frac{\eta(f(a),f(b))+\eta(f(b),f(a))}{b-a}\right]\int_a^b (x-a)\, g(x)\, dx
\end{aligned} \tag{3.9}$$

*and*

$$\begin{aligned}
\int_a^b f(x)\, g(x)\, dx \leq{}& \frac{1}{2}\left[f(a)+f(b)\right]\int_a^b g(x)\, dx \\
&+\frac{1}{2}\frac{\eta(f(a),f(b))}{b-a}\int_a^b (x-a)\, g(x)\, dx \\
&+\frac{1}{2}\frac{\eta(f(b),f(a))}{b-a}\int_a^b (b-x)\, g(x)\, dx.
\end{aligned} \tag{3.10}$$

*If $\eta(f(a),f(b))=-\eta(f(b),f(a))$, then from (3.8)-(3.10) we get*

$$\begin{aligned}
&\frac{1}{2}\int_a^b \left[f(x)+f(a+b-x)\right] g(x)\, dx \\
&\leq \min\left\{f(b)-\frac{1}{2}\eta(f(b),f(a)),\, f(a)+\frac{1}{2}\eta(f(b),f(a))\right\}\int_a^b g(x)\, dx \\
&\leq \frac{1}{2}\left[f(a)+f(b)\right]\int_a^b g(x)\, dx,
\end{aligned} \tag{3.11}$$

*and*

$$\int_a^b f(x) g(x) dx \leq \frac{1}{2}[f(a)+f(b)] \int_a^b g(x) dx + \frac{\eta(f(b),f(a))}{b-a} \int_a^b \left(\frac{b+a}{2}-x\right) g(x) dx. \tag{3.12}$$

*If $g$ is symmetric on $[a,b]$, then from (3.8)-(3.10) we get*

$$\begin{aligned}\int_a^b f(x) g(x) dx &\leq \min\left\{f(b)+\frac{1}{2}\eta(f(a),f(b)), f(a)+\frac{1}{2}\eta(f(b),f(a))\right\} \int_a^b g(x) dx \\ &\leq \frac{1}{2}[f(a)+f(b)] \int_a^b g(x) dx \\ &+ \frac{1}{4}[\eta(f(a),f(b))+\eta(f(b),f(a))] \int_a^b g(x) dx,\end{aligned} \tag{3.13}$$

*and*

$$\begin{aligned}\int_a^b f(x) g(x) dx &\leq \frac{1}{2}[f(a)+f(b)] \int_a^b g(x) dx \\ &+ \frac{1}{2}\left[\frac{\eta(f(a),f(b))+\eta(f(b),f(a))}{b-a}\right] \int_a^b (x-a) g(x) dx.\end{aligned} \tag{3.14}$$

*Since for symmetric functions $g$ we have*

$$\int_a^b (x-a) g(x) dx = \int_a^b (b-x) g(x) dx,$$

*then (3.14) is equivalent to (1.4).*

**Corollary 4.** *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $g:[a,b]\to[0,\infty)$ an integrable function, then*

$$\int_a^b f(x) g(x) dx \leq \frac{1}{2}[f(a)+f(b)] \int_a^b g(x) dx + \frac{f(b)-f(a)}{b-a} \int_a^b \left(\frac{b+a}{2}-x\right) g(x) dx. \tag{3.15}$$

The proof follow by (3.12) for the case of $\eta(u,v)=u-v$, $u,v\in\mathbb{R}$.

**Remark 3.** *If $f:[a,b]\to\mathbb{R}$ is a convex function and $g:[a,b]\to[0,\infty)$ an integrable function with*

$$\int_a^b \left(\frac{b+a}{2}-x\right) g(x) dx = 0, \tag{3.16}$$

*then we have Fejér's inequality*

$$\int_a^b f(x) g(x) dx \leq \frac{1}{2}[f(a)+f(b)] \int_a^b g(x) dx. \tag{3.17}$$

*If $g$ is symmetric, then $h:[a,b]\to\mathbb{R}$, $h(x):=\left(\frac{b+a}{2}-x\right)g(x)$ is antisymmetric on $[a,b]$ and the condition (3.16) is fulfilled, so Fejér's inequality holds under more general assumption than symmetry of function $g$.*

**Remark 4.** *Since, integrating by parts we have*

$$\int_a^b\left(\frac{b+a}{2}-x\right)g(x)\,dx=\int_a^b\left(\frac{b+a}{2}-x\right)d\left(\int_a^x g(t)\,dt\right)$$
$$=\frac{1}{2}(b-a)\int_a^b g(t)\,dt-\int_a^b\left(\int_a^x g(t)\,dt\right)dx,$$

*then (3.16) is equivalent to*

$$\int_a^b\left(\int_a^x g(t)\,dt\right)dx=\frac{1}{2}(b-a)\int_a^b g(t)\,dt. \tag{3.18}$$

The following result also holds

**Theorem 5.** *Let $f:C\to\mathbb{R}$ be a $\eta$-convex function on $C$ with $\eta$ bounded from above on $f(C)\times f(C)$. If $w:C\to[0,\infty)$ is Lebesgue integrable on $C$, then for any $x,y\in C$, $x\neq y$ we have the weighted inequalities:*

$$\begin{aligned}(3.19)\quad & f\left(\frac{x+y}{2}\right)\int_0^1 w((1-t)x+ty)\,dt\\
&\le\int_0^1 w((1-t)x+ty)\\
&\times\min\left\{f(tx+(1-t)y)+\frac{1}{2}\eta(f((1-t)x+ty),f(tx+(1-t)y)),\right.\\
&\left.f((1-t)x+ty)+\frac{1}{2}\eta(f(tx+(1-t)y),f((1-t)x+ty))\right\}dt\end{aligned}$$

$$\begin{aligned}&\le\min\left\{\int_0^1 w((1-t)x+ty)\,f(tx+(1-t)y)\,dt\right.\\
&+\frac{1}{2}\int_0^1 w((1-t)x+ty)\,\eta(f((1-t)x+ty),f(tx+(1-t)y))dt,\\
&\int_0^1 w((1-t)x+ty)\,f((1-t)x+ty)\,dt\\
&\left.+\frac{1}{2}\int_0^1 w((1-t)x+ty)\,\eta(f(tx+(1-t)y),f((1-t)x+ty))dt\right\}\\
&\le\int_0^1 w((1-t)x+ty)\,\frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2}dt\\
&+\frac{1}{4}\int_0^1 w((1-t)x+ty)\,[\eta(f((1-t)x+ty),f(tx+(1-t)y))\\
&+\eta(f(tx+(1-t)y),f((1-t)x+ty))]\,dt.\end{aligned}$$

*Proof.* From (2.16) we have

$$f\left(\frac{x+y}{2}\right)$$
$$\leq \min\left\{f\left(tx+(1-t)y\right)+\frac{1}{2}\eta(f((1-t)x+ty),f(tx+(1-t)y)),\right.$$
$$\left.f\left((1-t)x+ty\right)+\frac{1}{2}\eta(f(tx+(1-t)y),f((1-t)x+ty))\right\}$$

for any $x,y\in C$, $x\neq y$ and for any $t\in[0,1]$.

Multiplying with $w\left((1-t)x+ty\right)\geq 0$ and integrating over $t$ we get the first inequality in (3.19).

Now, if $f_1,f_2,h:[a,b]\to\mathbb{R}$ are integrable and $h\geq 0$, then

$$\int_a^b h(t)\min\{f_1(t),f_2(t)\}\,dt$$
$$=\frac{1}{2}\int_a^b h(t)\left[f_1(t)+f_2(t)-|f_1(t)-f_2(t)|\right]dt$$
$$=\frac{1}{2}\left[\int_a^b h(t)f_1(t)\,dt+\int_a^b h(t)f_2(t)\,dt-\int_a^b h(t)\left|f_1(t)-f_2(t)\right|dt\right]$$
$$\leq\frac{1}{2}\left[\int_a^b h(t)f_1(t)\,dt+\int_a^b h(t)f_2(t)\,dt-\left|\int_a^b h(t)f_1(t)\,dt-\int_a^b h(t)f_2(t)\,dt\right|\right]$$
$$=\min\left\{\int_a^b h(t)f_1(t)\,dt,\int_a^b h(t)f_2(t)\,dt\right\}.$$

Using this property, we have

$$\int_0^1 w\left((1-t)x+ty\right)$$
$$\times\min\left\{f\left(tx+(1-t)y\right)+\frac{1}{2}\eta(f((1-t)x+ty),f(tx+(1-t)y)),\right.$$
$$\left.f\left((1-t)x+ty\right)+\frac{1}{2}\eta(f(tx+(1-t)y),f((1-t)x+ty))\right\}dt$$

$$\leq \min\left\{\int_0^1 w\left((1-t)\,x+ty\right) f\left(tx+(1-t)\,y\right) dt\right.$$
$$+\frac{1}{2}\int_0^1 w\left((1-t)\,x+ty\right) \eta(f((1-t)\,x+ty), f(tx+(1-t)\,y))dt,$$
$$\int_0^1 w\left((1-t)\,x+ty\right) f\left((1-t)\,x+ty\right) dt$$
$$\left.+\frac{1}{2}\int_0^1 w\left((1-t)\,x+ty\right) \eta(f(tx+(1-t)\,y), f((1-t)\,x+ty))dt\right\}$$
$$\leq \int_0^1 w\left((1-t)\,x+ty\right) \frac{f\left(tx+(1-t)\,y\right)+f\left((1-t)\,x+ty\right)}{2} dt$$
$$+\frac{1}{4}\int_0^1 w\left((1-t)\,x+ty\right) [\eta(f((1-t)\,x+ty), f(tx+(1-t)\,y))$$
$$+\eta(f(tx+(1-t)\,y), f((1-t)\,x+ty))]\,dt,$$

where, for the last inequality we used the property that $\min\{c,d\} \leq \frac{c+d}{2}$. □

**Corollary 5.** *With the assumptions of Theorem 5 and if for some $x, y \in C$, $x \neq y$ the function $w$ is symmetric on the segment $[x, y]$, then we have the inequalities*

$$f\left(\frac{x+y}{2}\right)\int_0^1 w\left((1-t)\,x+ty\right) dt \tag{3.20}$$
$$\leq \int_0^1 w\left((1-t)\,x+ty\right)$$
$$\times \min\left\{f\left(tx+(1-t)\,y\right)+\frac{1}{2}\eta(f((1-t)\,x+ty), f(tx+(1-t)\,y)),\right.$$
$$\left.f\left((1-t)\,x+ty\right)+\frac{1}{2}\eta(f(tx+(1-t)\,y), f((1-t)\,x+ty))\right\} dt$$
$$\leq \int_0^1 w\left((1-t)\,x+ty\right) f\left(tx+(1-t)\,y\right) dt$$
$$+\frac{1}{2}\int_0^1 w\left((1-t)\,x+ty\right) \eta(f((1-t)\,x+ty), f(tx+(1-t)\,y))dt.$$

**Remark 5.** *If $f : [a, b] \to \mathbb{R}$ is a $\eta$-convex function, $\eta$ is bounded from above on $f([a, b]) \times f([a, b])$ and $g : [a, b] \to \mathbb{R}^+$ is integrable, then by Theorem 5 we have*

$$f\left(\frac{a+b}{2}\right)\int_a^b g\left(x\right) dx \tag{3.21}$$
$$\leq \int_a^b g\left(x\right) \min\left\{f\left(a+b-x\right)+\frac{1}{2}\eta(f(x), f(a+b-x),\right.$$
$$\left.f\left(x\right)+\frac{1}{2}\eta(f(a+b-x), f(x))\right\} dx$$

$$\leq \min\left\{\int_a^b g(x) f(a+b-x)\,dx + \frac{1}{2}\int_a^b g(x)\,\eta(f(x), f(a+b-x)dx,\right.$$

$$\left.\int_a^b g(x) f(x)\,dx + \frac{1}{2}\int_a^b g(x)\,\eta(f(a+b-x), f(x))dx\right\}$$

$$\leq \int_a^b g(x) \frac{f(a+b-x)+f(x)}{2}dx + \frac{1}{4}\int_a^b g(x)\left[\eta(f(x), f(a+b-x)) + \eta(f(a+b-x), f(x))\right]dx.$$

*Moreover, if $g$ is symmetric on $[a,b]$, then we have*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \tag{3.22}$$
$$\leq \int_a^b g(x) \min\left\{f(a+b-x) + \frac{1}{2}\eta(f(x), f(a+b-x), f(x) + \frac{1}{2}\eta(f(a+b-x), f(x))\right\}dx$$
$$\leq \int_a^b g(x) f(x)\,dx + \frac{1}{2}\int_a^b g(x)\,\eta(f(x), f(a+b-x)dx.$$

*This provide a refinement of (1.5).*

Now, if we use the inequality (3.8) for the function $g \equiv 1$, then we have the following Hermite-Hadamard type inequalities

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \min\left\{f(b) + \frac{1}{2}\eta(f(a), f(b)), f(a) + \frac{1}{2}\eta(f(b), f(a))\right\} \tag{3.23}$$
$$\leq \frac{1}{2}\left[f(a) + f(b)\right] + \frac{1}{4}\left[\eta(f(a), f(b)) + \eta(f(b), f(a))\right]$$

and from (3.22) we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \min\left\{f(a+b-x) + \frac{1}{2}\eta(f(x), f(a+b-x), f(x) + \frac{1}{2}\eta(f(a+b-x), f(x))\right\}dx \tag{3.24}$$
$$\leq \frac{1}{b-a}\int_a^b f(x)\,dx + \frac{1}{2(b-a)}\int_a^b \eta(f(x), f(a+b-x)dx,$$

which improve the inequality (1.3).

## 4. Applications for Trapezoid and Mid-Point Inequalities

We have the following identity for an absolutely continuous function $f : [a,b] \to \mathbb{R}$, see for instance [10]

$$\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt = \int_a^b \left(t - \frac{a+b}{2}\right) f'(t)\,dt. \tag{4.1}$$

This can be easily proved by using the integration by parts formula in the second integral.

**Proposition 1.** *Let $f : I \to \mathbb{R}$ be a differentiable function on $\mathring{I}$, the interior of $I$ and $[a,b] \subset \mathring{I}$. If the function $|f'|$ is a $\eta$-convex function, $\eta$ is bounded from above on $|f'|([a,b]) \times |f'|([a,b])$, then we have*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right| \\
&\le \frac{1}{4}(b-a)^2 \\
&\times \min\left\{|f'(b)|+\frac{1}{2}\eta(|f'(a)|,|f'(b)|),|f'(a)|+\frac{1}{2}\eta(|f'(b)|,|f'(a)|)\right\} \\
&\le \frac{1}{8}(b-a)^2 \\
&\times \left[|f'(a)|+|f'(b)|+\frac{1}{2}\left[\eta(|f'(a)|,|f'(b)|)+\eta(|f'(b)|,|f'(a)|)\right]\right].
\end{aligned} \tag{4.2}$$

*Proof.* Taking the modulus on (4.1) we have

$$\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right| \le \int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|\,dt. \tag{4.3}$$

Since $|f'|$ is a $\eta$-convex function and $g : [a,b] \to \mathbb{R}$, $g(t) := \left|t-\frac{a+b}{2}\right|$, then by the inequality (3.13) we have

$$\begin{aligned}
&\int_a^b |f'(t)| \left|t-\frac{a+b}{2}\right| dt \\
&\le \min\left\{|f'(b)|+\frac{1}{2}\eta(|f'(a)|,|f'(b)|),|f'(a)|+\frac{1}{2}\eta(|f'(b)|,|f'(a)|)\right\} \\
&\times \int_a^b \left|t-\frac{a+b}{2}\right| dt \\
&\le \frac{1}{2}\left[|f'(a)|+|f'(b)|\right]\int_a^b \left|t-\frac{a+b}{2}\right| dt \\
&+ \frac{1}{4}\left[\eta(|f'(a)|,|f'(b)|)+\eta(|f'(b)|,|f'(a)|)\right]\int_a^b \left|t-\frac{a+b}{2}\right| dt.
\end{aligned} \tag{4.4}$$

Taking into account that

$$\int_a^b \left|t-\frac{a+b}{2}\right| dt = \frac{1}{4}(b-a)^2,$$

then by (4.4) we get the desired result (4.2). □

We also have the following identity for absolutely continuous functions $f : [a,b] \to \mathbb{R}$, see for instance [9]

$$(b-a)f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt = \int_a^b k_1(t) f'(t)\,dt, \tag{4.5}$$

where $k : [a,b] \to \mathbb{R}$ is given by

$$k_1(t) := \begin{cases} t-a, & t \in \left[a, \frac{a+b}{2}\right), \\ t-b, & t \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

**Proposition 2.** *With the assumptions of Proposition 1 we have the inequalities*

$$\begin{aligned}(4.6)\qquad &\left|(b-a) f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{4}(b-a)^2 \\ &\times \min\left\{\left|f'(b)\right|+\frac{1}{2}\eta(\left|f'(a)\right|,\left|f'(b)\right|),\left|f'(a)\right|+\frac{1}{2}\eta(\left|f'(b)\right|,\left|f'(a)\right|)\right\} \\ &\leq \frac{1}{8}(b-a)^2 \\ &\times\left[\left|f'(a)\right|+\left|f'(b)\right|+\frac{1}{2}\left[\eta(\left|f'(a)\right|,\left|f'(b)\right|)+\eta(\left|f'(b)\right|,\left|f'(a)\right|)\right]\right].\end{aligned}$$

*Proof.* Taking the modulus in (4.5) we have

$$\left|(b-a) f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right| \leq \int_a^b\left|k_2(t)\right|\left|f'(t)\right| dt.$$

Observe that

$$\left|k_1(t)\right|=\begin{cases} t-a, & t\in\left[a,\frac{a+b}{2}\right), \\ b-t, & t\in\left[\frac{a+b}{2},b\right], \end{cases}$$

which is a symmetric on the interval $[a,b]$.

Since $|f'|$ is a $\eta$-convex function and $g:[a,b]\to\mathbb{R}$, $g(t):=|k(t)|$, then by the inequality (3.13) we have

$$\begin{aligned}(4.7)\qquad &\int_a^b\left|f'(t)\right|\left|k_1(t)\right| dt \\ &\leq \min\left\{\left|f'(b)\right|+\frac{1}{2}\eta(\left|f'(a)\right|,\left|f'(b)\right|),\left|f'(a)\right|+\frac{1}{2}\eta(\left|f'(b)\right|,\left|f'(a)\right|)\right\} \\ &\times\int_a^b\left|k_1(t)\right| dt \\ &\leq \frac{1}{2}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\int_a^b\left|k_1(t)\right| dt \\ &+\frac{1}{4}\left[\eta(\left|f'(a)\right|,\left|f'(b)\right|)+\eta(\left|f'(b)\right|,\left|f'(a)\right|)\right]\int_a^b\left|k_1(t)\right| dt.\end{aligned}$$

Since

$$\int_a^b\left|k_1(t)\right| dt=\frac{1}{4}(b-a)^2,$$

then by (4.7) we get the desired result (4.6). □

**Remark 6.** *If we use the following identities for twice differentiable functions*

$$\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt=\frac{1}{2}\int_a^b(t-a)(b-t) f''(t)\,dt,\ [10]$$

*and*

$$\int_a^b f(t)\,dt-(b-a) f\left(\frac{a+b}{2}\right)=\frac{1}{2}\int_a^b k_2(t) f''(t)\,dt,\ [11]$$

*where*

$$k_2(t) := \begin{cases} (t-a)^2, & t \in \left[a, \frac{a+b}{2}\right), \\ (t-b)^2, & t \in \left[\frac{a+b}{2}, b\right], \end{cases}$$

*then for twice differentiable functions $f$ for which $|f''|$ is a $\eta$-convex function, $\eta$ is bounded from above on $|f''|([a,b]) \times |f''|([a,b])$, then we have the inequalities*

$$(4.8)\quad \left| \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt \right| \le \frac{1}{12}(b-a)^2 \times \min\left\{ |f''(b)| + \frac{1}{2}\eta(|f''(a)|, |f''(b)|), |f''(a)| + \frac{1}{2}\eta(|f''(b)|, |f''(a)|) \right\} \le \frac{1}{24}(b-a)^2 \times \left[ |f''(a)| + |f''(b)| + \frac{1}{2}\left[\eta(|f''(a)|, |f''(b)|) + \eta(|f''(b)|, |f''(a)|)\right] \right]$$

*and*

$$(4.9)\quad \left| (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,dt \right| \le \frac{1}{24}(b-a)^2 \times \min\left\{ |f''(b)| + \frac{1}{2}\eta(|f''(a)|, |f''(b)|), |f''(a)| + \frac{1}{2}\eta(|f''(b)|, |f''(a)|) \right\} \le \frac{1}{48}(b-a)^2 \times \left[ |f''(a)| + |f''(b)| + \frac{1}{2}\left[\eta(|f''(a)|, |f''(b)|) + \eta(|f''(b)|, |f''(a)|)\right] \right].$$

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