Received 07/08/15

ON HERMITE-HADAMARD-FEJÉR INEQUALITY TYPE FOR CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this study, we have established some generalized integral inequalities of Hermite-Hadamard-Fejér type for generalized fractional integrals. The results presented here would provide generalizations of those given in earlier works.

1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function define on an interval I of real numbers, and $a, b \in I$ with $a < b$. Then the following inequalities hold:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

It was first discovered by Hermite in 1881 in the Journal Mathesis. This inequality (1.1) was nowhere mentioned in the mathematical literature untill 1893. In [4], Beckenbach, a leading expert on the theory of convex functions, wrote that the inequality (1.1) was proved by Hadamard in 1893. In 1974, Mitrinovič found Hermite and Hadamard's note in Mathesis. That is why, the inequality (1.1) was known as Hermite-Hadamard inequality. We note that HermiteHadamard's inequality may be regarded as a refinements of the concept of convexity and it follows easily from Jensen's inequality. This inequality (1.1) has been received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in $[4-16, 21]$.

The most well known inequalities connected with the integral mean of a convex functions are Hermite-Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejĕr inequality.

In this study, we have established some generalized fractional integral inequalities. The results presented here would provide generalizations of those given in earlier works.

2. Preliminares

Definition 1. ([5, 19])*Let $h(x)$ be an increasing and positive monotone function on $[0,\infty)$, also derivative $h'(x)$ is continuous on $[0,\infty)$ and $h(0) = 0$. The space $X_h^p(0,\infty)$ $(1 \leq p < \infty)$ of those real-valued Lebesque measurable functions f on*

Key words and phrases. Integral inequalities, Fractional integrals, Hermite-Hadamard Inequality, Preinvex functions, Hölder's inequality.

2010 Mathematics Subject Classification 26D15, 26A51, 26A33, 26A42.

$[0,\infty)$ for which

$$\|f\|_{X_h^p} = \left(\int_0^\infty |f(t)|^p h'(x)\,dt\right)^{\frac{1}{p}} < \infty, \quad 1 \le p \le \infty \tag{2.1}$$

and for the case $p=\infty$

$$\|f\|_{X_h^\infty} = ess \sup_{1\le t<\infty}\left[f(t)h'(x)\right]. \tag{2.2}$$

Definition 2. *In particular, when $h(x)=x$ $(1\le p<\infty)$ the space $X_h^p(0,\infty)$ coincides with the $L_p[0,\infty)-$space ($\|f\|_{X_h^\infty}=\|f\|_\infty$) and also if we take $h(x)=\dfrac{x^{k+1}}{k+1}$ $(k\ge 0)$ the space $X_h^p(0,\infty)$ coincides with the $L_{p,k}[0,\infty)-$space.*

Definition 3. *Let (a,b) be a finite interval of the real line $\mathbb{R}$ and $\alpha>0$. Also let $h(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $h'(x)$ on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function h on $[a,b]$ are defined by* $[1,5,19]$

$$\left(J^\alpha_{a^+,h}f\right)(x) := \tfrac{1}{\Gamma(\alpha)}\int_a^x [h(x)-h(t)]^{\alpha-1} h'(t)\, f(t)\,dt,\ x\ge a \tag{2.3}$$

and

$$\left(J^\alpha_{b^-,h}f\right)(x) := \tfrac{1}{\Gamma(\alpha)}\int_x^b [h(t)-h(x)]^{\alpha-1} h'(t)\, f(t)\,dt,\ x\le b. \tag{2.4}$$

Definition 4. *If we take $h(x)=x$, then the equalities* (2.3) *and* (2.4) *will be*

$$\left(J^\alpha_{a^+}f\right)(x) = \tfrac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt,\ x>a \tag{2.5}$$

and

$$\left(J^\alpha_{b^-}f\right)(x) = \tfrac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt,\ b>x. \tag{2.6}$$

These integrals are called right-sided Riemann-Liouville fractional integral and left-sided Riemann-Liouville fractional integral respectively $[1-3,17,20]$.

Theorem 1. *Let $f: I\to\mathbb{R}$ be a convex on I and let $a,b\in I$ with $a<b$. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \frac{1}{b-a}\int_a^b f(t)g(x)dt \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{2.7}$$

holds, where $f:[a,b]\to\mathbb{R}$ is nonnegative, integrable, and symmetric to $\dfrac{a+b}{2}$ [7].

3. Main Results

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and let $g:[a,b]\to\mathbb{R}$. If $f',\ g\in X_h^p[a,b]$, then the following identity for fractional*

integrals holds:

$$
\begin{aligned}
&f\left(h\left(\tfrac{a+b}{2}\right)\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]\\
&-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}\left(g\times(f\circ h)\right)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}\left(g\times(f\circ h)\right)(b)\right]\\
&=\frac{1}{\Gamma(\alpha)}\int_a^b k(t)df(h(t))
\end{aligned}
\tag{3.1}
$$

where

$$
k(t)=\begin{cases}\int_a^t (h(s)-h(a))^{\alpha-1}g(s)h'(s)ds & t\in\left[a,\frac{a+b}{2}\right],\\ \int_b^t (h(b)-h(s))^{\alpha-1}g(s)h'(s)ds & t\in\left[\frac{a+b}{2},a\right].\end{cases}
$$

Proof. It suffices to note that

$$
\begin{aligned}
I &=\int_a^b k(t)df(h(t))\\
&=\int_a^{\frac{a+b}{2}}\left(\int_a^t (h(s)-h(a))^{\alpha-1}g(s)h'(s)ds\right)df(h(t))\\
&+\int_{\frac{a+b}{2}}^b\left(\int_b^t (h(b)-h(s))^{\alpha-1}g(s)h'(s)ds\right)df(h(t))\\
&=I_1+I_2.
\end{aligned}
$$

By integration by parts, we get

$$
\begin{aligned}
I_1 &=\int_a^{\frac{a+b}{2}}\left(\int_a^t (h(s)-h(a))^{\alpha-1}g(s)h'(s)ds\right)df(h(t))\\
&=\left(\int_a^t (h(s)-h(a))^{\alpha-1}g(s)h'(s)ds\right)f(h(t))\Big|_a^{\frac{a+b}{2}}\\
&-\int_a^{\frac{a+b}{2}}(h(t)-h(a))^{\alpha-1}g(t)f(h(t))h'(t)dt\\
&=f\left(h\left(\tfrac{a+b}{2}\right)\right)\int_a^{\frac{a+b}{2}}(h(s)-h(a))^{\alpha-1}g(s)h'(s)ds\\
&-\int_a^{\frac{a+b}{2}}(h(t)-h(a))^{\alpha-1}g(t)f(h(t))h'(t)dt\\
&=\Gamma(\alpha)\left[f\left(h\left(\tfrac{a+b}{2}\right)\right)J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)-J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}\left(g\left(f\circ h\right)\right)(a)\right],
\end{aligned}
$$

and similarly

$$
I_2 =\Gamma(\alpha)\left[f\left(h\left(\tfrac{a+b}{2}\right)\right)J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)-J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}\left(g\left(f\circ h\right)\right)(b)\right].
$$

Thus, can write

$$
\begin{aligned}
I=I_1+I_2=\Gamma(\alpha)\Big\{&f\left(h\left(\tfrac{a+b}{2}\right)\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]\\
&-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}\left(g\left(f\circ h\right)\right)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}\left(g\left(f\circ h\right)\right)(b)\right]\Big\}.
\end{aligned}
$$

Multiplying the both sides $(\Gamma(\alpha))^{-1}$, we obtain (3.1) which complates the proof. □

Remark 1. *If we choose $h(x) = x$ in Lemma 1, we have*

$$f\left(\tfrac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(gf)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]=\frac{1}{\Gamma(\alpha)}\int_a^b k(t)df(t)$$

where

$$k(t)=\begin{cases}\int_a^t (s-a)^{\alpha-1}g(s)ds & t\in\left[a,\frac{a+b}{2}\right],\\ \int_b^t (b-s)^{\alpha-1}g(s)ds & t\in\left[\frac{a+b}{2},a\right].\end{cases}$$

Theorem 2. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in X_h^p[a,b]$ with $a < b$ and $g : [a,b] \to \mathbb{R}$ is continuous. If $|f'|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

(3.2)

$$\Bigg|f\left(h\left(\tfrac{a+b}{2}\right)\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]$$
$$-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(g\times(f\circ h))(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(g\times(f\circ h))(b)\right]\Bigg|$$
$$\le\frac{||g||_{X_h^\infty\left[a,\frac{a+b}{2}\right],\infty}}{(h(b)-h(a))\Gamma(\alpha+1)}\left\{|f'(h(a))|\left[\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+1}}{\alpha+1}(h(b)-h(a))-\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+2}}{\alpha+2}\right]\right.$$
$$\left.+|f'(h(b))|\left[\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+2}}{\alpha+2}\right]\right\}+\frac{||g||_{X_h^\infty\left[\frac{a+b}{2},b\right],\infty}}{(h(b)-h(a))\Gamma(\alpha+1)}$$
$$\times\left\{|f'(h(b))|\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+1}}{\alpha+1}(h(b)-h(a))-\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}\right]+|f'(h(a))|\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}\right]\right\}$$

with $\alpha > 0$.

Proof. If $|f'|$ is convex on $[a,b]$, we know that for $t \in [a,b]$

$$|f'(h(t))|=\left|f'\left(\frac{h(b)-h(t)}{h(b)-h(a)}h(a)+\frac{h(t)-h(a)}{h(b)-h(a)}h(b)\right)\right|\le\frac{h(b)-h(t)}{h(b)-h(a)}|f'(h(a))|+\frac{h(t)-h(a)}{h(b)-h(a)}|f'(h(b))|.$$

From Lemma 1, we have

$$\left|f\left(h\left(\frac{a+b}{2}\right)\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(g(f\circ h))(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(g(f\circ h))(b)\right]\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\left\{\int_a^{\frac{a+b}{2}}\left|\int_a^t (h(s)-h(a))^{\alpha-1}g(s)h'(s)ds\right| f'(h(t))\,h'(t)dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^b\left|\int_b^t (h(b)-h(s))^{\alpha-1}g(s)h'(s)ds\right| f'(h(t))\,h'(t)dt\right\}$$
$$\leq \frac{||g||_{X_h^\infty\left[a,\frac{a+b}{2}\right],\infty}}{(h(b)-h(a))\Gamma(\alpha)}\left\{\int_a^{\frac{a+b}{2}}\left|\int_a^t (h(s)-h(a))^{\alpha-1}h'(s)ds\right| (h(b)-h(t)\,|f'(h(a))|)\,h'(t)dt\right.$$
$$\left.+\int_a^{\frac{a+b}{2}}\left|\int_a^t (h(s)-h(a))^{\alpha-1}h'(s)ds\right| (h(t)-h(a)\,|f'(h(b))|)\,h'(t)dt\right\}$$
$$+\frac{||g||_{X_h^\infty\left[\frac{a+b}{2},b\right],\infty}}{(h(b)-h(a))\Gamma(\alpha)}\left\{\int_{\frac{a+b}{2}}^b\left|\int_b^t (h(b)-h(s))^{\alpha-1}h'(s)ds\right| (h(b)-h(t)\,|f'(h(a))|)\,h'(t)dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^b\left|\int_b^t (h(b)-h(s))^{\alpha-1}h'(s)ds\right| (h(t)-h(a)\,|f'(h(b))|)\,h'(t)dt\right\}.$$
$$\leq \frac{||g||_{X_h^\infty\left[a,\frac{a+b}{2}\right],\infty}}{(h(b)-h(a))\Gamma(\alpha+1)}\left\{\int_a^{\frac{a+b}{2}}(h(t)-h(a))^{\alpha}\,(h(b)-h(t)\,|f'(h(a))|)\,h'(t)dt\right.$$
$$\left.+\int_a^{\frac{a+b}{2}}(h(t)-h(a))^{\alpha+1}\,|f'(h(b))|\,h'(t)dt\right\}$$
$$+\frac{||g||_{X_h^\infty\left[\frac{a+b}{2},b\right],\infty}}{(h(b)-h(a))\Gamma(\alpha+1)}\left\{\int_{\frac{a+b}{2}}^b(h(b)-h(t))^{\alpha+1}\,|f'(h(a))|\,h'(t)dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^b(h(b)-h(t))^{\alpha-1}\,(h(t)-h(a)\,|f'(h(b))|)\,h'(t)dt\right\}.$$
$$\leq \frac{||g||_{X_h^\infty\left[a,\frac{a+b}{2}\right],\infty}}{(h(b)-h(a))\Gamma(\alpha+1)}\left\{|f'(h(a))|\left[\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+1}}{\alpha+1}(h(b)-h(a))-\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+2}}{\alpha+2}\right]\right.$$
$$\left.+|f'(h(b))|\left[\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+2}}{\alpha+2}\right]\right\}$$
$$+\frac{||g||_{X_h^\infty\left[\frac{a+b}{2},b\right],\infty}}{(h(b)-h(a))\Gamma(\alpha+1)}\left\{|f'(h(b))|\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+1}}{\alpha+1}(h(b)-h(a))-\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}\right]\right.$$
$$\left.+|f'(h(a))|\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}\right]\right\}.$$

This completes the proof. □

Remark 2. *If we choose $h(x)=x$ in Theorem 1, we have*

$$\begin{aligned}&f\left(\tfrac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]\\&-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]\\&\leq\frac{(b-a)^{\alpha+1}\,||g||_\infty}{2^{\alpha+1}\Gamma(\alpha+2)}\left\{|f'(a))|+|f'(b)|\right\},\end{aligned}$$

with $\alpha>0$.

Theorem 3. *Let $f:I\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in X_h^p[a,b]$ with $a<b$ and $g:[a,b]\to\mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a,b]$, $q\geq 1$ then*

the following inequality for fractional integrals holds:

$$
\begin{aligned}
&f\left(h\left(\tfrac{a+b}{2}\right)\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]\\
&-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}\left(g\times(f\circ h)\right)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}\left(g\times(f\circ h)\right)(b)\right]\\
&\leq \frac{\|g\|_{X_h^\infty\left[a,\frac{a+b}{2}\right],\infty}}{[(h(b)-h(a))]^{\frac{1}{q}}\Gamma(\alpha)}\left(\frac{h\left(\frac{a+b}{2}\right)-h(a)}{\alpha(\alpha+1)}\right)^{1-\frac{1}{q}}\left\{\left|f'\left(h(a)\right)\right|^q\right.\\
&\times\left[\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+1}(h(b)-h(a))}{\alpha+1}-\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+2}}{\alpha+2}\right]\\
&\left.+\frac{\left|f'(h(b))\right|^q(h(\frac{a+b}{2})-h(a))^{\alpha+2}}{\alpha+2}\right\}^{\frac{1}{q}}\\
&+\frac{\|g\|_{X_h^\infty\left[\frac{a+b}{2},b\right],\infty}}{[(h(b)-h(a))]^{\frac{1}{q}}\Gamma(\alpha)}\left(\frac{h(b)-h\left(\frac{a+b}{2}\right)}{\alpha(\alpha+1)}\right)^{1-\frac{1}{q}}\left\{\left|f'\left(h(b)\right)\right|^q\right.\\
&\times\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+1}(h(b)-h(a))}{\alpha+1}-\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}\right]\\
&\left.+\left|f'\left(h(a)\right)\right|^q\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}\right]\right\}^{\frac{1}{q}}.
\end{aligned}
\tag{3.3}
$$

with $\alpha>0$.

Proof. If $|f'|^q$ is convex on $[a,b]$, we know that for $t\in[a,b]$

$$
\begin{aligned}
|f'(h(t))| &= \left|f'\left(\frac{h(b)-h(t)}{h(b)-h(a)}h(a)+\frac{h(t)-h(a)}{h(b)-h(a)}h(b)\right)\right|\\
&\leq \frac{h(b)-h(t)}{h(b)-h(a)}\left|f'\left(h(a)\right)\right|^q+\frac{h(t)-h(a)}{h(b)-h(a)}\left|f'\left(h(b)\right)\right|^q.
\end{aligned}
$$

From Lemma 1, power mean inequality and the convexity of $|f'|^q$, it follows that

$$
\begin{aligned}
&f\left(h\left(\tfrac{a+b}{2}\right)\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}\left(g\left(f\circ h\right)\right)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}\left(g\left(f\circ h\right)\right)(b)\right]\\
&\leq\frac{1}{\Gamma(\alpha)}\left\{\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(h(s)-h(a))^{\alpha-1}g(s)h'(s)ds\right|h'(t)dt\right)^{1-1/q}\right.\\
&\left.\times\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(h(s)-h(a))^{\alpha-1}g(s)h'(s)ds\right|\left|f'\left(h(t)\right)\right|^q h'(t)dt\right)^{1/q}\right\}\\
&+\frac{1}{\Gamma(\alpha)}\left\{\left(\int_{\frac{a+b}{2}}^{b}\left|\int_b^t(h(b)-h(s))^{\alpha-1}g(s)h'(s)ds\right|h'(t)dt\right)^{1-1/q}\right.\\
&\left.\times\left(\int_{\frac{a+b}{2}}^{b}\left|\int_b^t(h(b)-h(s))^{\alpha-1}g(s)h'(s)ds\right|\left|f'\left(h(t)\right)\right|^q h'(t)dt\right)^{1/q}\right\}
\end{aligned}
$$

$$\leq \frac{||g||_{X_h^\infty\left[a,\frac{a+b}{2}\right],\infty}}{\Gamma(\alpha)}\left\{\left(\int\limits_a^{\frac{a+b}{2}}\left|\int\limits_a^t(h(s)-h(a))^{\alpha-1}h'(s)ds\right|h'(t)dt\right)^{1-1/q}\right.$$
$$\left.\times\left(\int\limits_a^{\frac{a+b}{2}}\left|\int\limits_a^t(h(s)-h(a))^{\alpha-1}h'(s)ds\right|\left|f'\left(h(t)\right)\right|^q h'(t)dt\right)^{1/q}\right\}$$
$$+\frac{||g||_{X_h^\infty\left[\frac{a+b}{2},b\right],\infty}}{\Gamma(\alpha)}\left\{\left(\int\limits_{\frac{a+b}{2}}^{b}\left|\int\limits_b^t(h(b)-h(s))^{\alpha-1}h'(s)ds\right|h'(t)dt\right)^{1-1/q}\right.$$
$$\left.\times\left(\int\limits_{\frac{a+b}{2}}^{b}\left|\int\limits_b^t(h(b)-h(s))^{\alpha-1}h'(s)ds\right|\left|f'\left(h(t)\right)\right|^q h'(t)dt\right)^{1/q}\right\}.$$
$$\leq \frac{||g||_{X_h^\infty\left[a,\frac{a+b}{2}\right],\infty}}{\left[(h(b)-h(a))\right]^{\frac{1}{q}}\Gamma(\alpha)}\left\{\int\limits_a^{\frac{a+b}{2}}\left|\int\limits_a^t(h(s)-h(a))^{\alpha-1}h'(s)ds\right|\left(h(b)-h(t)\left|f'\left(h(a)\right)\right|\right)h'(t)dt\right.$$
$$\left.+\int\limits_a^{\frac{a+b}{2}}\left|\int\limits_a^t(h(s)-h(a))^{\alpha-1}h'(s)ds\right|\left(h(t)-h(a)\left|f'\left(h(b)\right)\right|\right)h'(t)dt\right\}^{\frac{1}{q}}$$
$$+\frac{||g||_{X_h^\infty\left[\frac{a+b}{2},b\right],\infty}}{\left[(h(b)-h(a))\right]^{\frac{1}{q}}\Gamma(\alpha)}\left\{\int\limits_{\frac{a+b}{2}}^{b}\left|\int\limits_b^t(h(b)-h(s))^{\alpha-1}h'(s)ds\right|\left(h(b)-h(t)\left|f'\left(h(a)\right)\right|\right)h'(t)dt\right.$$
$$\left.+\int\limits_{\frac{a+b}{2}}^{b}\left|\int\limits_b^t(h(b)-h(s))^{\alpha-1}h'(s)ds\right|\left(h(t)-h(a)\left|f'\left(h(b)\right)\right|\right)h'(t)dt\right\}^{\frac{1}{q}}$$
$$\leq \frac{||g||_{X_h^\infty\left[a,\frac{a+b}{2}\right],\infty}}{\left[(h(b)-h(a))\right]^{\frac{1}{q}}\Gamma(\alpha)}\left(\frac{h\left(\frac{a+b}{2}\right)-h(a)}{\alpha(\alpha+1)}\right)^{1-1/q}\left\{\left[\frac{\left|f'(h(a))\right|^q(h(\frac{a+b}{2})-h(a))^{\alpha+1}(h(b)-h(a))}{\alpha+1}\right.\right.$$
$$\left.\left.-\frac{\left|f'(h(a))\right|^q(h(\frac{a+b}{2})-h(a))^{\alpha+2}}{\alpha+2}\right]+\left|f'\left(h(b)\right)\right|^q\left[\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+2}}{\alpha+2}\right]\right\}^{\frac{1}{q}}$$
$$+\frac{||g||_{X_h^\infty\left[\frac{a+b}{2},b\right],\infty}}{\left[(h(b)-h(a))\right]^{\frac{1}{q}}\Gamma(\alpha)}\left(\frac{h(b)-h\left(\frac{a+b}{2}\right)}{\alpha(\alpha+1)}\right)^{1-1/q}\left\{\left|f'\left(h(b)\right)\right|^q\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+1}(h(b)-h(a))}{\alpha+1}\right.\right.$$
$$\left.\left.-\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}\right]+\left|f'\left(h(a)\right)\right|^q\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}\right]\right\}^{\frac{1}{q}}.$$

□

Remark 3. *Under conditions of Theorem 3, when $h(x)=x$, we have*

$$f\left(\frac{a+b}{2}\right)\left[J_{(a)^-}^\alpha g(a)+J_{(b)^+}^\alpha g(b)\right]-\left[J_{(a)^-}^\alpha\left(fg\right)(a)+J_{(b)^+}^\alpha\left(fg\right)(b)\right]$$
$$\leq\frac{(b-a)^{\alpha+1}\left|\left|g\right|\right|_{[a,b],\infty}}{2^{\alpha+1+\frac{1}{q}}\left(\alpha+2\right)^{1\backslash q}\Gamma\left(\alpha+2\right)}\times\left\{\left(\left|f'\left(a\right)\right)\right|^q+\left(\alpha+1\right)\left|f'\left(b\right)\right|^q\right)^{1\backslash q}$$
$$+\left(\left(\alpha+1\right)\left|f'\left(a\right)\right)\right|^q+\left|f'\left(b\right)\right|^q\right)^{1\backslash q}\right\},$$

with $\alpha>0$.

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