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REFINING CBS INTEGRAL INEQUALITY FOR PARTITIONS OF WEIGHTS

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Abstract. In this paper we establish a refinement and some reverses for Cauchy-Bunyakovsky-Schwarz inequality for the general Lebesgue integral on measurable spaces and partitions of weights. Applications for discrete inequalities and weighted means of positive numbers are also given.

1. Introduction

The Cauchy-Bunyakovsky-Schwarz inequality, or for short, the CBS inequality, plays an important role in different branches of Modern Mathematics including Hilbert Spaces Theory, Probability & Statistics, Classical Real and Complex Analysis, Numerical Analysis, Qualitative Theory of Differential Equations and their applications.

Let (Ω,Σ,μ) be a measure space consisting of a set Ω, a σ-algebra of subsets of Ω denoted by Σ and μ a countably additive and positive measure on Σ with values in $\mathbb{R}\cup\{\infty\}$. Denote by $L_w^2(\Omega,\mu)$ the Hilbert space of all $\mathbb{C}$-valued functions f defined on Ω that are 2-w-integrable on Ω, i.e., $\int_\Omega w(x)|f(x)|^2 d\mu(x)<\infty$, where $w:\Omega\to[0,\infty)$ is a given μ-measurable function on Ω. We write for simplicity $\int_\Omega w|f|^2 d\mu$ instead of $\int_\Omega w(x)|f(x)|^2 d\mu(x)$.

The following inequality is well known in the literature as the integral Cauchy-Bunyakovsky-Schwarz inequality:

$$\left|\int_\Omega wfg\,d\mu\right|^2\le\int_\Omega w|f|^2 d\mu\int_\Omega w|g|^2 d\mu \tag{CBS}$$

provided that $f,g\in L_w^2(\Omega,\mu)$.

For the μ-integrable positive μ-a.e. weight w and a given $n\ge 2$ we consider the set $\mathfrak{P}_n(w)$ all possible n-tuples of μ-integrable positive μ-a.e. weights $\overline{w}=(w_1,....,w_n)$ with the property that $\sum_{i=1}^n w_i=w$. It is clear that $\sum_{i=1}^n\int_\Omega w_i d\mu=\int_\Omega w\,d\mu$ for any $\overline{w}=(w_1,....,w_n)\in\mathfrak{P}_n(w)$ and $\int_\Omega w_i d\mu>0$.

For $f,g\in L_w^2(\Omega,\mu)$ we consider the functional $\beta(|f|,|g|,\cdot):\mathfrak{P}_n(w)\to[0,\infty)$ defined by

$$\beta(|f|,|g|,\overline{w}):=\sum_{i=1}^n\left(\int_\Omega w_i|f|^2 d\mu\right)^{1/2}\left(\int_\Omega w_i|g|^2 d\mu\right)^{1/2}. \tag{1.1}$$

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In this paper we establish some inequalities concerning the functional $\beta(|f|,|g|,\cdot)$ that provide refinements and reverses for the CBS integral inequality (CBS). Applications for discrete inequalities and weighted means of positive numbers are also given.

For recent papers on CBS inequality, see [1], [2], [8], [9], [10], [13], [14], [15], [16], [18] and the references therein.

2. Some General Facts

The following refinement of the CBS inequality holds:

Theorem 1. *For $f,g\in L_w^2(\Omega,\mu)$ and $\overline{w}=(w_1,....,w_n)\in\mathfrak{P}_n(w)$ we have*

$$\left|\int_\Omega wfgd\mu\right|\le\beta(|f|,|g|,\overline{w})\le\left(\int_\Omega w|f|^2d\mu\right)^{1/2}\left(\int_\Omega w|g|^2d\mu\right)^{1/2}. \tag{2.1}$$

Proof. We have by CBS integral inequality in $L_w^2(\Omega,\mu)$ that

$$\begin{aligned}\left|\int_\Omega wfgd\mu\right|&\le\int_\Omega w|fg|d\mu=\int_\Omega\sum_{i=1}^n w_i|fg|d\mu=\sum_{i=1}^n\int_\Omega w_i|fg|d\mu\\&\le\sum_{i=1}^n\left(\int_\Omega w_i|f|^2d\mu\right)^{1/2}\left(\int_\Omega w_i|g|^2d\mu\right)^{1/2}=\beta(|f|,|g|,\overline{w}),\end{aligned}$$

which proves the first inequality in (2.1).

By the CBS discrete inequality we also have

$$\begin{aligned}\beta(|f|,|g|,\overline{w})&=\sum_{i=1}^n\left(\int_\Omega w_i|f|^2d\mu\right)^{1/2}\left(\int_\Omega w_i|g|^2d\mu\right)^{1/2}\\&\le\left[\sum_{i=1}^n\left(\left(\int_\Omega w_i|f|^2d\mu\right)^{1/2}\right)^2\right]^{1/2}\left[\sum_{i=1}^n\left(\left(\int_\Omega w_i|g|^2d\mu\right)^{1/2}\right)^2\right]^{1/2}\\&=\left[\sum_{i=1}^n\int_\Omega w_i|f|^2d\mu\right]^{1/2}\left[\sum_{i=1}^n\int_\Omega w_i|g|^2d\mu\right]^{1/2}\\&=\left[\int_\Omega\left(\sum_{i=1}^n w_i\right)|f|^2d\mu\right]^{1/2}\left[\int_\Omega\left(\sum_{i=1}^n w_i\right)|g|^2d\mu\right]^{1/2}\\&=\left[\int_\Omega w|f|^2d\mu\right]^{1/2}\left[\int_\Omega w|g|^2d\mu\right]^{1/2},\end{aligned}$$

which proves the second inequality in (2.1). □

We observe that $\mathfrak{P}_n(w)$ is a convex set. Indeed if $\overline{w}=(w_1,....,w_n)$, $\overline{p}=(p_1,....,p_n)\in\mathfrak{P}_n(w)$ then for $t\in[0,1]$ we have

$$\sum_{i=1}^n[(1-t)w_i+tp_i]=(1-t)\sum_{i=1}^n w_i+t\sum_{i=1}^n p_i=(1-t)w+tw=w,$$

which shows that $(1-t)\overline{w}+t\overline{p}\in\mathfrak{P}_n(w)$.

Theorem 2. *For $f,g\in L_w^2(\Omega,\mu)$ we have that $\beta(|f|,|g|,\cdot)$ is a concave mapping on $\mathfrak{P}_n(w)$.*

Proof. Let $\overline{w}=(w_1,....,w_n)$, $\overline{p}=(p_1,....,p_n)\in\mathfrak{P}_n(w)$ and $t\in[0,1]$. Then

$$\begin{aligned}(2.2)\quad &\beta(|f|,|g|,(1-t)\overline{w}+t\overline{p})\\ &=\sum_{i=1}^{n}\left(\int_\Omega((1-t)w_i+tp_i)|f|^2d\mu\right)^{1/2}\left(\int_\Omega((1-t)w_i+tp_i)|g|^2d\mu\right)^{1/2}\\ &=\sum_{i=1}^{n}\left(\left((1-t)\int_\Omega w_i|f|^2d\mu+t\int_\Omega p_i|f|^2d\mu\right)\right)^{1/2}\\ &\times\left(\left((1-t)\int_\Omega w_i|g|^2d\mu+t\int_\Omega p_i|g|^2d\mu\right)\right)^{1/2}.\end{aligned}$$

By the elementary inequality

$$\left[(1-t)a^2+tb^2\right]^{1/2}\left[(1-t)c^2+td^2\right]^{1/2}\ge(1-t)ac+tbd$$

that holds for the nonnegative numbers a,b,c,d and $t\in[0,1]$, we have

$$\begin{aligned}(2.3)\quad &\left(\left((1-t)\int_\Omega w_i|f|^2d\mu+t\int_\Omega p_i|f|^2d\mu\right)\right)^{1/2}\\ &\times\left(\left((1-t)\int_\Omega w_i|g|^2d\mu+t\int_\Omega p_i|g|^2d\mu\right)\right)^{1/2}\\ &=\left(\left((1-t)\left[\left(\int_\Omega w_i|f|^2d\mu\right)^{1/2}\right]^2+t\left[\left(\int_\Omega p_i|f|^2d\mu\right)^{1/2}\right]^2\right)\right)^{1/2}\\ &\times\left(\left((1-t)\left[\left(\int_\Omega w_i|g|^2d\mu\right)^{1/2}\right]^2+t\left[\left(\int_\Omega p_i|g|^2d\mu\right)^{1/2}\right]^2\right)\right)^{1/2}\\ &\ge(1-t)\left(\int_\Omega w_i|f|^2d\mu\right)^{1/2}\left(\int_\Omega w_i|g|^2d\mu\right)^{1/2}\\ &+t\left(\int_\Omega p_i|f|^2d\mu\right)^{1/2}\left(\int_\Omega p_i|g|^2d\mu\right)^{1/2}\end{aligned}$$

for any $i\in\{1,...,n\}$ and $t\in[0,1]$.

If we sum over i from 1 to n in the inequality (2.3) and use (2.2), then we get

$$\begin{aligned}&\beta(|f|,|g|,(1-t)\overline{w}+t\overline{p})\\ &\ge(1-t)\sum_{i=1}^{n}\left(\int_\Omega w_i|f|^2d\mu\right)^{1/2}\left(\int_\Omega w_i|g|^2d\mu\right)^{1/2}\\ &+t\sum_{i=1}^{n}\left(\int_\Omega p_i|f|^2d\mu\right)^{1/2}\left(\int_\Omega p_i|g|^2d\mu\right)^{1/2}\\ &=(1-t)\beta(|f|,|g|,\overline{w})+t\beta(|f|,|g|,\overline{p})\end{aligned}$$

and the concavity of the mapping $\beta(|f|,|g|,\cdot)$ is proven. □

In the next section we provide some lower bounds and reverse inequalities for the functional $\beta(|f|,|g|,\cdot)$ when some additional information for the functions involved is known.

3. Lower Bounds

The following result holds:

Theorem 3. *Let* $f, g \in L_w^2(\Omega, \mu)$ *and* $\overline{w} = (w_1, ..., w_n) \in \mathfrak{P}_n(w)$ *be such that there exists* $k, K, l, L > 0$ *with the property*

$$0 < k \le \left(\int_\Omega w_i |f|^2 \, d\mu\right)^{1/2} \le K < \infty \tag{3.1}$$

and

$$0 < l \le \left(\int_\Omega w_i |g|^2 \, d\mu\right)^{1/2} \le L < \infty \tag{3.2}$$

for each $i \in \{1, ..., n\}$. *Then*

$$\left(\int_\Omega w |f|^2 \, d\mu\right)^{1/2} \left(\int_\Omega w |g|^2 \, d\mu\right)^{1/2} \le \frac{kl + KL}{2\sqrt{klKL}} \beta(|f|, |g|, \overline{w}) \tag{3.3}$$

and

$$0 \le \frac{\int_\Omega w |f|^2 \, d\mu}{\beta(|f|, |g|, \overline{w})} - \frac{\beta(|f|, |g|, \overline{w})}{\int_\Omega w |g|^2 \, d\mu} \le \left(\sqrt{\frac{K}{l}} - \sqrt{\frac{k}{L}}\right)^2. \tag{3.4}$$

Proof. We use the Pólya-Szegö inequality that states that [17] (see also [4, p. 74]), if $0 < a \le a_i \le A < \infty$ and $0 < b \le b_i \le B < \infty$, $i \in \{1, ..., n\}$ then

$$\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2}{\left(\sum_{i=1}^n a_i b_i\right)^2} \le \frac{(ab + AB)^2}{4abAB}. \tag{3.5}$$

Now, if we take $a_i = \left(\int_\Omega w_i |f|^2 \, d\mu\right)^{1/2}$, $b_i = \left(\int_\Omega w_i |g|^2 \, d\mu\right)^{1/2}$, $i \in \{1, ..., n\}$, $a = k$, $A = K$, $b = l$ and $B = L$, then by (3.5) we get

$$\begin{aligned} &\left(\int_\Omega \sum_{i=1}^n w_i |f|^2 \, d\mu\right) \left(\int_\Omega \sum_{i=1}^n w_i |k|^2 \, d\mu\right) \\ &\le \frac{(kl + KL)^2}{4klKL} \left(\sum_{i=1}^n \left(\int_\Omega w_i |f|^2 \, d\mu\right)^{1/2} \left(\int_\Omega w_i |g|^2 \, d\mu\right)^{1/2}\right)^2 \end{aligned} \tag{3.6}$$

and the inequality (3.3) is proved.

We use now the Shisha-Mond inequality [19] (see also [4, p. 82]) that says that

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2 \le \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}}\right)^2 \sum_{i=1}^n a_i b_i \sum_{i=1}^n b_i^2 \tag{3.7}$$

provided $0 < a \le a_i \le A < \infty$ and $0 < b \le b_i \le B < \infty$, $i \in \{1, ..., n\}$.

Now, if we take $a_i = \left(\int_\Omega w_i |f|^2 d\mu\right)^{1/2}$, $b_i = \left(\int_\Omega w_i |g|^2 d\mu\right)^{1/2}$, $i \in \{1, ..., n\}$, $a = k$, $A = K$, $b = l$ and $B = L$, then by (3.7) we get

$$\sum_{i=1}^n \int_\Omega w_i |f|^2 d\mu \sum_{i=1}^n \int_\Omega w_i |g|^2 d\mu - \left(\sum_{i=1}^n \left(\int_\Omega w_i |f|^2 d\mu\right)^{1/2} \left(\int_\Omega w_i |g|^2 d\mu\right)^{1/2}\right)^2$$
$$\leq \left(\sqrt{\frac{K}{l}} - \sqrt{\frac{k}{L}}\right)^2 \sum_{i=1}^n \left(\int_\Omega w_i |f|^2 d\mu\right)^{1/2} \left(\int_\Omega w_i |g|^2 d\mu\right)^{1/2} \sum_{i=1}^n \int_\Omega w_i |g|^2 d\mu,$$

which is equivalent to (3.4). □

Corollary 1. *Let $f, g \in L_w^2(\Omega, \mu)$ and $\overline{w} = (w_1, ..., w_n) \in \mathfrak{P}_n(w)$ be such that there exists $m, M > 0$ with the property*

$$0 < m \leq w_i(x) \leq M < \infty \ \mu\text{-a.e. on } \Omega \tag{3.8}$$

for each $i \in \{1, ..., n\}$. Then

$$\left(\int_\Omega w |f|^2 d\mu\right)^{1/2} \left(\int_\Omega w |g|^2 d\mu\right)^{1/2} \leq \frac{m + M}{2\sqrt{mM}} \beta(|f|, |g|, \overline{w}) \tag{3.9}$$

and

$$0 \leq \frac{\int_\Omega w |f|^2 d\mu}{\beta(|f|, |g|, \overline{w})} - \frac{\beta(|f|, |g|, \overline{w})}{\int_\Omega w |g|^2 d\mu} \tag{3.10}$$
$$\leq \left(\frac{M^{1/4}}{m^{1/4}} - \frac{m^{1/4}}{M^{1/4}}\right)^2 \left(\frac{\int_\Omega |f|^2 d\mu}{\int_\Omega |g|^2 d\mu}\right)^{1/2}.$$

Proof. From (3.8) we have

$$0 < m \int_\Omega |f|^2 d\mu \leq \int_\Omega w_i |f|^2 d\mu \leq M \int_\Omega |f|^2 d\mu < \infty$$

giving that

$$0 < m^{1/2} \left(\int_\Omega |f|^2 d\mu\right)^{1/2} \leq \left(\int_\Omega w_i |f|^2 d\mu\right)^{1/2} \leq M^{1/2} \left(\int_\Omega |f|^2 d\mu\right)^{1/2} < \infty$$

and, similarly

$$0 < m^{1/2} \left(\int_\Omega |g|^2 d\mu\right)^{1/2} \leq \left(\int_\Omega w_i |g|^2 d\mu\right)^{1/2} \leq M^{1/2} \left(\int_\Omega |g|^2 d\mu\right)^{1/2} < \infty,$$

for any $i \in \{1, ..., n\}$.

Now, if we apply Theorem 3 for $k = m^{1/2} \left(\int_\Omega |f|^2 d\mu\right)^{1/2}$, $K = M^{1/2} \left(\int_\Omega |f|^2 d\mu\right)^{1/2}$, $l = m^{1/2} \left(\int_\Omega |g|^2 d\mu\right)^{1/2}$ and $L = M^{1/2} \left(\int_\Omega |g|^2 d\mu\right)^{1/2}$ we get the desired inequalities (3.9) and (3.10). □

Corollary 2. *Let $f, g \in L_w^2(\Omega, \mu)$ and $\overline{w} = (w_1, ..., w_n) \in \mathfrak{P}_n(w)$ be such that there exists $a, b, A, B > 0$ with the property*

$$0 < a \leq |f| \leq A < \infty \text{ and } 0 < b \leq |g| \leq B < \infty \ \mu\text{-a.e. on } \Omega. \tag{3.11}$$

Then

$$\left(\int_\Omega w\left|f\right|^2 d\mu\right)^{1/2}\left(\int_\Omega w\left|g\right|^2 d\mu\right)^{1/2}\leq\frac{abw_0^2+W_0^2AB}{2w_0W_0\sqrt{abAB}}\beta\left(\left|f\right|,\left|g\right|,\overline{w}\right), \tag{3.12}$$

and

$$0\leq\frac{\int_\Omega w\left|f\right|^2 d\mu}{\beta\left(\left|f\right|,\left|g\right|,\overline{w}\right)}-\frac{\beta\left(\left|f\right|,\left|g\right|,\overline{w}\right)}{\int_\Omega w\left|g\right|^2 d\mu}\leq\left(\sqrt{\frac{AW_0}{bw_0}}-\sqrt{\frac{aw_0}{BW_0}}\right)^2, \tag{3.13}$$

where

$$w_0:=\min_{i\in\{1,...,n\}}\left(\int_\Omega w_i d\mu\right)^{1/2}\text{ and }W_0:=\max_{i\in\{1,...,n\}}\left(\int_\Omega w_i d\mu\right)^{1/2}.$$

Proof. From (3.11) we have

$$a\left(\int_\Omega w_i d\mu\right)^{1/2}\leq\left(\int_\Omega w_i\left|f\right|^2 d\mu\right)^{1/2}\leq A\left(\int_\Omega w_i d\mu\right)^{1/2}<\infty,$$

which implies that

$$aw_0\leq\left(\int_\Omega w_i\left|f\right|^2 d\mu\right)^{1/2}\leq AW_0<\infty$$

for any $i\in\{1,...,n\}$.

Similarly, we have

$$bw_0\leq\left(\int_\Omega w_i\left|g\right|^2 d\mu\right)^{1/2}\leq BW_0<\infty,$$

for any $i\in\{1,...,n\}$.

By applying Theorem 3 for the choices $k=aw_0$, $K=AW_0$, $l=bw_0$ and $L=BW_0$ we get the desired results (3.12) and (3.13). □

The following result holds:

Theorem 4. *Let $f,g\in L_w^2(\Omega,\mu)$ and $\overline{w}=(w_1,....,w_n)\in\mathfrak{P}_n(w)$ be such that there exists $k,K,l,L>0$ with the property (3.1) and (3.2) for each $i\in\{1,...,n\}$. Then*

$$0\leq\int_\Omega w\left|f\right|^2 d\mu\int_\Omega w\left|g\right|^2 d\mu-\beta^2\left(\left|f\right|,\left|g\right|,\overline{w}\right)\leq\frac{1}{3}n^2\left(KL-kl\right)^2 \tag{3.14}$$

and

$$\int_\Omega w\left|g\right|^2 d\mu+\frac{lL}{kK}\int_\Omega w\left|f\right|^2 d\mu\leq\left(\frac{L}{k}+\frac{l}{K}\right)\beta\left(\left|f\right|,\left|g\right|,\overline{w}\right). \tag{3.15}$$

Proof. We use the following Ozeki's type inequality [12]

$$\sum_{i=1}^n a_i^2\sum_{i=1}^n b_i^2-\left(\sum_{i=1}^n a_ib_i\right)^2\leq\frac{1}{3}n^2\left(AB-ab\right)^2 \tag{3.16}$$

provided $0<a\leq a_i\leq A<\infty$ and $0<b\leq b_i\leq B<\infty$, $i\in\{1,...,n\}$.

Now, if we take $a_i = \left(\int_\Omega w_i |f|^2 d\mu\right)^{1/2}$, $b_i = \left(\int_\Omega w_i |g|^2 d\mu\right)^{1/2}$, $i \in \{1, ..., n\}$, $a = k$, $A = K$, $b = l$ and $B = L$, then by (3.16) we get

$$\begin{aligned} &\sum_{i=1}^{n} \int_\Omega w_i |f|^2 d\mu \sum_{i=1}^{n} \int_\Omega w_i |g|^2 d\mu \\ &- \left(\sum_{i=1}^{n} \left(\int_\Omega w_i |f|^2 d\mu\right)^{1/2} \left(\int_\Omega w_i |g|^2 d\mu\right)^{1/2}\right)^2 \\ &\le \frac{1}{3} n^2 (KL - kl)^2, \end{aligned} \tag{3.17}$$

which proves (3.14).

Further, we recall Diaz-Metcalf's inequality [3] (see also [4, p. 123])

$$\sum_{i=1}^{n} b_i^2 + \frac{bB}{aA} \sum_{i=1}^{n} a_i^2 \le \left(\frac{B}{a} + \frac{b}{A}\right) \sum_{i=1}^{n} a_i b_i \tag{3.18}$$

provided $0 < a \le a_i \le A < \infty$ and $0 < b \le b_i \le B < \infty$, $i \in \{1, ..., n\}$.

If we take $a_i = \left(\int_\Omega w_i |f|^2 d\mu\right)^{1/2}$, $b_i = \left(\int_\Omega w_i |g|^2 d\mu\right)^{1/2}$, $i \in \{1, ..., n\}$, $a = k$, $A = K$, $b = l$ and $B = L$, then by (3.18) we get

$$\begin{aligned} &\sum_{i=1}^{n} \int_\Omega w_i |g|^2 d\mu + \frac{bB}{aA} \sum_{i=1}^{n} \int_\Omega w_i |f|^2 d\mu \\ &\le \left(\frac{B}{a} + \frac{b}{A}\right) \sum_{i=1}^{n} \left(\int_\Omega w_i |f|^2 d\mu\right)^{1/2} \left(\int_\Omega w_i |g|^2 d\mu\right)^{1/2} \end{aligned} \tag{3.19}$$

that is equivalent to (3.15). □

Corollary 3. *Let $f, g \in L_w^2(\Omega, \mu)$ and $\overline{w} = (w_1, ..., w_n) \in \mathfrak{P}_n(w)$ be such that there exists $m, M > 0$ with the property (3.8). Then*

$$\begin{aligned} 0 &\le \int_\Omega w |f|^2 d\mu \int_\Omega w |g|^2 d\mu - \beta^2 (|f|, |g|, \overline{w}) \\ &\le \frac{1}{3} n^2 (M - m)^2 \int_\Omega |f|^2 d\mu \int_\Omega |g|^2 d\mu \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &\int_\Omega w |g|^2 d\mu + \frac{\int_\Omega |g|^2 d\mu}{\int_\Omega |f|^2 d\mu} \int_\Omega w |f|^2 d\mu \\ &\le \left(\frac{M^{1/2}}{m^{1/2}} + \frac{m^{1/2}}{M^{1/2}}\right) \beta (|f|, |g|, \overline{w}) \frac{\left(\int_\Omega |g|^2 d\mu\right)^{1/2}}{\left(\int_\Omega |f|^2 d\mu\right)^{1/2}}. \end{aligned} \tag{3.21}$$

Proof. Follows by the inequalities (3.14) and (3.15) for $k = m^{1/2} \left(\int_\Omega |f|^2 d\mu\right)^{1/2}$, $K = M^{1/2} \left(\int_\Omega |f|^2 d\mu\right)^{1/2}$, $l = m^{1/2} \left(\int_\Omega |g|^2 d\mu\right)^{1/2}$ and $L = M^{1/2} \left(\int_\Omega |g|^2 d\mu\right)^{1/2}$. □

We also have:

Corollary 4. *Let $f,g \in L_w^2(\Omega,\mu)$ and $\overline{w}=(w_1,....,w_n) \in \mathfrak{P}_n(w)$ be such that there exists $a,b,A,B>0$ with the property (3.11). Then*

$$0 \le \int_\Omega w|f|^2 d\mu \int_\Omega w|g|^2 d\mu - \beta^2(|f|,|g|,\overline{w}) \le \frac{1}{3}n^2\left(ABW_0^2 - abw_0^2\right)^2 \tag{3.22}$$

and

$$\int_\Omega w|g|^2 d\mu + \frac{bB}{aA}\int_\Omega w|f|^2 d\mu \le \left(\frac{BW_0}{aw_0}+\frac{bw_0}{AW_0}\right)\beta(|f|,|g|,\overline{w}). \tag{3.23}$$

Proof. Follows by the inequalities (3.14) and (3.15) for the choices $k=aw_0$, $K=AW_0$, $l=bw_0$ and $L=BW_0$. □

The following result holds:

Theorem 5. *Let $f,g \in L_w^2(\Omega,\mu)$ and $\overline{w}=(w_1,....,w_n) \in \mathfrak{P}_n(w)$ be such that there exists $p,P>0$ with the property*

$$0<p\le \frac{\left(\int_\Omega w_i|f|^2 d\mu\right)^{1/2}}{\left(\int_\Omega w_i|g|^2 d\mu\right)^{1/2}} \le P<\infty \tag{3.24}$$

for each $i\in\{1,...,n\}$. Then

$$\int_\Omega w|f|^2 d\mu \int_\Omega w|g|^2 d\mu \le \frac{(p+P)^2}{4pP}\beta^2(|f|,|g|,\overline{w}), \tag{3.25}$$

$$\begin{aligned} 0 &\le \left(\int_\Omega w|f|^2 d\mu\right)^{1/2}\left(\int_\Omega w|g|^2 d\mu\right)^{1/2} - \beta(|f|,|g|,\overline{w}) \\ &\le \frac{(P-p)^2}{4(p+P)}\int_\Omega w|g|^2 d\mu \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} 0 &\le \int_\Omega w|f|^2 d\mu \int_\Omega w|g|^2 d\mu - \beta^2(|f|,|g|,\overline{w}) \\ &\le \frac{1}{4}(P-p)^2\left(\int_\Omega w|g|^2 d\mu\right)^2. \end{aligned} \tag{3.27}$$

Proof. We use the following Cassels' inequality [20] (see also [4, p. 72])

$$\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2}{\left(\sum_{i=1}^n a_i b_i\right)^2} \le \frac{(c+C)^2}{4cC} \tag{3.28}$$

that holds provided

$$0<c\le\frac{a_i}{b_i}\le C<\infty \tag{3.29}$$

for any $i\in\{1,...,n\}$.

If we take $a_i=\left(\int_\Omega w_i|f|^2 d\mu\right)^{1/2}$, $b_i=\left(\int_\Omega w_i|g|^2 d\mu\right)^{1/2}$, $i\in\{1,...,n\}$, $c=p$ and $C=P$, then by (3.28) we get

$$\frac{\sum_{i=1}^n \int_\Omega w_i|f|^2 d\mu \sum_{i=1}^n \int_\Omega w_i|g|^2 d\mu}{\left(\sum_{i=1}^n \left(\int_\Omega w_i|f|^2 d\mu\right)^{1/2}\left(\int_\Omega w_i|g|^2 d\mu\right)^{1/2}\right)^2} \le \frac{(p+P)^2}{4pP},$$

which proves (3.25).

Further, we use the following Shisha-Mond inequality [19] (see also [4, p. 82])

$$\left(\sum_{i=1}^{n}a_i^2\right)^{1/2}\left(\sum_{i=1}^{n}b_i^2\right)^{1/2}-\sum_{i=1}^{n}a_ib_i\le\frac{(C-c)^2}{4(c+C)}\sum_{i=1}^{n}b_i^2 \tag{3.30}$$

that holds provided the condition (3.29) is valid.

Then by taking $a_i=\left(\int_\Omega w_i\left|f\right|^2d\mu\right)^{1/2}$, $b_i=\left(\int_\Omega w_i\left|g\right|^2d\mu\right)^{1/2}$, $i\in\{1,...,n\}$, $c=p$ and $C=P$ in (3.30) we get the desired result (3.26).

We use the following reverse of CBS inequality [4, p. 78]

$$\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2-\left(\sum_{i=1}^{n}a_ib_i\right)^2\le\frac{1}{4}(C-c)^2\sum_{i=1}^{n}b_i^2$$

provided the condition (3.29) is valid, for the choice $a_i=\left(\int_\Omega w_i\left|f\right|^2d\mu\right)^{1/2}$, $b_i=\left(\int_\Omega w_i\left|g\right|^2d\mu\right)^{1/2}$, $i\in\{1,...,n\}$, $c=p$ and $C=P$. Simple calculation yields the desired inequality (3.27). □

In the next section we provide some upper bounds and reverse inequalities for the functional $\beta(|f|,|g|,\cdot)$ when some additional information for the functions involved is known.

4. Upper Bounds

In [5] (see also [7, p. 14]) we have shown amongst other that, if $u,v\in L_w^2(\Omega,\mu)$ and there are $a,A\in\mathbb{C}$ with $\operatorname{Re}(\overline{a}A)>0$ and such that

$$\int_\Omega w\operatorname{Re}\left[(Au-v)(\overline{v}-\overline{au})\right]d\mu\ge0, \tag{4.1}$$

then

$$\begin{aligned}\left(\int_\Omega|u|^2wd\mu\right)^{1/2}\left(\int_\Omega|v|^2wd\mu\right)^{1/2}&\le\frac{1}{2}\frac{\int_\Omega w\operatorname{Re}\left(A\overline{v}u+\overline{a}v\overline{u}\right)d\mu}{\sqrt{\operatorname{Re}(\overline{a}A)}}\\&=\frac{1}{2}\frac{\int_\Omega w\operatorname{Re}\left(\overline{A\overline{v}u}+\overline{a}v\overline{u}\right)d\mu}{\sqrt{\operatorname{Re}(\overline{a}A)}}\\&=\frac{1}{2}\frac{\int_\Omega w\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)v\overline{u}\right)d\mu}{\sqrt{\operatorname{Re}(\overline{a}A)}}\\&=\frac{1}{2}\frac{\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\int_\Omega wv\overline{u}d\mu\right)}{\sqrt{\operatorname{Re}(\overline{a}A)}}.\end{aligned} \tag{4.2}$$

Now, if we replace u with $\overline{u}$ in (4.1), namely, we assume that

$$\int_\Omega w\operatorname{Re}\left[(A\overline{u}-v)(\overline{v}-\overline{a}u)\right]d\mu\ge0, \tag{4.3}$$

then we have the following inequality of interest:

$$\left(\int_\Omega|u|^2wd\mu\right)^{1/2}\left(\int_\Omega|v|^2wd\mu\right)^{1/2}\le\frac{1}{2}\frac{\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\int_\Omega wvud\mu\right)}{\sqrt{\operatorname{Re}(\overline{a}A)}}. \tag{4.4}$$

We observe that if

$$\operatorname{Re}\left[\left(A\overline{u}-v\right)\left(\overline{v}-\overline{a}u\right)\right]\geq 0\ \mu\text{-a.e. on }\Omega \tag{4.5}$$

then (4.3) is valid for any $w\geq 0$ μ-a.e. on Ω.

Moreover, if $A>a>0$ and u,v are real valued and such that $Au\geq v\geq au$ μ-a.e. on Ω, then (4.5) holds true for any $w\geq 0$ μ-a.e. on Ω.

Theorem 6. *Let $f,g\in L_{w}^{2}\left(\Omega,\mu\right)$ and $\overline{w}=\left(w_{1},....,w_{n}\right)\in\mathfrak{P}_{n}\left(w\right)$ be such that there exists $a,A\in\mathbb{C}$ with $\operatorname{Re}\left(\overline{a}A\right)>0$ and*

$$\int_{\Omega}w_{i}\operatorname{Re}\left[\left(A\overline{g}-f\right)\left(\overline{f}-\overline{a}g\right)\right]\geq 0, \tag{4.6}$$

for any $i\in\{1,...,n\}$, then we have

$$\begin{aligned}\beta\left(\left|f\right|,\left|g\right|,\overline{w}\right) &\leq\frac{1}{2}\frac{\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\int_{\Omega}wfgd\mu\right)}{\sqrt{\operatorname{Re}\left(\overline{a}A\right)}} \\ &\leq\frac{1}{2}\frac{\left|A+a\right|}{\sqrt{\operatorname{Re}\left(\overline{a}A\right)}}\left|\int_{\Omega}wfgd\mu\right|\leq\frac{1}{2}\frac{\left|A+a\right|}{\sqrt{\operatorname{Re}\left(\overline{a}A\right)}}\int_{\Omega}w\left|fg\right|d\mu.\end{aligned} \tag{4.7}$$

Proof. If the condition (4.6) is true, then by (4.4) we have

$$\left(\int_{\Omega}\left|f\right|^{2}w_{i}d\mu\right)^{1/2}\left(\int_{\Omega}\left|g\right|^{2}w_{i}d\mu\right)^{1/2}\leq\frac{1}{2}\frac{\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\int_{\Omega}w_{i}fgd\mu\right)}{\sqrt{\operatorname{Re}\left(\overline{a}A\right)}} \tag{4.8}$$

for any $i\in\{1,...,n\}$.

If we sum over i from 1 to n then we get

$$\sum_{i=1}^{n}\left(\int_{\Omega}\left|f\right|^{2}w_{i}d\mu\right)^{1/2}\left(\int_{\Omega}\left|g\right|^{2}w_{i}d\mu\right)^{1/2}\leq\frac{1}{2}\frac{\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\int_{\Omega}\sum_{i=1}^{n}w_{i}fgd\mu\right)}{\sqrt{\operatorname{Re}\left(\overline{a}A\right)}},$$

which proves the first inequality in (4.7).

Since

$$\begin{aligned}0 &\leq\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\int_{\Omega}wfgd\mu\right)=\left|\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\int_{\Omega}wfgd\mu\right)\right| \\ &\leq\left|\left(\overline{A}+\overline{a}\right)\int_{\Omega}wfgd\mu\right|=\left|\overline{A}+\overline{a}\right|\left|\int_{\Omega}wfgd\mu\right|=\left|A+a\right|\left|\int_{\Omega}wfgd\mu\right|,\end{aligned}$$

then the second inequality also holds.

The last part is obvious. □

Corollary 5. *If $A>a>0$, $\overline{w}=\left(w_{1},....,w_{n}\right)\in\mathfrak{P}_{n}\left(w\right)$ and f,g are positive valued and such that*

$$\int_{\Omega}w_{i}\left[\left(Ag-f\right)\left(f-ag\right)\right]\geq 0, \tag{4.9}$$

for any $i\in\{1,...,n\}$, then

$$\beta\left(f,g,\overline{w}\right)\leq\frac{1}{2}\frac{A+a}{\sqrt{Aa}}\int_{\Omega}wfgd\mu. \tag{4.10}$$

We have the following result as well:

Theorem 7. *Let $f,g\in L_w^2(\Omega,\mu)$ and $\overline{w}=(w_1,....,w_n)\in\mathfrak{P}_n(w)$ be such that there exists $a,A\in\mathbb{C}$ with $a+A\neq 0$ and (4.6) is valid, then we have*

$$\beta(|f|,|g|,\overline{w})\leq \operatorname{Re}\left[\frac{\overline{a}+\overline{A}}{|a+A|}\int_\Omega wfgd\mu\right]+\frac{1}{4}\frac{|A-a|^2}{|a+A|}\int_\Omega w|g|^2d\mu \leq\left|\int_\Omega wfgd\mu\right|+\frac{1}{4}\frac{|A-a|^2}{|a+A|}\int_\Omega w|g|^2d\mu. \tag{4.11}$$

Proof. We use the following inequality obtained in [6] (see also [7, p. 32])

$$\left(\int_\Omega |u|^2\ell d\mu\right)^{1/2}\left(\int_\Omega |v|^2\ell d\mu\right)^{1/2} \leq \operatorname{Re}\left[\frac{\overline{a}+\overline{A}}{|a+A|}\int_\Omega \ell fgd\mu\right]+\frac{1}{4}\frac{|A-a|^2}{|a+A|}\int_\Omega \ell|g|^2d\mu, \tag{4.12}$$

where ℓ in μ-integrable, $\ell\geq 0$ μ-a.e. on Ω and

$$\int_\Omega \ell\operatorname{Re}\left[(A\overline{g}-f)\left(\overline{f}-\overline{a}g\right)\right]d\mu\geq 0.$$

Now, if we write the inequality (4.12) for w_i with $i\in\{1,...,n\}$, then we get

$$\left(\int_\Omega |f|^2w_id\mu\right)^{1/2}\left(\int_\Omega |f|^2w_id\mu\right)^{1/2} \leq \operatorname{Re}\left[\frac{\overline{a}+\overline{A}}{|a+A|}\int_\Omega w_ifgd\mu\right]+\frac{1}{4}\frac{|A-a|^2}{|a+A|}\int_\Omega w_i|g|^2d\mu, \tag{4.13}$$

for any $i\in\{1,...,n\}$.

If we sum over i from 1 to n in (4.13), then we get

$$\sum_{i=1}^n\left(\int_\Omega |f|^2w_id\mu\right)^{1/2}\left(\int_\Omega |g|^2w_id\mu\right)^{1/2} \leq \operatorname{Re}\left[\frac{\overline{a}+\overline{A}}{|a+A|}\int_\Omega\sum_{i=1}^n w_ifgd\mu\right]+\frac{1}{4}\frac{|A-a|^2}{|a+A|}\int_\Omega\sum_{i=1}^n w_i|g|^2d\mu \tag{4.14}$$

that proves the first inequality in (4.11).

The second inequality follows by the fact that

$$\operatorname{Re}\left[\frac{\overline{a}+\overline{A}}{|a+A|}\int_\Omega wfgd\mu\right]\leq\left|\frac{\overline{a}+\overline{A}}{|a+A|}\int_\Omega wfgd\mu\right| =\left|\frac{\overline{a}+\overline{A}}{|a+A|}\right|\left|\int_\Omega wfgd\mu\right|=\left|\int_\Omega wfgd\mu\right|.$$

□

Corollary 6. *If $A>a>0$, $\overline{w}=(w_1,....,w_n)\in\mathfrak{P}_n(w)$ and f,g are positive valued and such that (4.9) is valid, then*

$$\beta(f,g,\overline{w})\leq\int_\Omega wfgd\mu+\frac{1}{4}\frac{(A-a)^2}{a+A}\int_\Omega wg^2d\mu. \tag{4.15}$$

5. Discrete Inequalities

When μ is the discrete measure on $\Omega = \{1, \dots, n\}$, then the corresponding discrete inequalities for complex (real) numbers can be stated as well. We give here some examples.

Consider the sequences of complex numbers $\overline{x} = (x_1, ..., x_m)$, $\overline{y} = (y_1, ..., y_m) \in \mathbb{C}^m$, $\overline{w} = (w_1, ..., w_m) \in \mathbb{R}^n$, with $w_k > 0$, $k \in \{1, ..., m\}$. Let $w_{ki} > 0$ for $k \in \{1, ..., m\}$, $i \in \{1, ..., n\}$ with $m, n \geq 2$ and

$$\sum_{i=1}^{n} w_{ki} = w_k \text{ for any } k \in \{1, ..., m\}. \tag{5.1}$$

We consider the functional associated with the matrix $W := \{w_{ki}\}_{k\in\{1,...,m\}, i\in\{1,...,n\}}$ that satisfy (5.1),

$$\beta(\overline{x}, \overline{y}, W) := \sum_{i=1}^{n} \left(\sum_{k=1}^{m} w_{ki} |x_k|^2 \right)^{1/2} \left(\sum_{k=1}^{m} w_{ki} |y_k|^2 \right)^{1/2}. \tag{5.2}$$

Using the inequality (2.1) we have the following refinement of the discrete CBS inequality

$$\begin{aligned} \left| \sum_{k=1}^{m} w_k x_k y_k \right| &\leq \sum_{i=1}^{n} \left(\sum_{k=1}^{m} w_{ki} |x_k|^2 \right)^{1/2} \left(\sum_{k=1}^{m} w_{ki} |y_k|^2 \right)^{1/2} \\ &\leq \left(\sum_{k=1}^{m} w_k |x_k|^2 \right)^{1/2} \left(\sum_{k=1}^{m} w_k |y_k|^2 \right)^{1/2}. \end{aligned} \tag{5.3}$$

Assume that there exists $c, C > 0$ with the property

$$0 < c \leq w_{ki} \leq C < \infty \tag{5.4}$$

for all $i \in \{1, ..., n\}$ and $k \in \{1, ..., m\}$. Then by (3.9) and (3.10) we have

$$\begin{aligned} &\left(\sum_{k=1}^{m} w_k |x_k|^2 \right)^{1/2} \left(\sum_{k=1}^{m} w_k |y_k|^2 \right)^{1/2} \\ &\leq \frac{c+C}{2\sqrt{cC}} \sum_{i=1}^{n} \left(\sum_{k=1}^{m} w_{ki} |x_k|^2 \right)^{1/2} \left(\sum_{k=1}^{m} w_{ki} |y_k|^2 \right)^{1/2} \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} 0 &\leq \frac{\sum_{k=1}^{m} w_k |x_k|^2}{\beta(\overline{x}, \overline{y}, W)} - \frac{\beta(\overline{x}, \overline{y}, W)}{\sum_{k=1}^{m} w_k |y_k|^2} \\ &\leq \left(\frac{C^{1/4}}{c^{1/4}} - \frac{c^{1/4}}{C^{1/4}} \right)^2 \left(\frac{\sum_{k=1}^{m} |x_k|^2}{\sum_{k=1}^{m} |y_k|^2} \right)^{1/2}. \end{aligned} \tag{5.6}$$

By (3.20) and (3.21) we also have

$$\begin{aligned} 0 &\leq \sum_{k=1}^{m} w_k |x_k|^2 \sum_{k=1}^{m} w_k |y_k|^2 - \beta^2(\overline{x}, \overline{y}, W) \\ &\leq \frac{1}{3} n^2 (C-c)^2 \sum_{k=1}^{m} |x_k|^2 \sum_{k=1}^{m} |y_k|^2 \end{aligned} \tag{5.7}$$

and

$$\sum_{k=1}^{m} w_k |y_k|^2 + \frac{\sum_{k=1}^{m} |y_k|^2}{\sum_{k=1}^{m} |x_k|^2} \sum_{k=1}^{m} w_k |x_k|^2 \tag{5.8}$$

$$\leq \left(\frac{C^{1/2}}{c^{1/2}} + \frac{c^{1/2}}{C^{1/2}} \right) \beta(\overline{x}, \overline{y}, W) \frac{\left(\sum_{k=1}^{m} |y_k|^2\right)^{1/2}}{\left(\sum_{k=1}^{m} |x_k|^2\right)^{1/2}}.$$

Consider

$$w_0 := \min_{i \in \{1,...,n\}} \left(\sum_{k=1}^{m} w_{ki} \right)^{1/2} \text{ and } W_0 := \max_{i \in \{1,...,n\}} \left(\sum_{k=1}^{m} w_{ki} \right)^{1/2}.$$

If there exists the constants a, A, b, B such that

$$0 < a \leq |x_k| \leq A < \infty \text{ and } 0 < b \leq |y_k| \leq B < \infty \text{ for any } k \in \{1, ..., m\}, \tag{5.9}$$

then by (3.12) and (3.13)

$$\left(\sum_{k=1}^{m} w_k |x_k|^2 \right)^{1/2} \left(\sum_{k=1}^{m} w_k |y_k|^2 \right)^{1/2} \tag{5.10}$$

$$\leq \frac{abw_0^2 + W_0^2 AB}{2w_0 W_0 \sqrt{abAB}} \sum_{i=1}^{n} \left(\sum_{k=1}^{m} w_{ki} |x_k|^2 \right)^{1/2} \left(\sum_{k=1}^{m} w_{ki} |y_k|^2 \right)^{1/2},$$

and

$$0 \leq \frac{\sum_{k=1}^{m} w_k |x_k|^2}{\beta(\overline{x}, \overline{y}, W)} - \frac{\beta(\overline{x}, \overline{y}, W)}{\sum_{k=1}^{m} w_k |y_k|^2} \leq \left(\sqrt{\frac{AW_0}{bw_0}} - \sqrt{\frac{aw_0}{BW_0}} \right)^2. \tag{5.11}$$

By the inequalities (3.22) and (3.23) we also have

$$0 \leq \sum_{k=1}^{m} w_k |x_k|^2 \sum_{k=1}^{m} w_k |y_k|^2 \tag{5.12}$$

$$- \left(\sum_{i=1}^{n} \left(\sum_{k=1}^{m} w_{ki} |x_k|^2 \right)^{1/2} \left(\sum_{k=1}^{m} w_{ki} |y_k|^2 \right)^{1/2} \right)^2$$

$$\leq \frac{1}{3} n^2 \left(ABW_0^2 - abw_0^2 \right)^2$$

and

$$\sum_{k=1}^{m} w_k |y_k|^2 + \frac{bB}{aA} \sum_{k=1}^{m} w_k |x_k|^2 \tag{5.13}$$

$$\leq \left(\frac{BW_0}{aw_0} + \frac{bw_0}{AW_0} \right) \sum_{i=1}^{n} \left(\sum_{k=1}^{m} w_{ki} |x_k|^2 \right)^{1/2} \left(\sum_{k=1}^{m} w_{ki} |y_k|^2 \right)^{1/2}.$$

Now, assume that there exist $a, A \in \mathbb{C}$, $W := \{w_{ki}\}_{k \in \{1,...,m\}, i \in \{1,...,n\}}$ satisfying the condition (5.1) and such that

$$\sum_{k=1}^{m} w_{ki} \operatorname{Re} \left[\left(A\overline{y_k} - x_k \right) \left(\overline{x_k} - \overline{a} y_k \right) \right] \geq 0 \text{ for any } i \in \{1, ..., n\}. \tag{5.14}$$

If $\operatorname{Re}(\overline{a}A) > 0$, then by (4.7) we have that

$$\sum_{i=1}^{n}\left(\sum_{k=1}^{m} w_{ki}\left|x_k\right|^2\right)^{1/2}\left(\sum_{k=1}^{m} w_{ki}\left|y_k\right|^2\right)^{1/2} \tag{5.15}$$
$$\leq \frac{1}{2}\frac{\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\sum_{k=1}^{m} w_k x_k y_k\right)}{\sqrt{\operatorname{Re}(\overline{a}A)}}$$
$$\leq \frac{1}{2}\frac{|A+a|}{\sqrt{\operatorname{Re}(\overline{a}A)}}\left|\sum_{k=1}^{m} w_k x_k y_k\right| \leq \frac{1}{2}\frac{|A+a|}{\sqrt{\operatorname{Re}(\overline{a}A)}}\sum_{k=1}^{m} w_k\left|x_k y_k\right|.$$

If $a+A \neq 0$ and (5.14) is satisfied then by (4.11) we also have that

$$\sum_{i=1}^{n}\left(\sum_{k=1}^{m} w_{ki}\left|x_k\right|^2\right)^{1/2}\left(\sum_{k=1}^{m} w_{ki}\left|y_k\right|^2\right)^{1/2} \tag{5.16}$$
$$\leq \operatorname{Re}\left[\frac{\overline{a}+\overline{A}}{|a+A|}\sum_{k=1}^{m} w_k x_k y_k\right]+\frac{1}{4}\frac{|A-a|^2}{|a+A|}\sum_{k=1}^{m} w_k\left|y_k\right|^2$$
$$\leq \left|\sum_{k=1}^{m} w_k x_k y_k\right|+\frac{1}{4}\frac{|A-a|^2}{|a+A|}\sum_{k=1}^{m} w_k\left|y_k\right|^2.$$

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