Received 30/07/15

REFINING CBS INEQUALITY FOR DIVISIONS OF MEASURABLE SPACE

S. S. DRAGOMIR[1,2]

Abstract. In this paper we establish a refinement and some reverses for CBS inequality for the general Lebesgue integral on divisions of measurable space. Applications for discrete inequalities and weighted means of positive numbers are also given.

1. Introduction

The Cauchy-Bunyakovsky-Schwarz inequality, or for short, the CBS inequality, plays an important role in different branches of Modern Mathematics including Hilbert Spaces Theory, Probability & Statistics, Classical Real and Complex Analysis, Numerical Analysis, Qualitative Theory of Differential Equations and their applications.

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a σ-algebra of subsets of Ω denoted by Σ and μ a countably additive and positive measure on Σ with values in $\mathbb{R} \cup \{\infty\}$. Denote by $L_w^2(\Omega, \mu)$ the Hilbert space of all $\mathbb{C}$-valued functions f defined on Ω that are 2-w-integrable on Ω, i.e., $\int_\Omega w(x)|f(x)|^2 d\mu(x) < \infty$, where $w : \Omega \to [0, \infty)$ is a given μ-measurable function on Ω. We write for simplicity $\int_\Omega w|f|^2 d\mu$ instead of $\int_\Omega w(x)|f(x)|^2 d\mu(x)$.

The following inequality is well known in the literature as the *integral Cauchy-Bunyakovsky-Schwarz inequality*:

$$\left|\int_\Omega wfg d\mu\right|^2 \leq \int_\Omega w|f|^2 d\mu \int_\Omega w|g|^2 d\mu \tag{CBS}$$

provided that $f, g \in L_w^2(\Omega, \mu)$.

We say that the family of measurable sets $F_n(\Omega) = \{\Omega_i\}_{i\in\{1,...,n\}}$ is a *n-division* for Ω if $\Omega = \bigcup_{i=1}^n \Omega_i$, $\Omega_i \cap \Omega_j = \emptyset$ for any $i, j \in \{1, ..., n\}$ with $i \neq j$ and $\mu(\Omega_i) > 0$ for any $i \in \{1, ..., n\}$. In this situation, if $f \in L_w(\Omega, \mu)$ then $f \in L_w(\Omega_i, \mu)$ for any $i \in \{1, ..., n\}$ and $\int_\Omega fw d\mu = \sum_{i=1}^n \int_{\Omega_i} fw d\mu$. Also, $\int_\Omega w d\mu = \sum_{i=1}^n \int_{\Omega_i} w d\mu$ with $\int_{\Omega_i} w d\mu > 0$ for any $i \in \{1, ..., n\}$.

For a given $n \geq 2$ we denote by $\mathfrak{D}_n(\Omega)$ the set of all n-divisions of Ω and consider the functional $\gamma(|f|, |g|, \cdot) : \mathfrak{D}_n(\Omega) \to \mathbb{R}$ defined by

$$\gamma(|f|, |g|, F_n(\Omega)) := \sum_{i=1}^n \left(\int_{\Omega_i} w|f|^2 d\mu\right)^{1/2} \left(\int_{\Omega_i} w|g|^2 d\mu\right)^{1/2}, \tag{1.1}$$

where $f, g \in L_w^2(\Omega, \mu)$.

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 28A25.

Key words and phrases. CBS inequality, Integrable functions, Lebesgue integral, Weighted means.

In this paper we establish some inequalities concerning the functional $\gamma(|f|,|g|,\cdot)$ that provide refinements and reverses for the CBS integral inequality (CBS). Applications for discrete inequalities and weighted means of positive numbers are also given.

For recent papers on CBS inequality, see [1], [2], [8], [9], [10], [13], [14], [15], [16], [18] and the references therein.

2. The Main Results

We state the following refinements of the CBS inequality:

Theorem 1. *For $f,g\in L_w^2(\Omega,\mu)$ and $F_n(\Omega)\in\mathfrak{D}_n(\Omega)$ we have*

$$\left|\int_\Omega wfgd\mu\right|\le\gamma(|f|,|g|,F_n(\Omega))\le\left(\int_\Omega w|f|^2d\mu\right)^{1/2}\left(\int_\Omega w|g|^2d\mu\right)^{1/2}. \tag{2.1}$$

Proof. Let $F_n(\Omega)=\{\Omega_i\}_{i\in\{1,\dots,n\}}$ be a n-division for Ω.

We have by CBS integral inequality in $L_w^2(\Omega_i,\mu)$, $i\in\{1,...,n\}$ that

$$\begin{aligned}\left|\int_\Omega wfgd\mu\right|&\le\int_\Omega w|fg|d\mu=\sum_{i=1}^n\int_{\Omega_i}w|fg|d\mu\\&\le\sum_{i=1}^n\left(\int_{\Omega_i}w|f|^2d\mu\right)^{1/2}\left(\int_{\Omega_i}w|g|^2d\mu\right)^{1/2}=\gamma(|f|,|g|,\overline{w}),\end{aligned}$$

which proves the first inequality in (2.1).

By the CBS discrete inequality we also have

$$\begin{aligned}\gamma(|f|,|g|,\overline{w})&=\sum_{i=1}^n\left(\int_{\Omega_i}w|f|^2d\mu\right)^{1/2}\left(\int_{\Omega_i}w|g|^2d\mu\right)^{1/2}\\&\le\left[\sum_{i=1}^n\left(\left(\int_{\Omega_i}w|f|^2d\mu\right)^{1/2}\right)^2\right]^{1/2}\left[\sum_{i=1}^n\left(\left(\int_{\Omega_i}w|g|^2d\mu\right)^{1/2}\right)^2\right]^{1/2}\\&=\left[\sum_{i=1}^n\int_{\Omega_i}w|f|^2d\mu\right]^{1/2}\left[\sum_{i=1}^n\int_{\Omega_i}w|g|^2d\mu\right]^{1/2}\\&=\left[\int_\Omega w|f|^2d\mu\right]^{1/2}\left[\int_\Omega w|g|^2d\mu\right]^{1/2},\end{aligned}$$

which proves the second inequality in (2.1). □

We can give now some lower bounds for $\gamma(|f|,|g|,\cdot)$.

The following result holds:

Theorem 2. *Let $f,g\in L_w^2(\Omega,\mu)$ and $F_n(\Omega)\in\mathfrak{D}_n(\Omega)$, $F_n(\Omega)=\{\Omega_i\}_{i\in\{1,\dots,n\}}$ be such that there exists $k,K,l,L>0$ with the property*

$$0<k\le\left(\int_{\Omega_i}w|f|^2d\mu\right)^{1/2}\le K<\infty \tag{2.2}$$

and

$$0<l\le\left(\int_{\Omega_i}w|g|^2d\mu\right)^{1/2}\le L<\infty \tag{2.3}$$

for each $i \in \{1, ..., n\}$. *Then*

$$\left(\int_\Omega w |f|^2 d\mu\right)^{1/2} \left(\int_\Omega w |g|^2 d\mu\right)^{1/2} \le \frac{kl + KL}{2\sqrt{klKL}} \gamma(|f|, |g|, F_n(\Omega)) \tag{2.4}$$

and

$$0 \le \frac{\int_\Omega w |f|^2 d\mu}{\gamma(|f|, |g|, F_n(\Omega))} - \frac{\gamma(|f|, |g|, F_n(\Omega))}{\int_\Omega w |g|^2 d\mu} \le \left(\sqrt{\frac{K}{l}} - \sqrt{\frac{k}{L}}\right)^2. \tag{2.5}$$

Proof. We use the Pólya-Szegö inequality that states that [17] (see also [4, p. 74]), if $0 < a \le a_i \le A < \infty$ and $0 < b \le b_i \le B < \infty$, $i \in \{1, ..., n\}$ then

$$\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2}{\left(\sum_{i=1}^n a_i b_i\right)^2} \le \frac{(ab + AB)^2}{4abAB}. \tag{2.6}$$

Now, if we take $a_i = \left(\int_{\Omega_i} w |f|^2 d\mu\right)^{1/2}$, $b_i = \left(\int_{\Omega_i} w |g|^2 d\mu\right)^{1/2}$, $i \in \{1, ..., n\}$, $a = k$, $A = K$, $b = l$ and $B = L$, then by (2.6) we get

$$\begin{aligned} &\left(\sum_{i=1}^n \int_{\Omega_i} w |f|^2 d\mu\right) \left(\sum_{i=1}^n \int_{\Omega_i} w |k|^2 d\mu\right) \\ &\le \frac{(kl + KL)^2}{4klKL} \left(\sum_{i=1}^n \left(\int_{\Omega_i} w |f|^2 d\mu\right)^{1/2} \left(\int_{\Omega_i} w |g|^2 d\mu\right)^{1/2}\right)^2 \end{aligned} \tag{2.7}$$

and the inequality (2.4) is proved.

We use now the Shisha-Mond inequality [19] (see also [4, p. 82]) that says that

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2 \le \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}}\right)^2 \sum_{i=1}^n a_i b_i \sum_{i=1}^n b_i^2 \tag{2.8}$$

provided $0 < a \le a_i \le A < \infty$ and $0 < b \le b_i \le B < \infty$, $i \in \{1, ..., n\}$.

Now, if we take $a_i = \left(\int_{\Omega_i} w |f|^2 d\mu\right)^{1/2}$, $b_i = \left(\int_{\Omega_i} w |g|^2 d\mu\right)^{1/2}$, $i \in \{1, ..., n\}$, $a = k$, $A = K$, $b = l$ and $B = L$, then by (2.8) we get

$$\begin{aligned} &\sum_{i=1}^n \int_{\Omega_i} w |f|^2 d\mu \sum_{i=1}^n \int_{\Omega_i} w |g|^2 d\mu - \left(\sum_{i=1}^n \left(\int_{\Omega_i} w |f|^2 d\mu\right)^{1/2} \left(\int_{\Omega_i} w |g|^2 d\mu\right)^{1/2}\right)^2 \\ &\le \left(\sqrt{\frac{K}{l}} - \sqrt{\frac{k}{L}}\right)^2 \sum_{i=1}^n \left(\int_{\Omega_i} w |f|^2 d\mu\right)^{1/2} \left(\int_{\Omega_i} w |g|^2 d\mu\right)^{1/2} \sum_{i=1}^n \int_{\Omega_i} w |g|^2 d\mu, \end{aligned}$$

which is equivalent to (2.5). □

Corollary 1. *Let* $f, g \in L_w^2(\Omega, \mu)$ *and* $F_n(\Omega) \in \mathfrak{D}_n(\Omega)$, $F_n(\Omega) = \{\Omega_i\}_{i \in \{1, ..., n\}}$ *be such that there exists* $m, M > 0$ *with the property*

$$0 < m \le w \le M < \infty \;\; \mu\text{-a.e. on } \Omega. \tag{2.9}$$

Then

$$\left(\int_{\Omega} w|f|^{2} d\mu\right)^{1/2}\left(\int_{\Omega} w|g|^{2} d\mu\right)^{1/2} \leq \frac{m f_{0} g_{0}+M F_{0} G_{0}}{2 m^{1/2} M^{1/2} \sqrt{f_{0} g_{0} F_{0} G_{0}}} \gamma\left(|f|,|g|, F_{n}(\Omega)\right) \tag{2.10}$$

and

$$0 \leq \frac{\int_{\Omega} w|f|^{2} d\mu}{\gamma\left(|f|,|g|, F_{n}(\Omega)\right)}-\frac{\gamma\left(|f|,|g|, F_{n}(\Omega)\right)}{\int_{\Omega} w|g|^{2} d\mu} \leq\left(\sqrt{\frac{M^{1/2} F_{0}}{m^{1/2} g_{0}}}-\sqrt{\frac{m^{1/2} f_{0}}{M^{1/2} G_{0}}}\right)^{2} \tag{2.11}$$

where

$$f_{0}:=\min_{i \in\{1, \ldots, n\}}\left(\int_{\Omega_{i}}|f|^{2} d\mu\right)^{1/2}, \; F_{0}:=\max_{i \in\{1, \ldots, n\}}\left(\int_{\Omega_{i}}|f|^{2} d\mu\right)^{1/2}$$
$$g_{0}:=\min_{i \in\{1, \ldots, n\}}\left(\int_{\Omega_{i}}|g|^{2} d\mu\right)^{1/2}, \; G_{0}:=\max_{i \in\{1, \ldots, n\}}\left(\int_{\Omega_{i}}|g|^{2} d\mu\right)^{1/2}. \tag{2.12}$$

Proof. Let $F_{n}(\Omega)=\{\Omega_{i}\}_{i \in\{1, \ldots, n\}}$ be a n-division for Ω.

From (2.9) we have

$$0<m \int_{\Omega_{i}}|f|^{2} d\mu \leq \int_{\Omega_{i}} w|f|^{2} d\mu \leq M \int_{\Omega_{i}}|f|^{2} d\mu<\infty$$

for any $i \in\{1, \ldots, n\}$, giving that

$$0<m \min_{i \in\{1, \ldots, n\}}\left\{\int_{\Omega_{i}}|f|^{2} d\mu\right\} \leq \int_{\Omega_{i}} w|f|^{2} d\mu \leq M \max_{i \in\{1, \ldots, n\}}\left\{\int_{\Omega_{i}}|f|^{2} d\mu\right\}<\infty,$$

which is equivalent to

$$0<m^{1/2} \min_{i \in\{1, \ldots, n\}}\left(\int_{\Omega_{i}}|f|^{2} d\mu\right)^{1/2} \leq\left(\int_{\Omega_{i}} w|f|^{2} d\mu\right)^{1/2} \leq M^{1/2} \max_{i \in\{1, \ldots, n\}}\left(\int_{\Omega_{i}}|f|^{2} d\mu\right)^{1/2}<\infty$$

and, similarly

$$0<m^{1/2} \min_{i \in\{1, \ldots, n\}}\left(\int_{\Omega_{i}}|g|^{2} d\mu\right)^{1/2} \leq\left(\int_{\Omega_{i}} w|g|^{2} d\mu\right)^{1/2} \leq M^{1/2} \max_{i \in\{1, \ldots, n\}}\left(\int_{\Omega_{i}}|g|^{2} d\mu\right)^{1/2}<\infty$$

for any $i \in\{1, \ldots, n\}$.

Now, if we apply Theorem 2 for $k=m^{1/2} f_{0}$, $K=M^{1/2} F_{0}$, $l=m^{1/2} g_{0}$ and $L=M^{1/2} G_{0}$ we get the desired inequalities (2.10) and (2.11). □

Corollary 2. *Let $f, g \in L_{w}^{2}(\Omega, \mu)$ and $F_{n}(\Omega) \in \mathfrak{D}_{n}(\Omega)$, $F_{n}(\Omega)=\{\Omega_{i}\}_{i \in\{1, \ldots, n\}}$ be such that there exist $a, b, A, B>0$ with the property*

$$0<a \leq|f| \leq A<\infty \text{ and } 0<b \leq|g| \leq B<\infty \;\mu\text{-a.e. on } \Omega. \tag{2.13}$$

Then

$$\left(\int_{\Omega} w\left|f\right|^{2} d\mu\right)^{1/2}\left(\int_{\Omega} w\left|g\right|^{2} d\mu\right)^{1/2} \leq \frac{abw_0^2+W_0^2AB}{2w_0W_0\sqrt{abAB}}\gamma\left(\left|f\right|,\left|g\right|,F_n\left(\Omega\right)\right), \tag{2.14}$$

and

$$\begin{aligned} 0 &\leq \frac{\int_{\Omega} w\left|f\right|^{2} d\mu}{\gamma\left(\left|f\right|,\left|g\right|,F_n\left(\Omega\right)\right)}-\frac{\gamma\left(\left|f\right|,\left|g\right|,F_n\left(\Omega\right)\right)}{\int_{\Omega} w\left|g\right|^{2} d\mu} \\ &\leq \left(\sqrt{\frac{AW_0}{bw_0}}-\sqrt{\frac{aw_0}{BW_0}}\right)^2, \end{aligned} \tag{2.15}$$

where

$$w_0 := \min_{i\in\{1,\dots,n\}}\left(\int_{\Omega_i} wd\mu\right)^{1/2} \text{ and } W_0 := \max_{i\in\{1,\dots,n\}}\left(\int_{\Omega_i} w_i d\mu\right)^{1/2}. \tag{2.16}$$

Proof. From (2.13) we have

$$a\left(\int_{\Omega_i} wd\mu\right)^{1/2} \leq \left(\int_{\Omega_i} w\left|f\right|^{2} d\mu\right)^{1/2} \leq A\left(\int_{\Omega_i} wd\mu\right)^{1/2} < \infty,$$

which implies that

$$aw_0 \leq \left(\int_{\Omega_i} w\left|f\right|^{2} d\mu\right)^{1/2} \leq AW_0 < \infty$$

for any $i \in \{1,...,n\}$.

Similarly, we have

$$bw_0 \leq \left(\int_{\Omega_i} w\left|g\right|^{2} d\mu\right)^{1/2} \leq BW_0 < \infty,$$

for any $i \in \{1,...,n\}$.

Now, if we apply Theorem 2 for $k = aw_0$, $K = AW_0$, $l = bw_0$ and $L = BW_0$ we get the desired inequalities (2.14) and (2.15). □

The following result holds:

Theorem 3. *Let $f, g \in L_w^2(\Omega,\mu)$ and $F_n(\Omega) \in \mathfrak{D}_n(\Omega)$, $F_n(\Omega) = \{\Omega_i\}_{i\in\{1,...,n\}}$ be such that there exist $k, K, l, L > 0$ with the property (2.2) and (2.3) for each $i \in \{1,...,n\}$. Then*

$$\begin{aligned} 0 &\leq \int_{\Omega} w\left|f\right|^{2} d\mu\int_{\Omega} w\left|g\right|^{2} d\mu-\gamma^2\left(\left|f\right|,\left|g\right|,F_n\left(\Omega\right)\right) \\ &\leq \frac{1}{3}n^2\left(KL-kl\right)^2 \end{aligned} \tag{2.17}$$

and

$$\int_{\Omega} w\left|g\right|^{2} d\mu+\frac{lL}{kK}\int_{\Omega} w\left|f\right|^{2} d\mu \leq \left(\frac{L}{k}+\frac{l}{K}\right)\gamma\left(\left|f\right|,\left|g\right|,F_n\left(\Omega\right)\right). \tag{2.18}$$

Proof. We use the following Ozeki's type inequality [12]

$$\sum_{i=1}^{n} a_i^2\sum_{i=1}^{n} b_i^2-\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \frac{1}{3}n^2\left(AB-ab\right)^2 \tag{2.19}$$

provided $0 < a \leq a_i \leq A < \infty$ and $0 < b \leq b_i \leq B < \infty$, $i \in \{1,...,n\}$.

Now, if we take $a_i = \left(\int_{\Omega_i} w\left|f\right|^2 d\mu\right)^{1/2}$, $b_i = \left(\int_{\Omega_i} w\left|g\right|^2 d\mu\right)^{1/2}$, $i \in \{1,...,n\}$, $a = k$, $A = K$, $b = l$ and $B = L$, then by (2.19) we get

$$\begin{aligned}(2.20)\qquad & \sum_{i=1}^{n}\int_{\Omega_i} w\left|f\right|^2 d\mu \sum_{i=1}^{n}\int_{\Omega_i} w\left|g\right|^2 d\mu \\ & -\left(\sum_{i=1}^{n}\left(\int_{\Omega_i} w\left|f\right|^2 d\mu\right)^{1/2}\left(\int_{\Omega_i} w\left|g\right|^2 d\mu\right)^{1/2}\right)^2 \\ & \leq \frac{1}{3}n^2\left(KL-kl\right)^2,\end{aligned}$$

which proves (2.17).

Further, we recall Diaz-Metcalf's inequality [3] (see also [4, p. 123])

$$(2.21)\qquad \sum_{i=1}^{n} b_i^2 + \frac{bB}{aA}\sum_{i=1}^{n} a_i^2 \leq \left(\frac{B}{a}+\frac{b}{A}\right)\sum_{i=1}^{n} a_i b_i$$

provided $0 < a \leq a_i \leq A < \infty$ and $0 < b \leq b_i \leq B < \infty$, $i \in \{1,...,n\}$.

If we take $a_i = \left(\int_{\Omega_i} w\left|f\right|^2 d\mu\right)^{1/2}$, $b_i = \left(\int_{\Omega_i} w\left|g\right|^2 d\mu\right)^{1/2}$, $i \in \{1,...,n\}$, $a = k$, $A = K$, $b = l$ and $B = L$, then by (2.21) we get

$$\begin{aligned}(2.22)\qquad & \sum_{i=1}^{n}\int_{\Omega_i} w\left|g\right|^2 d\mu + \frac{bB}{aA}\sum_{i=1}^{n}\int_{\Omega_i} w\left|f\right|^2 d\mu \\ & \leq \left(\frac{B}{a}+\frac{b}{A}\right)\sum_{i=1}^{n}\left(\int_{\Omega_i} w\left|f\right|^2 d\mu\right)^{1/2}\left(\int_{\Omega_i} w\left|g\right|^2 d\mu\right)^{1/2}\end{aligned}$$

that is equivalent to (2.18). □

Corollary 3. *Let $f, g \in L_w^2(\Omega,\mu)$ and $F_n(\Omega) \in \mathfrak{D}_n(\Omega)$, $F_n(\Omega) = \{\Omega_i\}_{i\in\{1,...,n\}}$ be such that there exists $m, M > 0$ with the property (2.9). Then*

$$\begin{aligned}(2.23)\qquad 0 & \leq \int_{\Omega} w\left|f\right|^2 d\mu \int_{\Omega} w\left|g\right|^2 d\mu - \gamma^2\left(\left|f\right|,\left|g\right|,F_n(\Omega)\right) \\ & \leq \frac{1}{3}n^2\left(MF_0G_0 - mf_0g_0\right)^2\end{aligned}$$

and

$$\begin{aligned}(2.24)\qquad & \int_{\Omega} w\left|g\right|^2 d\mu + \frac{g_0G_0}{f_0F_0}\int_{\Omega} w\left|f\right|^2 d\mu \\ & \leq \left(\frac{M^{1/2}G_0}{m^{1/2}f_0}+\frac{m^{1/2}g_0}{M^{1/2}F_0}\right)\gamma\left(\left|f\right|,\left|g\right|,F_n(\Omega)\right),\end{aligned}$$

where f_0, F_0, g_0 and G_0 are defined by (2.12).

Proof. Follows by the inequalities (2.17) and (2.18) for or $k = m^{1/2}f_0$, $K = M^{1/2}F_0$, $l = m^{1/2}g_0$ and $L = M^{1/2}G_0$. □

We also have:

Corollary 4. *Let $f,g\in L_w^2(\Omega,\mu)$ and $F_n(\Omega)\in\mathfrak{D}_n(\Omega)$, $F_n(\Omega)=\{\Omega_i\}_{i\in\{1,...,n\}}$ be such that there exists $a,b,A,B>0$ with the property (2.13). Then*

$$0\le\int_\Omega w|f|^2d\mu\int_\Omega w|g|^2d\mu-\gamma^2(|f|,|g|,F_n(\Omega))\le\frac{1}{3}n^2\left(ABW_0^2-abw_0^2\right)^2 \tag{2.25}$$

and

$$\int_\Omega w|g|^2d\mu+\frac{bB}{aA}\int_\Omega w|f|^2d\mu\le\left(\frac{BW_0}{aw_0}+\frac{bw_0}{AW_0}\right)\gamma(|f|,|g|,F_n(\Omega)), \tag{2.26}$$

where w_0 and W_0 are defined by (2.16).

Proof. Follows by the inequalities (2.17) and (2.18) for the choices $k=aw_0$, $K=AW_0$, $l=bw_0$ and $L=BW_0$. □

The following result holds:

Theorem 4. *Let $f,g\in L_w^2(\Omega,\mu)$ and $F_n(\Omega)\in\mathfrak{D}_n(\Omega)$, $F_n(\Omega)=\{\Omega_i\}_{i\in\{1,...,n\}}$ be such that there exists $p,P>0$ with the property*

$$0<p\le\frac{\left(\int_{\Omega_i}w|f|^2d\mu\right)^{1/2}}{\left(\int_{\Omega_i}w|g|^2d\mu\right)^{1/2}}\le P<\infty \tag{2.27}$$

for each $i\in\{1,...,n\}$. Then

$$\int_\Omega w|f|^2d\mu\int_\Omega w|g|^2d\mu\le\frac{(p+P)^2}{4pP}\gamma^2(|f|,|g|,F_n(\Omega)), \tag{2.28}$$

$$0\le\left(\int_\Omega w|f|^2d\mu\right)^{1/2}\left(\int_\Omega w|g|^2d\mu\right)^{1/2}-\gamma(|f|,|g|,F_n(\Omega))\le\frac{(P-p)^2}{4(p+P)}\int_\Omega w|g|^2d\mu \tag{2.29}$$

and

$$0\le\int_\Omega w|f|^2d\mu\int_\Omega w|g|^2d\mu-\gamma^2(|f|,|g|,F_n(\Omega))\le\frac{1}{4}(P-p)^2\left(\int_\Omega w|g|^2d\mu\right)^2. \tag{2.30}$$

Proof. We use the following Cassels' inequality [20] (see also [4, p. 72])

$$\frac{\sum_{i=1}^n a_i^2\sum_{i=1}^n b_i^2}{\left(\sum_{i=1}^n a_ib_i\right)^2}\le\frac{(c+C)^2}{4cC} \tag{2.31}$$

that holds provided

$$0<c\le\frac{a_i}{b_i}\le C<\infty \tag{2.32}$$

for any $i\in\{1,...,n\}$.

If we take $a_i=\left(\int_{\Omega_i}w\left|f\right|^2d\mu\right)^{1/2}$, $b_i=\left(\int_{\Omega_i}w\left|g\right|^2d\mu\right)^{1/2}$, $i\in\{1,...,n\}$, $c=p$ and $C=P$, then by (2.31) we get

$$\frac{\sum_{i=1}^n\int_{\Omega_i}w\left|f\right|^2d\mu\sum_{i=1}^n\int_{\Omega_i}w\left|g\right|^2d\mu}{\left(\sum_{i=1}^n\left(\int_{\Omega_i}w\left|f\right|^2d\mu\right)^{1/2}\left(\int_{\Omega_i}w\left|g\right|^2d\mu\right)^{1/2}\right)^2}\le\frac{(p+P)^2}{4pP},$$

which proves (2.28).

Further, we use the following Shisha-Mond inequality [19] (see also [4, p. 82])

$$\left(\sum_{i=1}^n a_i^2\right)^{1/2}\left(\sum_{i=1}^n b_i^2\right)^{1/2}-\sum_{i=1}^n a_ib_i\le\frac{(C-c)^2}{4(c+C)}\sum_{i=1}^n b_i^2 \tag{2.33}$$

that holds provided the condition (2.32) is valid.

Then by taking $a_i=\left(\int_{\Omega_i}w\left|f\right|^2d\mu\right)^{1/2}$, $b_i=\left(\int_{\Omega_i}w\left|g\right|^2d\mu\right)^{1/2}$, $i\in\{1,...,n\}$, $c=p$ and $C=P$ in (2.33) we get the desired result (2.29).

We use the following reverse of CBS inequality [4, p. 78]

$$\sum_{i=1}^n a_i^2\sum_{i=1}^n b_i^2-\left(\sum_{i=1}^n a_ib_i\right)^2\le\frac{1}{4}(C-c)^2\sum_{i=1}^n b_i^2$$

provided the condition (2.32) is valid, for the choice $a_i=\left(\int_{\Omega_i}w\left|f\right|^2d\mu\right)^{1/2}$, $b_i=\left(\int_{\Omega_i}w\left|g\right|^2d\mu\right)^{1/2}$, $i\in\{1,...,n\}$, $c=p$ and $C=P$. Simple calculation yields the desired inequality (2.30). □

In [5] (see also [7, p. 14]) we have shown amongst other that, if $u,v\in L_w^2(\Omega,\mu)$ and there are $a,A\in\mathbb{C}$ with $\operatorname{Re}(\overline{a}A)>0$ and such that

$$\int_\Omega w\operatorname{Re}\left[(Au-v)\left(\overline{v}-\overline{au}\right)\right]d\mu\ge0, \tag{2.34}$$

then

$$\left(\int_\Omega\left|u\right|^2wd\mu\right)^{1/2}\left(\int_\Omega\left|v\right|^2wd\mu\right)^{1/2}\le\frac{1}{2}\frac{\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\int_\Omega wv\overline{u}d\mu\right)}{\sqrt{\operatorname{Re}(\overline{a}A)}}. \tag{2.35}$$

Now, if we replace u with $\overline{u}$ in (2.34), namely, we assume that

$$\int_\Omega w\operatorname{Re}\left[(A\overline{u}-v)\left(\overline{v}-\overline{a}u\right)\right]d\mu\ge0, \tag{2.36}$$

then we have the following inequality of interest:

$$\left(\int_\Omega\left|u\right|^2wd\mu\right)^{1/2}\left(\int_\Omega\left|v\right|^2wd\mu\right)^{1/2}\le\frac{1}{2}\frac{\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\int_\Omega wvud\mu\right)}{\sqrt{\operatorname{Re}(\overline{a}A)}}. \tag{2.37}$$

We observe that if

$$\operatorname{Re}\left[(A\overline{u}-v)\left(\overline{v}-\overline{a}u\right)\right]\ge0\ \mu\text{-a.e. on }\Omega \tag{2.38}$$

then (2.36) is valid for any $w\ge0$ μ-a.e. on Ω.

Moreover, if $A>a>0$ and u,v are real valued and such that $Au\ge v\ge au$ μ-a.e. on Ω, then (2.38) holds true for any $w\ge0$ μ-a.e. on Ω.

Theorem 5. *Let $f,g\in L_w^2(\Omega,\mu)$ and $F_n(\Omega)\in\mathfrak{D}_n(\Omega)$ be such that there exists $a,A\in\mathbb{C}$ with $\operatorname{Re}(\overline{a}A)>0$ and*

$$\int_{\Omega_i} w\operatorname{Re}\left[(A\overline{g}-f)\left(\overline{f}-\overline{a}g\right)\right]d\mu\geq 0 \tag{2.39}$$

for any $i\in\{1,...,n\}$, then we have

$$\begin{aligned}\gamma(|f|,|g|,F_n(\Omega)) &\leq \frac{1}{2}\frac{\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\int_\Omega wfgd\mu\right)}{\sqrt{\operatorname{Re}(\overline{a}A)}} \\ &\leq \frac{1}{2}\frac{|A+a|}{\sqrt{\operatorname{Re}(\overline{a}A)}}\left|\int_\Omega wfgd\mu\right| \leq \frac{1}{2}\frac{|A+a|}{\sqrt{\operatorname{Re}(\overline{a}A)}}\int_\Omega w|fg|\,d\mu.\end{aligned} \tag{2.40}$$

Proof. Let $F_n(\Omega)\in\mathfrak{D}_n(\Omega)$, $F_n(\Omega)=\{\Omega_i\}_{i\in\{1,...,n\}}$.

If the condition (2.39) is true, then by (2.37) we have

$$\left(\int_{\Omega_i}|f|^2 wd\mu\right)^{1/2}\left(\int_{\Omega_i}|g|^2 wd\mu\right)^{1/2}\leq\frac{1}{2}\frac{\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\int_{\Omega_i}wfgd\mu\right)}{\sqrt{\operatorname{Re}(\overline{a}A)}} \tag{2.41}$$

for any $i\in\{1,...,n\}$.

If we sum over i from 1 to n then we get

$$\sum_{i=1}^n\left(\int_{\Omega_i}|f|^2 w_id\mu\right)^{1/2}\left(\int_{\Omega_i}|g|^2 w_id\mu\right)^{1/2}\leq\frac{1}{2}\frac{\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\sum_{i=1}^n\int_{\Omega_i}wfgd\mu\right)}{\sqrt{\operatorname{Re}(\overline{a}A)}},$$

which proves the first inequality in (2.40).

Since

$$\begin{aligned}0&\leq\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\int_\Omega wfgd\mu\right)=\left|\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\int_\Omega wfgd\mu\right)\right| \\ &\leq\left|\left(\overline{A}+\overline{a}\right)\int_\Omega wfgd\mu\right|=\left|\overline{A}+\overline{a}\right|\left|\int_\Omega wfgd\mu\right|=|A+a|\left|\int_\Omega wfgd\mu\right|,\end{aligned}$$

then the second inequality also holds.

The last part is obvious. □

Corollary 5. *If $A>a>0$, $F_n(\Omega)\in\mathfrak{D}_n(\Omega)$ and f,g are positive valued and such that*

$$\int_{\Omega_i}w\left[(Ag-f)(f-ag)\right]d\mu\geq 0, \tag{2.42}$$

for any $i\in\{1,...,n\}$, then

$$\gamma(|f|,|g|,F_n(\Omega))\leq\frac{1}{2}\frac{A+a}{\sqrt{Aa}}\int_\Omega wfgd\mu. \tag{2.43}$$

We have the following result as well:

Theorem 6. *Let $f,g\in L_w^2(\Omega,\mu)$, $F_n(\Omega)\in\mathfrak{D}_n(\Omega)$ be such that there exists $a,A\in\mathbb{C}$ with $a+A\neq 0$ and (2.39) is valid, then we have*

$$\begin{aligned}\gamma(|f|,|g|,F_n(\Omega))&\leq\operatorname{Re}\left[\frac{\overline{a}+\overline{A}}{|a+A|}\int_\Omega wfgd\mu\right]+\frac{1}{4}\frac{|A-a|^2}{|a+A|}\int_\Omega w|g|^2\,d\mu \\ &\leq\left|\int_\Omega wfgd\mu\right|+\frac{1}{4}\frac{|A-a|^2}{|a+A|}\int_\Omega w|g|^2\,d\mu.\end{aligned} \tag{2.44}$$

Proof. Let $F_n(\Omega) \in \mathfrak{D}_n(\Omega)$, $F_n(\Omega) = \{\Omega_i\}_{i\in\{1,\dots,n\}}$.

We use the following inequality obtained in [6] (see also [7, p. 32])

$$\left(\int_\Omega |f|^2 \ell d\mu\right)^{1/2} \left(\int_\Omega |f|^2 \ell d\mu\right)^{1/2} \le \operatorname{Re}\left[\frac{\overline{a}+\overline{A}}{|a+A|}\int_\Omega \ell f g d\mu\right] + \frac{1}{4}\frac{|A-a|^2}{|a+A|}\int_\Omega \ell |g|^2 d\mu, \tag{2.45}$$

where ℓ in μ-integrable, $\ell \ge 0$ μ-a.e. on Ω and

$$\int_\Omega w \operatorname{Re}\left[(A\overline{g}-f)\left(\overline{f}-\overline{a}g\right)\right] \ge 0.$$

Now, if we write the inequality (2.45) on Ω_i with $i \in \{1,\dots,n\}$, then we get

$$\left(\int_{\Omega_i} |f|^2 w d\mu\right)^{1/2} \left(\int_{\Omega_i} |g|^2 w d\mu\right)^{1/2} \le \operatorname{Re}\left[\frac{\overline{a}+\overline{A}}{|a+A|}\int_{\Omega_i} w f g d\mu\right] + \frac{1}{4}\frac{|A-a|^2}{|a+A|}\int_{\Omega_i} w |g|^2 d\mu, \tag{2.46}$$

for any $i \in \{1,\dots,n\}$.

If we sum over i from 1 to n in (2.46), then we get

$$\sum_{i=1}^n \left(\int_{\Omega_i} |f|^2 w_i d\mu\right)^{1/2} \left(\int_{\Omega_i} |g|^2 w_i d\mu\right)^{1/2} \le \operatorname{Re}\left[\frac{\overline{a}+\overline{A}}{|a+A|}\sum_{i=1}^n\int_{\Omega_i} w f g d\mu\right] + \frac{1}{4}\frac{|A-a|^2}{|a+A|}\sum_{i=1}^n\int_{\Omega_i} w |g|^2 d\mu \tag{2.47}$$

that proves the first inequality in (2.44).

The second inequality follows by the fact that

$$\operatorname{Re}\left[\frac{\overline{a}+\overline{A}}{|a+A|}\int_\Omega w f g d\mu\right] \le \left|\frac{\overline{a}+\overline{A}}{|a+A|}\int_\Omega w f g d\mu\right| = \left|\frac{\overline{a}+\overline{A}}{|a+A|}\right|\left|\int_\Omega w f g d\mu\right| = \left|\int_\Omega w f g d\mu\right|.$$

□

Corollary 6. *If $A > a > 0$, $F_n(\Omega) \in \mathfrak{D}_n(\Omega)$ and f, g are positive valued and such that (2.42) is valid for any $i \in \{1,\dots,n\}$, then*

$$\gamma(|f|, |g|, F_n(\Omega)) \le \int_\Omega w f g d\mu + \frac{1}{4}\frac{(A-a)^2}{a+A}\int_\Omega w g^2 d\mu. \tag{2.48}$$

3. Discrete Inequalities

For a nonempty finite family of indices J and positive weights w_j, $j \in J$ we denote $W_J := \sum_{j\in J} w_j$. Assume that, for $n \ge 2$, the family J of indices containing more than n elements and $F_n(J) = \{J_i\}_{i\in\{1,\dots,n\}}$ is a n-division for J, namely $J = \bigcup_{i=1}^n J_i$ and $J_i \cap J_j = \emptyset$ for any $i, j \in \{1,\dots,n\}$ with $i \ne j$.

For a given $n \geq 2$ we denote by $\mathfrak{D}_n(J)$ the set of all n-divisions of J and consider the functional $\gamma(x, y, \cdot) : \mathfrak{D}_n(J) \to \mathbb{R}$ defined by

$$\gamma(x, y, F_n(J)) := \sum_{i=1}^{n} \left(\sum_{j \in J_i} w_j |x_j|^2 \right)^{1/2} \left(\sum_{j \in J_i} w_j |y_j|^2 \right)^{1/2}, \tag{3.1}$$

where $x = \{x_j\}_{j \in J}$ and $y = \{y_j\}_{j \in J} \subset \mathbb{C}$.

From (2.1) we have

$$\left| \sum_{j \in J} w_j x_j y_j \right| \leq \gamma(x, y, F_n(J)) \leq \left(\sum_{j \in J} w_j |x_j|^2 \right)^{1/2} \left(\sum_{j \in J} w_j |y_j|^2 \right)^{1/2}. \tag{3.2}$$

If $F_n(J) = \{J_i\}_{i \in \{1,\ldots,n\}}$ is a n-division for J and there exists k, K, l, L so that

$$0 < k \leq \left(\sum_{j \in J_i} w_j |x_j|^2 \right)^{1/2} \leq K < \infty \tag{3.3}$$

and

$$0 < l \leq \left(\sum_{j \in J_i} w_j |y_j|^2 \right)^{1/2} \leq L < \infty \tag{3.4}$$

for each $i \in \{1, ..., n\}$, then by Theorem 2 we have

$$\begin{aligned} &\left(\sum_{j \in J} w_j |x_j|^2 \right)^{1/2} \left(\sum_{j \in J} w_j |y_j|^2 \right)^{1/2} \\ &\leq \frac{kl + KL}{2\sqrt{klKL}} \sum_{i=1}^{n} \left(\sum_{j \in J_i} w_j |x_j|^2 \right)^{1/2} \left(\sum_{j \in J_i} w_j |y_j|^2 \right)^{1/2} \end{aligned} \tag{3.5}$$

and

$$0 \leq \frac{\sum_{j \in J} w_j |x_j|^2}{\gamma(x, y, F_n(J))} - \frac{\gamma(x, y, F_n(J))}{\sum_{j \in J} w_j |y_j|^2} \leq \left(\sqrt{\frac{K}{l}} - \sqrt{\frac{k}{L}} \right)^2. \tag{3.6}$$

From Theorem 3 we also have

$$\begin{aligned} 0 &\leq \sum_{j \in J} w_j |x_j|^2 \sum_{j \in J} w_j |y_j|^2 \\ &\quad - \left(\sum_{i=1}^{n} \left(\sum_{j \in J_i} w_j |x_j|^2 \right)^{1/2} \left(\sum_{j \in J_i} w_j |y_j|^2 \right)^{1/2} \right)^2 \\ &\leq \frac{1}{3} n^2 (KL - kl)^2 \end{aligned} \tag{3.7}$$

and

$$\sum_{j\in J} w_j |y_j|^2 + \frac{lL}{kK}\sum_{j\in J} w_j |x_j|^2 \leq \left(\frac{L}{k}+\frac{l}{K}\right)\sum_{i=1}^{n}\left(\sum_{j\in J_i} w_j |x_j|^2\right)^{1/2}\left(\sum_{j\in J_i} w_j |y_j|^2\right)^{1/2}. \tag{3.8}$$

If $F_n(J) = \{J_i\}_{i\in\{1,\dots,n\}}$ is a n-division for J and there exists $a, A \in \mathbb{C}$ with $\operatorname{Re}(\overline{a}A) > 0$ and

$$\sum_{j\in J_i} w_j \operatorname{Re}\left[\left(A\overline{y_j} - x_j\right)\left(\overline{x_j} - \overline{a}y_j\right)\right] \geq 0 \tag{3.9}$$

for any $i \in \{1, ..., n\}$, then by Theorem 5 we have

$$\begin{aligned} &\sum_{i=1}^{n}\left(\sum_{j\in J_i} w_j |x_j|^2\right)^{1/2}\left(\sum_{j\in J_i} w_j |y_j|^2\right)^{1/2} \\ &\leq \frac{1}{2}\frac{\operatorname{Re}\left(\left(\overline{A}+\overline{a}\right)\sum_{j\in J} w_j x_j y_j\right)}{\sqrt{\operatorname{Re}(\overline{a}A)}} \\ &\leq \frac{1}{2}\frac{|A+a|}{\sqrt{\operatorname{Re}(\overline{a}A)}}\left|\sum_{j\in J} w_j x_j y_j\right| \leq \frac{1}{2}\frac{|A+a|}{\sqrt{\operatorname{Re}(\overline{a}A)}}\sum_{j\in J} w_j |x_j y_j|. \end{aligned} \tag{3.10}$$

If $F_n(J) = \{J_i\}_{i\in\{1,\dots,n\}}$ is a n-division for J and there exists $a, A \in \mathbb{C}$ with $a + A \neq 0$ and (3.9) is valid, then by Theorem 6 we also have

$$\begin{aligned} &\sum_{i=1}^{n}\left(\sum_{j\in J_i} w_j |x_j|^2\right)^{1/2}\left(\sum_{j\in J_i} w_j |y_j|^2\right)^{1/2} \\ &\leq \operatorname{Re}\left[\frac{\overline{a}+\overline{A}}{|a+A|}\sum_{j\in J} w_j x_j y_j\right] + \frac{1}{4}\frac{|A-a|^2}{|a+A|}\sum_{j\in J_i} w_j |y_j|^2 \\ &\leq \left|\sum_{j\in J} w_j x_j y_j\right| + \frac{1}{4}\frac{|A-a|^2}{|a+A|}\sum_{j\in J_i} w_j |y_j|^2. \end{aligned} \tag{3.11}$$

References

[1] J. M. Aldaz, Strengthened Cauchy-Schwarz and Hölder inequalities. *J. Inequal. Pure Appl. Math.* **10** (2009), no. 4, Article 116, 6 pp

[2] N. S. Barnett, S. S. Dragomir and I. Gomm, On some integral inequalities related to the Cauchy-Bunyakovsky-Schwarz inequality. *Appl. Math. Lett.* **23** (2010), no. 9, 1008–1012.

[3] J. B. Diaz and F. T. Metcalf, Stronger forms of a class of inequalities of G. Pólya-G. Szegö and L.V. Kantorovich, *Bull. Amer. Math. Soc.*, **69** (1963), 415-418.

[4] S. S. Dragomir, A survey on Cauchy-Bunyakovsky-Schwarz type discrete inequalities. *J. Inequal. Pure Appl. Math.* **4** (2003), no. 3, Article 63, 142 pp.

[5] S. S. Dragomir, Reverses of Schwarz, triangle and Bessel inequalities in inner product spaces. J. *Inequal. Pure Appl. Math.* **5** (2004), no. 3, Article 76, 18 pp.

[6] S. S. Dragomir, New reverses of Schwarz, triangle and Bessel inequalities in inner product spaces. *Aust. J. Math. Anal. Appl.* **1** (2004), no. 1, Art. 1, 18 pp.

[7] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces.* Nova Science Publishers, Inc., Hauppauge, NY, 2005. viii+249 pp. ISBN: 1-59454-202-3

[8] S. S. Dragomir, Refinements of the CauchyBunyakovsky-Schwarz inequality for functions of selfadjoint operators in Hilbert spaces. *Linear Multilinear Algebra* **59** (2011), no. 7, 711–717.

[9] M. Hajja, A generalized Cauchy-Schwarz inequality. *Math. Inequal. Appl.* **18** (2015), no. 3, 893–899.

[10] N. J. A. Harvey, A generalization of the Cauchy-Schwarz inequality involving four vectors. *J. Math. Inequal.* **9** (2015), no. 2, 489–491

[11] A. Ibrahim and S. S. Dragomir, A survey on CauchyBunyakovsky-Schwarz inequality for power series. in *Analytic Number Theory, Approximation Theory, and Special Functions,* 247–295, Springer, New York, 2014.

[12] S. Izumino and J. Pečarić, A weighted version of Ozeki's inequality, *Sci. Math. Japonicae*, **56**(3) (2002), 511-526.

[13] M. Jeong, Inequalities via power series and Cauchy-Schwarz inequality. *J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math.* **19** (2012), no. 3, 305–313.

[14] C. Lupu and D. Schwarz, Another look at some new Cauchy-Schwarz type inner product inequalities. *Appl. Math. Comput.* **231** (2014), 463–477

[15] M. Masjed-Jamei and N. Hussain, More results on a functional generalization of the Cauchy-Schwarz inequality. *J. Inequal. Appl.* **2012**, 2012:239, 9 pp.

[16] I. Pinelis, On the Hölder and Cauchy–Schwarz inequalities. *Amer. Math. Monthly* **122** (2015), no. 6, 593–595.

[17] G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Volume 1: Series, Integral Calculus, Theory of Functions (in English), translated from german by D. Aeppli, corrected printing of the revised translation of the fourth German edition, Springer Verlag, New York, 1972.

[18] R. Sharma, R. Bhandari and M. Gupta, Inequalities related to the Cauchy-Schwarz inequality. *Sankhya A* **74** (2012), no. 1, 101–111.

[19] O. Shisha and B. Mond, Bounds on differences of means, *Inequalities*, Academic Press Inc., New York, 1967, pp. 293-308.

[20] G. S. Watson, G. Alpargu and G. P. H. Styan, Some comments on six inequalities associated with the inefficiency of ordinary least squares with one regressor, *Linear Algebra and its Appl.*, **264** (1997), 13-54.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa