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A Note on the Wang-Zhang and Schwarz Inequalities

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Abstract

In this note we show that the Wang-Zhang inequality can be naturally applied to obtain an elegant reverse for the classical Schwarz inequality in complex inner product spaces.

1 Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex inner product space and $x, y \in H$ two nonzero vectors. One can define the *angle* between the vectors x, y either by

$$\cos \Phi_{x,y} = \frac{\operatorname{Re} \langle x, y \rangle}{\|x\| \|y\|} \text{ or by } \cos \Psi_{x,y} = \frac{|\langle x, y \rangle|}{\|x\| \|y\|}.$$

The function $\Psi_{x,y}$ is a natural metric on complex projective space [6].

In 1969 M. K. Kreĭn [5] obtained the following inequality for angles between two vectors

$$\Phi_{x,y} \leq \Phi_{x,z} + \Phi_{x,z} \tag{1}$$

for any $x, y, z \in H \setminus \{0\}$.

By using the representation

$$\Psi_{x,y} = \inf_{\alpha,\beta \in \mathbb{C} \setminus \{0\}} \Phi_{\alpha x, \beta y} = \inf_{\alpha \in \mathbb{C} \setminus \{0\}} \Phi_{\alpha x, y} = \inf_{\beta \in \mathbb{C} \setminus \{0\}} \Phi_{x, \beta y} \tag{2}$$

and Kreĭn's inequality (1), M. Lin [6] has shown recently that the following triangle inequality is also valid

$$\Psi_{x,y} \leq \Psi_{x,z} + \Psi_{y,z} \tag{3}$$

for any $x, y, z \in H \setminus \{0\}$.

The following inequality has been obtained by Wang and Zhang in [9] (see also [11, p. 195])

$$\sqrt{1 - \frac{|\langle x, y\rangle|^2}{\|x\|^2 \|y\|^2}} \leq \sqrt{1 - \frac{|\langle x, z\rangle|^2}{\|x\|^2 \|z\|^2}} + \sqrt{1 - \frac{|\langle y, z\rangle|^2}{\|y\|^2 \|z\|^2}} \tag{4}$$

for any $x, y, z \in H \setminus \{0\}$. Using the above notations it can be written as [6]

$$\sin \Psi_{x,y} \leq \sin \Psi_{x,z} + \sin \Psi_{x,z} \tag{5}$$

for any $x, y, z \in H \setminus \{0\}$. It also provides another triangle type inequality complementing the Kreĭn and Lin inequalities above.

In this note we show that the Wang-Zhang inequality can be naturally applied to obtain an elegant reverse for the classical Schwarz inequality in complex inner product spaces.

2 Reverse of Schwarz Inequality

In the sequel we assume that $(H, \langle \cdot, \cdot \rangle)$ is a complex inner product space. The inequality

$$|\langle x, y\rangle|^2 \leq \|x\|^2 \|y\|^2 \text{ for } x, y \in H \tag{6}$$

is well know in the literature as the *Schwarz inequality*. The equality holds in (6) iff x and y are linearly dependent.

Theorem 1 *Let $x, y, z \in H$ with $\|z\| = 1$ and $\alpha, \beta \in \mathbb{C}$, $r, s > 0$ such that*

$$\|x - \alpha z\| \leq r \text{ and } \|y - \beta z\| \leq s. \tag{7}$$

Then

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq (r \|y\| + s \|x\|)^2. \tag{8}$$

Proof. If we multiply (4) by $\|x\| \|y\| \|z\| > 0$, then we get

$$\begin{aligned} &\|z\| \sqrt{\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2} \\ &\leq \|y\| \sqrt{\|x\|^2 \|z\|^2 - |\langle x, z\rangle|^2} + \|x\| \sqrt{\|y\|^2 \|z\|^2 - |\langle y, z\rangle|^2} \end{aligned} \tag{9}$$

for any $x, y, z \in H \setminus \{0\}$.

We observe that, if either $x = 0$ or $y = 0$, then the inequality (9) reduces to an equality.

Let $z \in H$ with $\|z\| = 1$, and since (see for instance [2, Lemma 2.4])

$$\|x\|^2 - |\langle x, z\rangle|^2 = \inf_{\lambda \in \mathbb{C}} \|x - \lambda z\|^2 \text{ and } \|y\|^2 - |\langle y, z\rangle|^2 = \inf_{\mu \in \mathbb{C}} \|y - \mu z\|^2$$

then by (9) we have

$$\sqrt{\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2} \leq \|y\| \inf_{\lambda \in \mathbb{C}} \|x - \lambda z\| + \|x\| \inf_{\mu \in \mathbb{C}} \|y - \mu z\|, \tag{10}$$

for any $x, y, z \in H$ with $\|z\| = 1$.

Since, by (7)

$$\inf_{\lambda \in \mathbb{C}} \|x - \lambda z\| \leq \|x - \alpha z\| \leq r \text{ and } \inf_{\mu \in \mathbb{C}} \|y - \mu z\| \leq \|y - \beta z\| \leq s,$$

then by (10) we obtain the desired result (8). ■

Corollary 2 *Let $x, y, z \in H$ with $\|z\| = 1$ and $\lambda, \Lambda, \gamma, \Gamma \in \mathbb{C}$ with $\lambda \neq \Lambda$, $\gamma \neq \Gamma$ and such that either*

$$\mathrm{Re}\langle \Lambda z - x, x - \lambda z\rangle \geq 0 \text{ and } \mathrm{Re}\langle \Gamma z - y, y - \gamma z\rangle \geq 0 \tag{11}$$

or, equivalently

$$\left\| x - \frac{\lambda + \Lambda}{2} z \right\| \leq \frac{1}{2}|\Lambda - \lambda| \text{ and } \left\| y - \frac{\gamma + \Gamma}{2} z \right\| \leq \frac{1}{2}|\Gamma - \gamma|$$

are valid. Then

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4}\left(|\Lambda - \lambda| \|y\| + |\Gamma - \gamma| \|x\|\right)^2. \tag{12}$$

Proof. Follows by Theorem 1 on observing that

$$\mathrm{Re}\langle \Delta e - u, u - \delta e\rangle = \frac{1}{4}|\Delta - \delta|^2 - \left\| u - \frac{\delta + \Delta}{2} e \right\|^2$$

for any $\delta, \Delta \in \mathbb{C}$ with $\delta \neq \Delta$ and $u, e \in H$ with $\|e\| = 1$. ■

We give an example for n-tuples of complex numbers.

Let $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ and $z = (z_1, ..., z_n)$ be n-tuples of complex numbers, $p = (p_1, ..., p_n)$ a probability distribution, i.e. $p_i > 0$ $i \in \{1, ..., n\}$ and $\sum_{i=1}^n p_i = 1$, with $\sum_{i=1}^n p_i |z_i|^2 = 1$ and $\lambda, \Lambda, \gamma, \Gamma \in \mathbb{C}$ with $\lambda \neq \Lambda$, $\gamma \neq \Gamma$ and such that

$$\mathrm{Re}\left[(\Lambda z_i - x_i)\left(\overline{x}_i - \overline{\lambda}\overline{z}_i\right)\right] \geq 0 \text{ and } \mathrm{Re}\left[(\Gamma z_i - \overline{y}_i)(\overline{y}_i - \overline{\gamma z_i})\right] \geq 0$$

or, equivalently

$$\left| x_i - \frac{\lambda + \Lambda}{2} z_i \right| \leq \frac{1}{2}|\Lambda - \lambda| \text{ and } \left| y_i - \frac{\gamma + \Gamma}{2} z_i \right| \leq \frac{1}{2}|\Gamma - \gamma|$$

for any $i \in \{1, ..., n\}$. Then

$$\sum_{i=1}^{n} p_i \operatorname{Re}\left[(\Lambda z_i - x_i)\left(\overline{x}_i - \overline{\lambda}\overline{z}_i\right)\right] \geq 0 \text{ and } \sum_{i=1}^{n} p_i \operatorname{Re}\left[(\Gamma z_i - \overline{y}_i)\left(\overline{y}_i - \overline{\gamma z_i}\right)\right] \geq 0$$

and by applying Corollary 2 for the inner product $\langle \cdot, \cdot \rangle_p : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ with $\langle x, y \rangle_p = \sum_{i=1}^{n} p_i x_i \overline{y}_i$, we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2 - \left| \sum_{i=1}^{n} p_i x_i \overline{y}_i \right|^2 \qquad (13) \\ &\leq \frac{1}{4} \left[|\Lambda - \lambda| \left(\sum_{i=1}^{n} p_i |y_i|^2 \right)^{1/2} + |\Gamma - \gamma| \left(\sum_{i=1}^{n} p_i |x_i|^2 \right)^{1/2} \right]^2 . \end{aligned}$$

If $0 < a \leq a_i \leq A < \infty$ and $0 < b \leq b_i \leq B < \infty$ for any $i \in \{1, ..., n\}$ then by (13) we have for any $p = (p_1, ..., p_n)$ a probability distribution that

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 - \left(\sum_{i=1}^{n} p_i a_i b_i \right)^2 \qquad (14) \\ &\leq \frac{1}{4} \left[(A - a) \left(\sum_{i=1}^{n} p_i b_i^2 \right)^{1/2} + (B - b) \left(\sum_{i=1}^{n} p_i a_i^2 \right)^{1/2} \right]^2 . \end{aligned}$$

The interested reader may compare this new result with the classical reverses of Schwarz inequality obtained by Diaz and Metcalf [1], Ozeki [4], G. Pólya and G. Szegö [7], Shisha and Mond [8] and Cassels [10].

For other reverses of Schwarz inequality in complex inner product spaces see the monograph [3] and the references therein.

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