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WEIGHTED REVERSE INEQUALITIES OF JENSEN TYPE FOR FUNCTIONS OF SELFADJOINT OPERATORS

S. S. DRAGOMIR[1,2]

Abstract. On making use of the representation in terms of the Riemann-Stieltjes integral of spectral families for selfadjoint operators in Hilbert spaces, we establish here some weighted reverse inequalities of Jensen's type for convex functions of operators. Some applications for simple functions of operators are also provided.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for μ -a.e. (almost every) $x\in\Omega$, consider the Lebesgue space

$$L_w(\Omega,\mu):=\{f:\Omega\to\mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x)|f(x)|\,d\mu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega wd\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, the author obtained in 2002 [6] the following result:

Theorem 1. *Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:\Omega\to[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)f\in L_w(\Omega,\mu)$, where $w\geq 0$ μ-a.e. (almost everywhere) on Ω with $\int_\Omega wd\mu=1$. Then we have the inequality:*

$$\begin{aligned}0&\leq\int_\Omega(\Phi\circ f)\,wd\mu-\Phi\left(\int_\Omega fwd\mu\right)\\&\leq\int_\Omega(\Phi'\circ f)\,fwd\mu-\int_\Omega(\Phi'\circ f)\,wd\mu\int_\Omega wfd\mu\\&\leq\frac{1}{2}\left[\Phi'(M)-\Phi'(m)\right]\int_\Omega w\left|f-\int_\Omega fwd\mu\right|d\mu.\end{aligned}\tag{1.1}$$

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If $\mu(\Omega) < \infty$ and $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L(\Omega, \mu)$, then we have the inequality:

$$
\begin{aligned}
(1.2)\qquad 0 &\leq \frac{1}{\mu(\Omega)} \int_\Omega (\Phi \circ f)\, d\mu - \Phi\left(\frac{1}{\mu(\Omega)} \int_\Omega f d\mu\right) \\
&\leq \frac{1}{\mu(\Omega)} \int_\Omega (\Phi' \circ f) f d\mu - \frac{1}{\mu(\Omega)} \int_\Omega (\Phi' \circ f)\, d\mu \frac{1}{\mu(\Omega)} \int_\Omega f d\mu \\
&\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \frac{1}{\mu(\Omega)} \int_\Omega \left| f - \frac{1}{\mu(\Omega)} \int_\Omega f d\mu \right| d\mu.
\end{aligned}
$$

The following discrete inequality is of interest as well.

Corollary 1. *Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable convex function on (m, M). If $x_i \in [m, M]$ and $w_i \geq 0$ $(i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has the counterpart of Jensen's weighted discrete inequality:*

$$
\begin{aligned}
(1.3)\qquad 0 &\leq \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n w_i x_i\right) \\
&\leq \sum_{i=1}^n w_i \Phi'(x_i)\, x_i - \sum_{i=1}^n w_i \Phi'(x_i) \sum_{i=1}^n w_i x_i \\
&\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \sum_{i=1}^n w_i \left| x_i - \sum_{j=1}^n w_j x_j \right|.
\end{aligned}
$$

Remark 1. *We notice that the inequality between the first and the second term in (1.3) was proved in 1994 by Dragomir & Ionescu, see* [24].

If $f,\ g : \Omega \to \mathbb{R}$ are μ-measurable functions and $f,\ g,\ fg \in L_w(\Omega, \mu)$, then we may consider the *Čebyšev functional*

$$
(1.4)\qquad T_w(f, g) := \int_\Omega w f g d\mu - \int_\Omega w f d\mu \int_\Omega w g d\mu.
$$

The following result is known in the literature as the *Grüss inequality*

$$
(1.5)\qquad |T_w(f, g)| \leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta),
$$

provided

$$
(1.6)\qquad -\infty < \gamma \leq f(x) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(x) \leq \Delta < \infty
$$

for μ –a.e. (almost every) $x \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

If we assume that $-\infty < \gamma \leq f(x) \leq \Gamma < \infty$ for μ –a.e. $x \in \Omega$, then by the Grüss inequality for $g = f$ and by the Schwarz's integral inequality, we have

$$
(1.7)\qquad \int_\Omega w \left| f - \int_\Omega w f d\mu \right| d\mu \leq \left[\int_\Omega w f^2 d\mu - \left(\int_\Omega w f d\mu\right)^2\right]^{\frac{1}{2}} \leq \frac{1}{2} (\Gamma - \gamma).
$$

In what follows, we assume that $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ –a.e. $x \in \Omega$, is a μ-measurable function with $\int_\Omega w d\mu = 1$.

On making use of the results (1.1) and (1.7), we can state the following string of reverse inequalities:

Lemma 1. *Let $\Phi : I \to \mathbb{R}$ be a differentiable convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ is the interior of I. If $f : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \leq f(x) \leq M < \infty \quad \text{for } \mu\text{-a.e. } x \in \Omega \tag{1.8}$$

and $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L_w(\Omega, \mu)$, then

$$\begin{aligned} 0 &\leq \int_\Omega (\Phi \circ f) w d\mu - \Phi\left(\int_\Omega f w d\mu\right) \\ &\leq \int_\Omega (\Phi' \circ f) f w d\mu - \int_\Omega (\Phi' \circ f) w d\mu \int_\Omega f w d\mu \\ &\leq \frac{1}{2}[\Phi'(M) - \Phi'(m)] \int_\Omega \left| f - \int_\Omega w f d\mu \right| w d\mu \\ &\leq \frac{1}{2}[\Phi'(M) - \Phi'(m)] \left[\int_\Omega f^2 w d\mu - \left(\int_\Omega f w d\mu\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}[\Phi'(M) - \Phi'(m)](M - m). \end{aligned} \tag{1.9}$$

Remark 2. *We notice that the inequality between the first, second and last term from (1.9) was proved in the general case of positive linear functionals in 2001 by the author in* [5].

If the differentiability condition is removed, then Φ' can be replaced in the inequality (1.9) above with a section φ of the subdifferential $\partial\Phi$. We omit the details.

The following reverse of the Jensen's inequality holds [20], [21]:

Lemma 2. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$. If $f : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds (1.8) and $f, \Phi \circ f \in L_w(\Omega, \mu)$, then*

$$\begin{aligned} 0 &\leq \int_\Omega (\Phi \circ f) w d\mu - \Phi\left(\int_\Omega f w d\mu\right) \\ &\leq \frac{\left(M - \int_\Omega f w d\mu\right)\left(\int_\Omega f w d\mu - m\right)}{M - m} \sup_{t \in (m, M)} \Psi_\Phi(t; m, M) \\ &\leq \left(M - \int_\Omega f w d\mu\right)\left(\int_\Omega f w d\mu - m\right) \frac{\Phi'_-(M) - \Phi'_+(m)}{M - m} \\ &\leq \frac{1}{4}(M - m)\left[\Phi'_-(M) - \Phi'_+(m)\right], \end{aligned} \tag{1.10}$$

where $\Psi_\Phi(\cdot; m, M) : (m, M) \to \mathbb{R}$ is defined by

$$\Psi_\Phi(t; m, M) = \frac{\Phi(M) - \Phi(t)}{M - t} - \frac{\Phi(t) - \Phi(m)}{t - m}.$$

We also have the inequality

$$\begin{aligned} 0 &\leq \int_\Omega (\Phi \circ f) w d\mu - \Phi\left(\int_\Omega f w d\mu\right) \leq \frac{1}{4}(M - m)\Psi_\Phi\left(\int_\Omega f w d\mu; m, M\right) \\ &\leq \frac{1}{4}(M - m)\left[\Phi'_-(M) - \Phi'_+(m)\right], \end{aligned} \tag{1.11}$$

provided that $\int_\Omega f w d\mu \in (m, M)$.

We also have:

Lemma 3. *With the assumptions of Lemma 2, we have the inequalities*

$$
\begin{aligned}
(1.12) \qquad 0 &\leq \int_\Omega w(\Phi \circ f)\, d\mu(x) - \Phi\left(\int_\Omega fwd\mu\right) \\
&\leq 2\max\left\{\frac{M-\int_\Omega fwd\mu}{M-m}, \frac{\int_\Omega fwd\mu - m}{M-m}\right\} \\
&\times \left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\
&\leq \frac{1}{2}\max\left\{M-\int_\Omega fwd\mu, \int_\Omega fwd\mu - m\right\}\left[\Phi'_-(M) - \Phi'_+(m)\right].
\end{aligned}
$$

Since

$$
\frac{M-\int_\Omega fwd\mu}{M-m}, \frac{\int_\Omega fwd\mu - m}{M-m} \leq 1
$$

we also have the simpler inequality

$$
\begin{aligned}
(1.13) \qquad 0 &\leq \int_\Omega w(\Phi \circ f)\, d\mu(x) - \Phi\left(\int_\Omega fwd\mu\right) \\
&\leq 2\left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right].
\end{aligned}
$$

The discrete case of this inequality was obtained in [41].

For a real function $g : [m, M] \to \mathbb{R}$ and two distinct points $\alpha, \beta \in [m, M]$ we recall that the *divided difference* of g in these points is defined by

$$
[\alpha, \beta; g] := \frac{g(\beta) - g(\alpha)}{\beta - \alpha}.
$$

The following result holds [22]:

Lemma 4. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$. If $f : \Omega \to \mathbb{R}$ is μ-measurable, satisfying the bounds (1.8) and f, $\Phi \circ f \in L_w(\Omega, \mu)$, then by assuming that $\int_\Omega wfd\mu \neq m, M$, we have*

$$
\begin{aligned}
(1.14) \qquad &\left|\int_\Omega \left|\Phi(f) - \Phi\left(\int_\Omega wfd\mu\right)\right| sgn\left[f - \int_\Omega wfd\mu\right] wd\mu\right| \\
&\leq \int_\Omega (\Phi \circ f)\, wd\mu - \Phi\left(\int_\Omega wfd\mu\right) \\
&\leq \frac{1}{2}\left(\left[\int_\Omega wfd\mu, M; \Phi\right] - \left[m, \int_\Omega wfd\mu; \Phi\right]\right) D_w(f) \\
&\leq \frac{1}{2}\left(\left[\int_\Omega wfd\mu, M; \Phi\right] - \left[m, \int_\Omega wfd\mu; \Phi\right]\right) D_{w,2}(f) \\
&\leq \frac{1}{4}\left(\left[\int_\Omega wfd\mu, M; \Phi\right] - \left[m, \int_\Omega wfd\mu; \Phi\right]\right) (M-m),
\end{aligned}
$$

where

$$
D_w(f) := \int_\Omega w\left|f - \int_\Omega wfd\mu\right| d\mu
$$

and

$$D_{w,2}(f) := \left[\int_\Omega wf^2 d\mu - \left(\int_\Omega wf d\mu\right)^2\right]^{\frac{1}{2}}.$$

The constant $\frac{1}{2}$ *in the second inequality from (1.10) is best possible.*

For recent results related to Jensen's inequality, see [1]-[7], [26]-[42] and the references therein.

2. Inequalities for Selfadjoint Operators

Let A be a selfadjoint operator on the complex Hilbert space $(H, \langle ., . \rangle)$ with the spectrum $Sp(A)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f : [m, M] \to \mathbb{R}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral* (see for instance [26, p. 257]):

$$\langle f(A)x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y\rangle, \tag{2.1}$$

and

$$\|f(A)x\|^2 = \int_{m-0}^{M} |f(\lambda)|^2\, d\|E_\lambda x\|^2, \tag{2.2}$$

for any $x, y \in H$.

The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and $g_{x,y}(m-0) = 0$ while $g_{x,y}(M) = \langle x, y\rangle$ for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$ for any $x \in H$.

The following result that provides an operator version for the Jensen inequality:

Theorem 2 (Mond-Pečarić, 1993, [35])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If Φ is a convex function on $[m, M]$, then*

$$\Phi(\langle Ax, x\rangle) \le \langle \Phi(A)x, x\rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 2 we have the following Hölder-McCarthy inequality:

Theorem 3 (Hölder-McCarthy, 1967, [32])**.** *Let A be a selfadjoint positive operator on a Hilbert space H. Then for all $x \in H$ with $\|x\| = 1$,*

(i) $\langle A^r x, x\rangle \ge \langle Ax, x\rangle^r$ *for all* $r > 1$;
(ii) $\langle A^r x, x\rangle \le \langle Ax, x\rangle^r$ *for all* $0 < r < 1$;
(iii) If A is invertible, then $\langle A^r x, x\rangle \ge \langle Ax, x\rangle^r$ *for all* $r < 0$.

The following reverse for the (MP) inequality that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [25, p. 57]:

Theorem 4. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If Φ is a convex function on $[m, M]$, then*

$$\langle \Phi(A)x, x\rangle \le \frac{M - \langle Ax, x\rangle}{M - m}\Phi(m) + \frac{\langle Ax, x\rangle - m}{M - m}\Phi(M) \tag{LR}$$

for each $x \in H$ *with* $\|x\| = 1$.

In [23] we obtained the following weighted version of (MP) and (LR).

Theorem 5. *Let* A *be a selfadjoint operator on the Hilbert space* H *and assume that* $Sp(A) \subseteq [m, M]$ *for some scalars* m, M *with* $m < M$. *If* $\Phi : [k, K] \subset \mathbb{R} \to \mathbb{R}$ *is a continuous convex function on the interval* $[k, K]$, $w : [m, M] \to [0, \infty)$ *is continuous on* $[m, M]$, $f : [m, M] \subset \mathbb{R} \to \mathbb{R}$ *is a continuous function on the interval* $[m, M]$ *and with the property that*

$$k \leq f(t) \leq K \text{ for any } t \in [m, M], \tag{2.3}$$

then

$$\begin{aligned} \Phi\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right) &\leq \frac{\langle w(A)(\Phi \circ f)(A) x, x\rangle}{\langle w(A) x, x\rangle} \\ &\leq \frac{\left(K - \frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)\Phi(k) + \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle} - k\right)\Phi(K)}{K - k}, \end{aligned} \tag{2.4}$$

for any $x \in H$ *with* $\langle w(A) x, x\rangle \neq 0$.

For various particular instances of (2.4) that are of interest being related to Hölder-McCarthy's inequalities mentioned above, see [23].

For classical and recent result concerning inequalities for continuos functions of selfadjoint operators, see [35], [34], [36], [31], [25], [6], [10], [13], [18], [11], [15], [19], [17], [16], [14], [9], and [12].

3. Further Reverse Inequalities

We have the following new results:

Theorem 6. *Let* A *be a selfadjoint operator on the Hilbert space* H *such that* $Sp(A) \subseteq [m, M]$ *for some scalars* m, M *with* $m < M$. *Assume that* $\Phi : [k, K] \subset \mathbb{R} \to \mathbb{R}$ *is a continuous convex function on the interval* $[k, K]$, $w : [m, M] \to [0, \infty)$ *is continuous on* $[m, M]$, $f : [m, M] \subset \mathbb{R} \to \mathbb{R}$ *is a continuous function on the interval* $[m, M]$ *and satisfies the property (2.3)*

(i) If Φ *is continuously differentiable on* (k, K), *then we have*

$$\begin{aligned} 0 &\leq \frac{\langle w(A)(\Phi \circ f)(A) x, x\rangle}{\langle w(A) x, x\rangle} - \Phi\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right) \\ &\leq \frac{\langle (\Phi' \circ f)(A) f(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} \\ &\quad - \frac{\langle (\Phi' \circ f)(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} \frac{\langle f(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} \\ &\leq \frac{1}{2}\left[\Phi'_-(K) - \Phi'_+(k)\right] \frac{\left\langle \left| f(A) - \frac{\langle f(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} 1_H \right| x, x\right\rangle}{\langle w(A) x, x\rangle} \\ &\leq \frac{1}{2}\left[\Phi'_-(K) - \Phi'_+(k)\right]\left[\frac{\langle f^2(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}\left[\Phi'_-(K) - \Phi'_+(k)\right](K - k) \end{aligned} \tag{3.1}$$

for any $x \in H$ *with* $\langle w(A) x, x\rangle \neq 0$.

(ii) If we consider the function $\Psi_{\Phi}(\cdot;k,K):(k,K)\to\mathbb{R}$ *defined by*

$$\Psi_{\Phi}(t;k,K)=\frac{\Phi(K)-\Phi(t)}{K-t}-\frac{\Phi(t)-\Phi(k)}{t-k},$$

then

$$
\begin{aligned}
(3.2)\quad 0 &\le \frac{\langle w(A)(\Phi\circ f)(A)x,x\rangle}{\langle w(A)x,x\rangle}-\Phi\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)\\
&\le \frac{\left(K-\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}-k\right)}{K-k}\sup_{t\in(k,K)}\Psi_{\Phi}(t;k,K)\\
&\le \left(K-\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}-k\right)\frac{\Phi'_{-}(K)-\Phi'_{+}(k)}{K-k}\\
&\le \frac{1}{4}\left[\Phi'_{-}(K)-\Phi'_{+}(k)\right](K-k)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.3)\quad 0 &\le \frac{\langle w(A)(\Phi\circ f)(A)x,x\rangle}{\langle w(A)x,x\rangle}-\Phi\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)\\
&\le \frac{1}{4}(K-k)\Psi_{\Phi}\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle};k,K\right)\\
&\le \frac{1}{4}\left[\Phi'_{-}(K)-\Phi'_{+}(k)\right](K-k)
\end{aligned}
$$

for any $x\in H$ *with* $\langle w(A)x,x\rangle\neq 0$.

(iii) We have the inequalities

$$
\begin{aligned}
(3.4)\quad 0 &\le \frac{\langle w(A)(\Phi\circ f)(A)x,x\rangle}{\langle w(A)x,x\rangle}-\Phi\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)\\
&\le 2\max\left\{\frac{K-\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}}{K-k},\frac{\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}-k}{K-k}\right\}\\
&\quad\times\left[\frac{\Phi(k)+\Phi(K)}{2}-\Phi\left(\frac{k+K}{2}\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(3.5)\quad 0 &\le \frac{\langle w(A)(\Phi\circ f)(A)x,x\rangle}{\langle w(A)x,x\rangle}-\Phi\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)\\
&\le 2\left[\frac{\Phi(k)+\Phi(K)}{2}-\Phi\left(\frac{k+K}{2}\right)\right]
\end{aligned}
$$

for any $x\in H$ *with* $\langle w(A)x,x\rangle\neq 0$.

(iv) We also have the inequalities

$$0 \leq \frac{\langle w(A)(\Phi \circ f)(A)x, x\rangle}{\langle w(A)x, x\rangle} - \Phi\left(\frac{\langle w(A) f(A)x, x\rangle}{\langle w(A)x, x\rangle}\right) \tag{3.6}$$
$$\leq \frac{1}{2}\Psi_\Phi\left(\frac{\langle w(A) f(A)x, x\rangle}{\langle w(A)x, x\rangle}; k, K\right)\frac{\left\langle \left| f(A) - \frac{\langle f(A)w(A)x,x\rangle}{\langle w(A)x,x\rangle}1_H \right| x, x\right\rangle}{\langle w(A)x, x\rangle}$$
$$\leq \frac{1}{2}\Psi_\Phi\left(\frac{\langle w(A) f(A)x, x\rangle}{\langle w(A)x, x\rangle}; k, K\right)$$
$$\times\left[\frac{\langle f^2(A)w(A)x, x\rangle}{\langle w(A)x, x\rangle} - \left(\frac{\langle w(A) f(A)x, x\rangle}{\langle w(A)x, x\rangle}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\Psi_\Phi\left(\frac{\langle w(A) f(A)x, x\rangle}{\langle w(A)x, x\rangle}; k, K\right)(K - k)$$

for any $x \in H$ *with* $\langle w(A)x, x\rangle \neq 0$.

Proof. (i) Let $\{E_\lambda\}_\lambda$ be the spectral family of the operator A. Let $\varepsilon > 0$ and write the inequality (1.9) on the interval $[m - \varepsilon, M]$ and for the monotonic nondecreasing function $g(t) = \langle E_t x, x\rangle$, $x \in H$ with $\langle w(A)x, x\rangle \neq 0$, to get

$$0 \leq \frac{\int_{m-\varepsilon}^M (\Phi \circ f)(t) w(t) d\langle E_t x, x\rangle}{\int_{m-\varepsilon}^M w(t) d\langle E_t x, x\rangle} - \Phi\left(\frac{\int_{m-\varepsilon}^M f(t) w(t) d\langle E_t x, x\rangle}{\int_{m-\varepsilon}^M w(t) d\langle E_t x, x\rangle}\right) \tag{3.7}$$
$$\leq \frac{\int_{m-\varepsilon}^M (\Phi' \circ f)(t) f(t) w(t) d\langle E_t x, x\rangle}{\int_{m-\varepsilon}^M w(t) d\langle E_t x, x\rangle}$$
$$- \frac{\int_{m-\varepsilon}^M (\Phi' \circ f)(t) w(t) d\langle E_t x, x\rangle}{\int_{m-\varepsilon}^M w(t) d\langle E_t x, x\rangle}\frac{\int_{m-\varepsilon}^M f(t) w(t) d\langle E_t x, x\rangle}{\int_{m-\varepsilon}^M w(t) d\langle E_t x, x\rangle}$$
$$\leq \frac{1}{2}\frac{\left[\Phi'_-(K) - \Phi'_+(k)\right]}{\int_{m-\varepsilon}^M w(t) d\langle E_t x, x\rangle}$$
$$\times \int_{m-\varepsilon}^M \left| f(t) - \frac{\int_{m-\varepsilon}^M f(s) w(s) d\langle E_s x, x\rangle}{\int_{m-\varepsilon}^M w(s) d\langle E_s x, x\rangle}\right| w(t) d\langle E_t x, x\rangle$$
$$\leq \frac{1}{2}\left[\Phi'_-(K) - \Phi'_+(k)\right]$$
$$\times\left[\frac{\int_{m-\varepsilon}^M f^2(t) w(t) d\langle E_t x, x\rangle}{\int_{m-\varepsilon}^M w(s) d\langle E_s x, x\rangle} - \left(\frac{\int_{m-\varepsilon}^M f(s) w(s) d\langle E_s x, x\rangle}{\int_{m-\varepsilon}^M w(s) d\langle E_s x, x\rangle}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left[\Phi'_-(K) - \Phi'_+(k)\right](K - k).$$

Letting $\varepsilon \to 0+$ and using the spectral representation theorem summarized in (2.1) we get the required inequality (3.1).

(ii) Follows by Lemma 2, (iii) follows by Lemma 3 while (iv) follows by Lemma 4. The details are omitted. □

We have the following generalization and reverse for the Hölder-McCarthy inequality:

Corollary 2. *Let A be a selfadjoint operator on the Hilbert space H such that $Sp(A) \subseteq [m,M]$ for some scalars m,M with $m<M$. Assume that $w:[m,M]\to[0,\infty)$ is continuous on $[m,M]$, $f:[m,M]\subset\mathbb{R}\to\mathbb{R}$ is a continuous function on the interval $[m,M]$ and satisfies the property (2.3) with $k>0$. Assume also that $p\in(-\infty,0)\cup(1,\infty)$.*

(i) We have

$$
\begin{aligned}
(3.8)\qquad 0 &\le \frac{\langle w(A) f^p(A) x,x\rangle}{\langle w(A)x,x\rangle} - \left(\frac{\langle w(A) f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)^p \\
&\le p\left[\frac{\langle f^p(A) w(A)x,x\rangle}{\langle w(A)x,x\rangle} - \frac{\langle f^{p-1}(A) w(A)x,x\rangle}{\langle w(A)x,x\rangle}\frac{\langle f(A) w(A)x,x\rangle}{\langle w(A)x,x\rangle}\right] \\
&\le \frac{1}{2}p\left(K^{p-1}-k^{p-1}\right)\frac{\left\langle \left| f(A) - \frac{\langle f(A)w(A)x,x\rangle}{\langle w(A)x,x\rangle}1_H\right| x,x\right\rangle}{\langle w(A)x,x\rangle} \\
&\le \frac{1}{2}p\left(K^{p-1}-k^{p-1}\right)\left[\frac{\langle f^2(A) w(A)x,x\rangle}{\langle w(A)x,x\rangle} - \left(\frac{\langle w(A) f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)^2\right]^{\frac{1}{2}} \\
&\le \frac{1}{4}p\left(K^{p-1}-k^{p-1}\right)(K-k)
\end{aligned}
$$

for any $x\in H$ with $\langle w(A)x,x\rangle \neq 0$.

(ii) If we consider the function $\Psi_p(\cdot;k,K):(k,K)\to\mathbb{R}$ defined by

$$\Psi_p(t;k,K) = \frac{K^p-t^p}{K-t} - \frac{t^p-k^p}{t-k},$$

then

$$
\begin{aligned}
(3.9)\qquad 0 &\le \frac{\langle w(A) f^p(A) x,x\rangle}{\langle w(A)x,x\rangle} - \left(\frac{\langle w(A) f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)^p \\
&\le \frac{\left(K - \frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle} - k\right)}{K-k} \sup_{t\in(k,K)} \Psi_p(t;k,K) \\
&\le p\frac{K^{p-1}-k^{p-1}}{K-k}\left(K - \frac{\langle w(A) f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)\left(\frac{\langle w(A) f(A)x,x\rangle}{\langle w(A)x,x\rangle} - k\right) \\
&\le \frac{1}{4}p\left(K^{p-1}-k^{p-1}\right)(K-k)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.10)\qquad 0 &\le \frac{\langle w(A) f^p(A) x,x\rangle}{\langle w(A)x,x\rangle} - \left(\frac{\langle w(A) f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)^p \\
&\le \frac{1}{4}(K-k)\,\Psi_p\left(\frac{\langle w(A) f(A)x,x\rangle}{\langle w(A)x,x\rangle};k,K\right) \\
&\le \frac{1}{4}p\left(K^{p-1}-k^{p-1}\right)(K-k)
\end{aligned}
$$

for any $x\in H$ with $\langle w(A)x,x\rangle \neq 0$.

(iii) We have the inequalities

$$0 \le \frac{\langle w(A) f^p(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^p \le 2\max\left\{\frac{K - \frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}}{K-k}, \frac{\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle} - k}{K-k}\right\} \times \left[\frac{k^p + K^p}{2} - \left(\frac{k+K}{2}\right)^p\right] \tag{3.11}$$

and

$$0 \le \frac{\langle w(A) f^p(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^p \le 2\left[\frac{k^p + K^p}{2} - \left(\frac{k+K}{2}\right)^p\right] \tag{3.12}$$

for any $x \in H$ *with* $\langle w(A) x, x\rangle \neq 0$.

(iv) We also have the inequalities

$$\begin{aligned} 0 &\le \frac{\langle w(A) f^p(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^p \\ &\le \frac{1}{2}\Psi_p\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}; k, K\right) \frac{\left\langle \left| f(A) - \frac{\langle f(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} 1_H \right| x, x\right\rangle}{\langle w(A) x, x\rangle} \\ &\le \frac{1}{2}\Psi_p\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}; k, K\right) \\ &\quad \times \left[\frac{\langle f^2(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{4}\Psi_p\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}; k, K\right)(K-k) \end{aligned} \tag{3.13}$$

for any $x \in H$ *with* $\langle w(A) x, x\rangle \neq 0$.

If $p \in (0,1)$, then by taking $\Phi(t) = -t^p$ we can get similar inequalities. However the details are omitted.

If we take $\Phi(t) = -\ln t$, $t > 0$ in Theorem 6 then we get the following logarithmic inequalities:

Corollary 3. *Let* A *be a selfadjoint operator on the Hilbert space* H *such that* $Sp(A) \subseteq [m, M]$ *for some scalars* m, M *with* $m < M$. *Assume that* $w : [m, M] \to [0, \infty)$ *is continuous on* $[m, M]$, $f : [m, M] \subset \mathbb{R} \to \mathbb{R}$ *is a continuous function on the interval* $[m, M]$ *and satisfies the property (2.3) with* $k > 0$.

(i) We have

$$0 \leq \ln\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right) - \frac{\langle w(A) \ln f(A) x, x\rangle}{\langle w(A) x, x\rangle} \tag{3.14}$$
$$\leq \frac{\langle f^{-1}(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} \frac{\langle f(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} - 1$$
$$\leq \frac{1}{2} \frac{K-k}{kK} \frac{\left\langle \left| f(A) - \frac{\langle f(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} 1_H \right| x, x \right\rangle}{\langle w(A) x, x\rangle}$$
$$\leq \frac{1}{2} \frac{K-k}{kK} \left[\frac{\langle f^2(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4} \frac{(K-k)^2}{kK}$$

for any $x \in H$ *with* $\langle w(A) x, x\rangle \neq 0$,

(ii) If we consider the function $\Psi_{-\ln}(\cdot; k, K) : (k, K) \to \mathbb{R}$ *defined by*

$$\Psi_{-\ln}(t; k, K) = \frac{\ln t - \ln k}{t - k} - \frac{\ln K - \ln t}{K - t},$$

then

$$0 \leq \ln\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right) - \frac{\langle w(A) \ln f(A) x, x\rangle}{\langle w(A) x, x\rangle} \tag{3.15}$$
$$\leq \frac{\left(K - \frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle} - k\right)}{K-k} \sup_{t \in (k,K)} \Psi_{-\ln}(t; k, K)$$
$$\leq \frac{1}{Kk}\left(K - \frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle} - k\right) \leq \frac{1}{4} \frac{(K-k)^2}{kK}$$

and

$$0 \leq \ln\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right) - \frac{\langle w(A) \ln f(A) x, x\rangle}{\langle w(A) x, x\rangle} \tag{3.16}$$
$$\leq \frac{1}{4}(K-k)\, \Psi_{-\ln}\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}; k, K\right) \leq \frac{1}{4} \frac{(K-k)^2}{kK}$$

for any $x \in H$ *with* $\langle w(A) x, x\rangle \neq 0$.

(iii) We have the inequalities

$$0 \leq \ln\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right) - \frac{\langle w(A) \ln f(A) x, x\rangle}{\langle w(A) x, x\rangle} \tag{3.17}$$
$$\leq 2 \max\left\{\frac{K - \frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}}{K-k}, \frac{\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle} - k}{K-k}\right\} \ln\left(\frac{k+K}{2\sqrt{kK}}\right)$$

and

$$0 \leq \ln\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right) - \frac{\langle w(A) \ln f(A) x, x\rangle}{\langle w(A) x, x\rangle} \leq \ln\left(\frac{k+K}{2\sqrt{kK}}\right)^2 \tag{3.18}$$

for any $x \in H$ *with* $\langle w(A) x, x\rangle \neq 0$.

(iv) We also have the inequalities

$$0 \le \ln\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right) - \frac{\langle w(A) \ln f(A) x, x\rangle}{\langle w(A) x, x\rangle} \tag{3.19}$$
$$\le \frac{1}{2}\Psi_{-\ln}\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}; k, K\right) \frac{\left\langle \left| f(A) - \frac{\langle f(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} 1_H \right| x, x\right\rangle}{\langle w(A) x, x\rangle}$$
$$\le \frac{1}{2}\Psi_{-\ln}\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}; k, K\right)$$
$$\times \left[\frac{\langle f^2(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^2\right]^{\frac{1}{2}}$$
$$\le \frac{1}{4}\Psi_{-\ln}\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}; k, K\right)(K - k)$$

for any $x \in H$ *with* $\langle w(A) x, x\rangle \ne 0$.

4. Some Examples

If we choose $w(t) = 1$ and $f(t) = t$ with $t \in [m, M] \subset [0, \infty)$ then we get from Corollary 2 that

$$0 \le \langle A^p x, x\rangle - \langle Ax, x\rangle^p \le p\left[\langle A^p x, x\rangle - \langle A^{p-1} x, x\rangle \langle Ax, x\rangle\right] \tag{4.1}$$
$$\le \frac{1}{2} p\left(M^{p-1} - m^{p-1}\right) \langle |A - \langle Ax, x\rangle 1_H| x, x\rangle$$
$$\le \frac{1}{2} p\left(M^{p-1} - m^{p-1}\right)\left[\langle A^2 x, x\rangle - \langle Ax, x\rangle^2\right]^{\frac{1}{2}}$$
$$\le \frac{1}{4} p\left(M^{p-1} - m^{p-1}\right)(M - m),$$

$$0 \le \langle A^p x, x\rangle - \langle Ax, x\rangle^p \tag{4.2}$$
$$\le \frac{(M - \langle Ax, x\rangle)(\langle Ax, x\rangle - m)}{M - m} \sup_{t \in (m, M)} \Psi_p(t; m, M)$$
$$\le p\frac{M^{p-1} - m^{p-1}}{M - m}(M - \langle Ax, x\rangle)(\langle Ax, x\rangle - m)$$
$$\le \frac{1}{4} p\left(M^{p-1} - m^{p-1}\right)(M - m),$$

$$0 \le \langle A^p x, x\rangle - \langle Ax, x\rangle^p \le \frac{1}{4}(M - m)\Psi_p(\langle Ax, x\rangle; m, M) \tag{4.3}$$
$$\le \frac{1}{4} p\left(M^{p-1} - m^{p-1}\right)(M - m),$$

$$0 \le \langle A^p x, x\rangle - \langle Ax, x\rangle^p \tag{4.4}$$
$$\le 2\max\left\{\frac{M - \langle Ax, x\rangle}{M - m}, \frac{\langle Ax, x\rangle - m}{M - m}\right\}\left[\frac{m^p + M^p}{2} - \left(\frac{m + M}{2}\right)^p\right],$$

$$0 \le \langle A^p x, x\rangle - \langle Ax, x\rangle^p \le 2\left[\frac{m^p + M^p}{2} - \left(\frac{m + M}{2}\right)^p\right] \tag{4.5}$$

and

$$0 \le \langle A^p x, x\rangle - \langle Ax, x\rangle^p \le \frac{1}{2}\Psi_p(\langle Ax, x\rangle; m, M)\langle |A - \langle Ax, x\rangle 1_H| x, x\rangle \tag{4.6}$$
$$\le \frac{1}{2}\Psi_p(\langle Ax, x\rangle; m, M)\left[\langle A^2 x, x\rangle - \langle Ax, x\rangle^2\right]^{\frac{1}{2}}$$
$$\le \frac{1}{4}\Psi_p(\langle Ax, x\rangle; m, M)(M - m)$$

for any $x \in H$, $\|x\| = 1$.

If we choose $w(t) = t^q$, $q \neq 0$ and $f(t) = t$ with $t \in [m, M] \subset [0, \infty)$ then we get from Corollary 2 that

$$0 \le \frac{\langle A^{p+q}x, x\rangle}{\langle A^q x, x\rangle} - \left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)^p \tag{4.7}$$
$$\le p\left[\frac{\langle A^{p+q}x, x\rangle}{\langle A^q x, x\rangle} - \frac{\langle A^{p+q-1}x, x\rangle}{\langle A^q x, x\rangle}\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right]$$
$$\le \frac{1}{2}p\left(M^{p-1} - m^{p-1}\right)\frac{\left\langle \left|A - \frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}1_H\right| x, x\right\rangle}{\langle A^q x, x\rangle}$$
$$\le \frac{1}{2}p\left(M^{p-1} - m^{p-1}\right)\left[\frac{\langle A^{q+2}x, x\rangle}{\langle A^q x, x\rangle} - \left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)^2\right]^{\frac{1}{2}}$$
$$\le \frac{1}{4}p\left(M^{p-1} - m^{p-1}\right)(M - m),$$

$$0 \le \frac{\langle A^{p+q}x, x\rangle}{\langle A^q x, x\rangle} - \left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)^p \tag{4.8}$$
$$\le \frac{\left(M - \frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)\left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle} - m\right)}{M - m}\sup_{t\in(m,M)}\Psi_p(t; m, M)$$
$$\le p\frac{M^{p-1} - m^{p-1}}{M - m}\left(M - \frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)\left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle} - m\right)$$
$$\le \frac{1}{4}p\left(M^{p-1} - m^{p-1}\right)(M - m),$$

$$0 \le \frac{\langle A^{p+q}x, x\rangle}{\langle A^q x, x\rangle} - \left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)^p \le \frac{1}{4}(M - m)\Psi_p\left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}; m, M\right) \tag{4.9}$$
$$\le \frac{1}{4}p\left(M^{p-1} - m^{p-1}\right)(M - m),$$

$$0 \le \frac{\langle A^{p+q}x,x\rangle}{\langle A^{q}x,x\rangle} - \left(\frac{\langle A^{q+1}x,x\rangle}{\langle A^{q}x,x\rangle}\right)^{p} \le 2\max\left\{\frac{M-\frac{\langle A^{q+1}x,x\rangle}{\langle A^{q}x,x\rangle}}{M-m}, \frac{\frac{\langle A^{q+1}x,x\rangle}{\langle A^{q}x,x\rangle}-m}{M-m}\right\}\left[\frac{m^{p}+M^{p}}{2} - \left(\frac{m+M}{2}\right)^{p}\right], \tag{4.10}$$

$$0 \le \frac{\langle A^{p+q}x,x\rangle}{\langle A^{q}x,x\rangle} - \left(\frac{\langle A^{q+1}x,x\rangle}{\langle A^{q}x,x\rangle}\right)^{p} \le 2\left[\frac{m^{p}+M^{p}}{2} - \left(\frac{m+M}{2}\right)^{p}\right] \tag{4.11}$$

and

$$\begin{aligned} 0 &\le \frac{\langle A^{p+q}x,x\rangle}{\langle A^{q}x,x\rangle} - \left(\frac{\langle A^{q+1}x,x\rangle}{\langle A^{q}x,x\rangle}\right)^{p} \\ &\le \frac{1}{2}\Psi_{p}\left(\frac{\langle A^{q+1}x,x\rangle}{\langle A^{q}x,x\rangle}; m, M\right)\frac{\left\langle \left| A - \frac{\langle A^{q+1}x,x\rangle}{\langle A^{q}x,x\rangle} 1_{H}\right| x,x\right\rangle}{\langle A^{q}x,x\rangle} \\ &\le \frac{1}{2}\Psi_{p}\left(\frac{\langle A^{q+1}x,x\rangle}{\langle A^{q}x,x\rangle}; m, M\right)\left[\frac{\langle A^{q+2}x,x\rangle}{\langle A^{q}x,x\rangle} - \left(\frac{\langle A^{q+1}x,x\rangle}{\langle A^{q}x,x\rangle}\right)^{2}\right]^{\frac{1}{2}} \\ &\le \frac{1}{4}\Psi_{p}\left(\frac{\langle A^{q+1}x,x\rangle}{\langle A^{q}x,x\rangle}; m, M\right)(M-m) \end{aligned} \tag{4.12}$$

for any $x \in H \setminus \{0\}$.

If we choose $w(t) = 1$ and $f(t) = t$ with $t \in [m, M] \subset [0, \infty)$ then we get from Corollary 3 that

$$\begin{aligned} 0 &\le \ln\langle Ax,x\rangle - \langle \ln Ax,x\rangle \le \langle A^{-1}x,x\rangle\langle Ax,x\rangle - 1 \\ &\le \frac{1}{2}\frac{M-m}{mM}\langle |A - \langle Ax,x\rangle 1_{H}| x,x\rangle \\ &\le \frac{1}{2}\frac{M-m}{mM}\left[\langle A^{2}x,x\rangle - \langle Ax,x\rangle^{2}\right]^{\frac{1}{2}} \le \frac{1}{4}\frac{(M-m)^{2}}{mM}, \end{aligned} \tag{4.13}$$

$$\begin{aligned} 0 &\le \ln\langle Ax,x\rangle - \langle \ln Ax,x\rangle \\ &\le \frac{(M-\langle Ax,x\rangle)(\langle Ax,x\rangle - m)}{M-m}\sup_{t\in(m,M)}\Psi_{-\ln}(t; m, M) \\ &\le \frac{1}{Mm}(M-\langle Ax,x\rangle)(\langle Ax,x\rangle - m) \le \frac{1}{4}\frac{(M-m)^{2}}{mM}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} 0 &\le \ln\langle Ax,x\rangle - \langle \ln Ax,x\rangle \le \frac{1}{4}(M-m)\Psi_{-\ln}(\langle Ax,x\rangle; m, M) \\ &\le \frac{1}{4}\frac{(M-m)^{2}}{mM}, \end{aligned} \tag{4.15}$$

$$\begin{aligned} 0 &\le \ln\langle Ax,x\rangle - \langle \ln Ax,x\rangle \\ &\le 2\max\left\{\frac{M-\langle Ax,x\rangle}{M-m}, \frac{\langle Ax,x\rangle - m}{M-m}\right\}\ln\left(\frac{m+M}{2\sqrt{mM}}\right), \end{aligned} \tag{4.16}$$

$$0 \le \ln \langle Ax, x\rangle - \langle \ln Ax, x\rangle \le \ln \left(\frac{m+M}{2\sqrt{mM}}\right)^2 \tag{4.17}$$

and

$$\begin{aligned} 0 &\le \ln \langle Ax, x\rangle - \langle \ln Ax, x\rangle \\ &\le \frac{1}{2}\Psi_{-\ln}\left(\langle Ax, x\rangle; m, M\right)\langle \left|f(A) - \langle Ax, x\rangle 1_H\right| x, x\rangle \\ &\le \frac{1}{2}\Psi_{-\ln}\left(\langle Ax, x\rangle; m, M\right)\left[\left\langle A^2x, x\right\rangle - \langle Ax, x\rangle^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{4}\Psi_{-\ln}\left(\langle Ax, x\rangle; m, M\right)(M-m) \end{aligned} \tag{4.18}$$

for any $x \in H$ with $\|x\| = 1$.

If we choose $w(t) = t^q$, $q \neq 0$ and $f(t) = t$ with $t \in [m, M] \subset [0, \infty)$ then we get from Corollary 3 that

$$\begin{aligned} 0 &\le \ln\left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}\right) - \frac{\langle A^q \ln Ax, x\rangle}{\langle A^q x, x\rangle} \\ &\le \frac{\left\langle A^{q-1}x, x\right\rangle}{\langle A^q x, x\rangle}\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle} - 1 \le \frac{1}{2}\frac{K-k}{kK}\frac{\left\langle \left|A - \frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}1_H\right| x, x\right\rangle}{\langle A^q x, x\rangle} \\ &\le \frac{1}{2}\frac{K-k}{kK}\left[\frac{\left\langle A^{q+2}x, x\right\rangle}{\langle A^q x, x\rangle} - \left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}\right)^2\right]^{\frac{1}{2}} \le \frac{1}{4}\frac{(K-k)^2}{kK}, \end{aligned} \tag{4.19}$$

$$\begin{aligned} 0 &\le \ln\left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}\right) - \frac{\langle A^q \ln Ax, x\rangle}{\langle A^q x, x\rangle} \\ &\le \frac{\left(K - \frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}\right)\left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle} - k\right)}{K-k}\sup_{t \in (k, K)}\Psi_{-\ln}(t; k, K) \\ &\le \frac{1}{Kk}\left(K - \frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}\right)\left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle} - k\right) \le \frac{1}{4}\frac{(K-k)^2}{kK}, \end{aligned} \tag{4.20}$$

$$\begin{aligned} 0 &\le \ln\left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}\right) - \frac{\langle A^q \ln Ax, x\rangle}{\langle A^q x, x\rangle} \\ &\le \frac{1}{4}(K-k)\Psi_{-\ln}\left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}; k, K\right) \le \frac{1}{4}\frac{(K-k)^2}{kK}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} 0 &\le \ln\left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}\right) - \frac{\langle A^q \ln Ax, x\rangle}{\langle A^q x, x\rangle} \\ &\le 2\max\left\{\frac{K - \frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}}{K-k}, \frac{\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle} - k}{K-k}\right\}\ln\left(\frac{k+K}{2\sqrt{kK}}\right), \end{aligned} \tag{4.22}$$

$$0 \le \ln\left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right) - \frac{\langle A^q \ln Ax, x\rangle}{\langle A^q x, x\rangle} \le \ln\left(\frac{k+K}{2\sqrt{kK}}\right)^2, \tag{4.23}$$

and

$$\begin{aligned}
0 &\le \ln\left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right) - \frac{\langle A^q \ln Ax, x\rangle}{\langle A^q x, x\rangle} \\
&\le \frac{1}{2}\Psi_{-\ln}\left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}; k, K\right) \frac{\left\langle \left| A - \frac{\langle A^{q+1}x,x\rangle}{\langle A^q x,x\rangle} 1_H \right| x, x \right\rangle}{\langle A^q x, x\rangle} \\
&\le \frac{1}{2}\Psi_{-\ln}\left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}; k, K\right) \left[\frac{\langle A^{q+2}x, x\rangle}{\langle A^q x, x\rangle} - \left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)^2\right]^{\frac{1}{2}} \\
&\le \frac{1}{4}\Psi_{-\ln}\left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}; k, K\right)(K-k)
\end{aligned} \tag{4.24}$$

for any $x \in H \setminus \{0\}$.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa