Received 05/08/15

A Note on Numerical Radius and the Kreĭn-Lin Inequality

S. S. Dragomir[1,2]
[1]Mathematics, College of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia.
sever.dragomir@vu.edu.au
http://rgmia.org/dragomir
[2]School of Computer Science & Applied Mathematics,
University of the Witwatersrand, Private Bag 3,
Johannesburg 2050, South Africa

August 5, 2015

Abstract

In this note we show that the Kreĭn-Lin triangle inequality can be naturally applied to obtain an elegant reverse for a classical numerical radius power inequality for bounded linear operators on complex Hilbert space due to C. Pearcy.

1 Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex inner product space and $x, y \in H$ two nonzero vectors. One can define the *angle* between the vectors x, y either by the standard formula

$$\cos \Phi_{x,y} = \frac{\operatorname{Re} \langle x, y \rangle}{\|x\| \|y\|}$$

or by

$$\cos \Psi_{x,y} = \frac{|\langle x, y \rangle|}{\|x\| \|y\|}.$$

The function $\Psi_{x,y}$ is a natural metric on complex projective space [6].

In 1969 M. K. Kreĭn [5] obtained the following inequality for angles between two vectors

$$\Phi_{x,y} \leq \Phi_{x,z} + \Phi_{x,z} \tag{1}$$

for any $x, y, z \in H \setminus \{0\}$.

By using the representation

$$\Psi_{x,y} = \inf_{\alpha,\beta\in\mathbb{C}\setminus\{0\}} \Phi_{\alpha x,\beta y} = \inf_{\alpha\in\mathbb{C}\setminus\{0\}} \Phi_{\alpha x,y} = \inf_{\beta\in\mathbb{C}\setminus\{0\}} \Phi_{x,\beta y} \tag{2}$$

and Kreĭn's inequality (1), M. Lin [6] has shown recently that the following triangle inequality is also valid

$$\Psi_{x,y} \leq \Psi_{x,z} + \Psi_{y,z} \tag{3}$$

for any $x, y, z \in H \setminus \{0\}$.

In this note we show that the Kreĭn-Lin triangle inequality (3) can be naturally applied to obtain an elegant reverse for a classical numerical radius power inequality for bounded linear operators on complex Hilbert space due to C. Pearcy [7].

2 A Reverse Inequality

Let $(H; \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [3, p. 1]:

$$W(T) = \{\langle Tx, x\rangle, \ x \in H, \ \|x\| = 1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is given by [3, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{4}$$

It is well known that $w(\cdot)$ is a *norm* on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$. This norm is equivalent to the operator norm. In fact, the following more precise result holds [3, p. 9]:

$$w(T) \leq \|T\| \leq 2w(T), \tag{5}$$

for any $T \in B(H)$

For other results on numerical radii, see [4, Chapter 11], [3] and the recent monograph [2].

The following result is well known in the literature [7]:

$$w(T^n) \leq w^n(T) \tag{6}$$

for each positive integer n and any operator $T \in B(H)$.

The following elegant reverse inequalities for $n = 2$ can be derived from the Kreĭn-Lin triangle inequality (3).

Theorem 1 *For any $T \in B(H)$ we have*

$$w^2(T) \leq w\left(T^2\right) + \inf_{\lambda\in\mathbb{C}} \|T - \lambda I\|^2. \tag{7}$$

Proof. The inequality (3) is equivalent to

$$\cos \Psi_{x,y} \geq \cos \left(\Psi_{x,z} + \Psi_{y,z} \right) = \cos \Psi_{x,z} \cos \Psi_{y,z} - \sin \Psi_{x,z} \sin \Psi_{y,z}, \quad (8)$$

or to

$$\frac{|\langle x, y \rangle|}{\|x\| \|y\|} + \sqrt{1 - \frac{|\langle x, z \rangle|^2}{\|x\|^2 \|z\|^2}} \sqrt{1 - \frac{|\langle y, z \rangle|^2}{\|y\|^2 \|z\|^2}} \geq \frac{|\langle x, z \rangle|}{\|x\| \|z\|} \frac{|\langle y, z \rangle|}{\|y\| \|z\|} \quad (9)$$

for any $x, y, z \in H \setminus \{0\}$.

If we multiply (10) by $\|x\| \|z\|^2 \|y\| > 0$ then we get

$$\begin{aligned} &|\langle x, y \rangle| \|z\|^2 + \sqrt{\|x\|^2 \|z\|^2 - |\langle x, z \rangle|^2} \sqrt{\|y\|^2 \|z\|^2 - |\langle y, z \rangle|^2} \\ &\geq |\langle x, z \rangle| |\langle y, z \rangle|. \end{aligned} \quad (10)$$

We notice that the inequality (10) remains true, becoming equality, if either $x = 0$ or $y = 0$ or $z = 0$.

We know that for any $u, e \in H$ with $\|e\| = 1$ we have the representation (see for instance [1, Lemma 2.4])

$$\|u\|^2 - |\langle u, e \rangle|^2 = \|u - \langle u, e \rangle e\|^2 = \inf_{\lambda \in \mathbb{C}} \|u - \lambda e\|^2.$$

Then by (10) we have for any $x, y, z \in H$ with $\|z\| = 1$ that

$$|\langle x, y \rangle| + \inf_{\lambda \in \mathbb{C}} \|x - \lambda z\| \inf_{\mu \in \mathbb{C}} \|y - \mu z\| \geq |\langle x, z \rangle| |\langle y, z \rangle|. \quad (11)$$

By taking $x = Tz$ and $y = T^* z$ in (11) we get

$$\begin{aligned} |\langle Tz, z \rangle| |\langle T^* z, z \rangle| &\leq |\langle Tz, T^* z \rangle| + \inf_{\lambda \in \mathbb{C}} \|Tz - \lambda z\| \inf_{\mu \in \mathbb{C}} \|T^* z - \mu z\| \\ &\leq |\langle Tz, T^* z \rangle| + \|Tz - \lambda z\| \|T^* z - \mu z\| \end{aligned}$$

for any $z \in H$ with $\|z\| = 1$ and $\lambda, \mu \in \mathbb{C}$.

Therefore

$$|\langle Tz, z \rangle|^2 \leq \left| \left\langle T^2 z, z \right\rangle \right| + \|Tz - \lambda z\| \|T^* z - \mu z\|$$

for any $z \in H$ with $\|z\| = 1$ and $\lambda, \mu \in \mathbb{C}$.

By taking the supremum over $z \in H$ with $\|z\| = 1$, we deduce

$$w^2 (T) \leq w \left(T^2 \right) + \|T - \lambda I\| \|T^* - \mu I\| \quad (12)$$

for any $\lambda, \mu \in \mathbb{C}$.

Finally, by taking the infimum in (12) over $\lambda, \mu \in \mathbb{C}$ and since

$$\inf_{\mu \in \mathbb{C}} \|T^* - \mu I\| = \inf_{\mu \in \mathbb{C}} \|T - \overline{\mu} I\| = \inf_{\lambda \in \mathbb{C}} \|T - \lambda I\|,$$

we deduce the desired result (7). ■

Corollary 2 *Let $T \in B(H)$. If there exist $\omega \in \mathbb{C}$ and $r > 0$ such that $\|T - \omega I\| \leq r$, then*

$$w^2 (T) \leq w \left(T^2 \right) + r^2. \quad (13)$$

References

[1] S. S. Dragomir, Some Grüss type inequalities in inner product spaces. *J. Inequal. Pure Appl. Math.* **4** (2003), No. 2, Article 42, 10 pp. [http://www.emis.de/journals/JIPAM/images/032_03_JIPAM/032_03.pdf].

[2] S. S. Dragomir, *Inequalities for the Numerical Radius of Linear Operators in Hilbert Spaces.* Springer Briefs in Mathematics. Springer, 2013. x+120 pp. ISBN: 978-3-319-01447-0; 978-3-319-01448-7.

[3] K. E. Gustafson and D. K. M. Rao, *Numerical Range,* Springer-Verlag, New York, Inc., 1997.

[4] P. R. Halmos, *A Hilbert Space Problem Book,* Springer-Verlag, New York, Heidelberg, Berlin, Second edition, 1982.

[5] M. K. Kreĭn, Angular localization of the spectrum of a multiplicative integral in a Hilbert space, *Funct. Anal. Appl.* **3** (1969), 89–90.

[6] M. Lin, Remarks on Kreĭn's inequality, *The Math. Intelligencer*, **34** (2012), No.1, 3-4.

[7] C. Pearcy, An elementary proof of the power inequality for the numerical radius, *Michigan Math. J.* **13** (1966), 289–291.