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SOME RESULTS FOR ISOTONIC FUNCTIONALS VIA AN INEQUALITY DUE TO TOMINAGA

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Abstract. In this paper we obtain some inequalities for isotonic functionals via a reverse of Young's inequality due to Tominaga.

1. Introduction

Let L be a *linear class* of real-valued functions $g : E \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1$, $t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalised if*
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2], [12] and [13]). For other inequalities for isotonic functionals see [1], [4]-[11] and [14]-[17].

We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second $(p_k \geq 0, k \in E)$.

As is known to all, the famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0, 1]$, then

$$a^{1-\nu} b^{\nu} \leq (1 - \nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called ν-weighted arithmetic-geometric mean inequality.

Tominaga [18] had proved a reverse Young inequality with the Specht's ratio [16] as follows:

$$(1 - \nu) a + \nu b \leq S\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}. \tag{1.2}$$

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We recall that *Specht's ratio* is defined by

$$S(h):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty)\\ \\ 1 \text{ if } h=1.\end{cases}$$

It is well known that $\lim_{h\to1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

Let $a,b\in[m,M]\subset(0,\infty)$, then $\frac{m}{M}\le\frac{a}{b}\le\frac{M}{m}$ with $\frac{m}{M}<1<\frac{M}{m}$. If $\frac{a}{b}\in\left[\frac{m}{M},1\right)$ then $S\left(\frac{a}{b}\right)\le S\left(\frac{m}{M}\right)=S\left(\frac{M}{m}\right)$. If $\frac{a}{b}\in\left(1,\frac{M}{m}\right]$ then also $S\left(\frac{a}{b}\right)\le S\left(\frac{M}{m}\right)$. Therefore for any $a,b\in[m,M]$ we have

$$(1-\nu)a+\nu b\le S\left(\frac{M}{m}\right)a^{1-\nu}b^{\nu}. \tag{1.3}$$

In this paper we obtain some inequalities for isotonic functionals via a reverse of Young's inequality due to Tominaga. Reverses of Callebaut, Hölder and Hölder's related inequalities are also provided. Some examples for integrals and n-tuples of real numbers are given as well.

2. A Reverse of Callebaut's Inequality

We start with the following result:

Theorem 1. *Let $A,B:L\to\mathbb{R}$ be two normalised isotonic functionals. If $f,g:E\to\mathbb{R}$ are such that $f\ge0$, $g>0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$ and*

$$0<m\le\frac{f}{g}\le M<\infty \tag{2.1}$$

for some constants m,M, then

$$\begin{aligned}&A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right)\\&\le(1-\nu)A\left(f^2\right)B\left(g^2\right)+\nu A\left(g^2\right)B\left(f^2\right)\\&\le S\left(\left(\frac{M}{m}\right)^2\right)A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right).\end{aligned} \tag{2.2}$$

Proof. For any $x,y\in E$ we have

$$m^2\le\frac{f^2(x)}{g^2(x)},\frac{f^2(y)}{g^2(y)}\le M^2.$$

If we use the inequalities (1.1) and (1.3) for

$$a=\frac{f^2(x)}{g^2(x)},\ b=\frac{f^2(y)}{g^2(y)},$$

then we get

$$\begin{aligned}\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}&\le(1-\nu)\frac{f^2(x)}{g^2(x)}+\nu\frac{f^2(y)}{g^2(y)}\\&\le S\left(\left(\frac{M}{m}\right)^2\right)\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}\end{aligned} \tag{2.3}$$

for any $x, y \in E$.

Now, if we multiply (2.3) by $g^2(x)\, g^2(y) > 0$ then we get

$$\begin{aligned} f^{2(1-\nu)}(x)\, g^{2\nu}(x)\, f^{2\nu}(y)\, g^{2(1-\nu)}(y) \\ \le (1-\nu) f^2(x)\, g^2(y) + \nu g^2(x)\, f^2(y) \\ \le S\left(\left(\frac{M}{m}\right)^2\right) f^{2(1-\nu)}(x)\, g^{2\nu}(x)\, f^{2\nu}(y)\, g^{2(1-\nu)}(y) \end{aligned} \tag{2.4}$$

for any $x, y \in E$.

Fix $y \in E$. Then by (2.4) we have in the order of L that

$$\begin{aligned} f^{2\nu}(y)\, g^{2(1-\nu)}(y)\, f^{2(1-\nu)} g^{2\nu} &\le (1-\nu)\, g^2(y)\, f^2 + \nu f^2(y)\, g^2 \\ &\le S\left(\left(\frac{M}{m}\right)^2\right) f^{2\nu}(y)\, g^{2(1-\nu)}(y)\, f^{2(1-\nu)} g^{2\nu}. \end{aligned} \tag{2.5}$$

If we take the functional A in (2.5) we get

$$\begin{aligned} f^{2\nu}(y)\, g^{2(1-\nu)}(y)\, A\left(f^{2(1-\nu)} g^{2\nu}\right) \\ \le (1-\nu)\, g^2(y)\, A\left(f^2\right) + \nu f^2(y)\, A\left(g^2\right) \\ \le S\left(\left(\frac{M}{m}\right)^2\right) f^{2\nu}(y)\, g^{2(1-\nu)}(y)\, A\left(f^{2(1-\nu)} g^{2\nu}\right), \end{aligned} \tag{2.6}$$

for any $y \in E$.

This inequality can be written in the order of L as

$$\begin{aligned} A\left(f^{2(1-\nu)} g^{2\nu}\right) f^{2\nu} g^{2(1-\nu)} &\le (1-\nu)\, A\left(f^2\right) g^2 + \nu A\left(g^2\right) f^2 \\ &\le S\left(\left(\frac{M}{m}\right)^2\right) A\left(f^{2(1-\nu)} g^{2\nu}\right) f^{2\nu} g^{2(1-\nu)}. \end{aligned} \tag{2.7}$$

Now, if we take the functional B in (2.7), then we get the desired result (2.2). □

The following reverse of Cauchy-Bunyakovsky-Schwarz inequality for isotonic functionals holds:

Corollary 1. *Let $A, B : L \to \mathbb{R}$ be two normalised isotonic functionals. If $f, g : E \to \mathbb{R}$ are such that $f \ge 0$, $g > 0$, f^2, g^2, $fg \in L$ and the condition (2.1) holds true, then*

$$\begin{aligned} A(fg)\, B(fg) &\le \frac{1}{2}\left[A\left(f^2\right) B\left(g^2\right) + A\left(g^2\right) B\left(f^2\right)\right] \\ &\le S\left(\left(\frac{M}{m}\right)^2\right) A(fg)\, B(fg)\,. \end{aligned} \tag{2.8}$$

In particular,

$$A^2(fg) \le A\left(f^2\right) A\left(g^2\right) \le S\left(\left(\frac{M}{m}\right)^2\right) A^2(fg)\,. \tag{2.9}$$

The following reverse Callebaut type inequality holds:

Corollary 2. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional. If $f, g : E \to \mathbb{R}$ are such that $f \geq 0$, $g > 0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L$ for some $\nu \in [0,1]$ and the condition (2.1) is valid, then*

$$\begin{aligned} A\left(f^{2(1-\nu)}g^{2\nu}\right) A\left(f^{2\nu}g^{2(1-\nu)}\right) & \\ \leq A\left(f^2\right) A\left(g^2\right) & \\ \leq S\left(\left(\frac{M}{m}\right)^2\right) A\left(f^{2(1-\nu)}g^{2\nu}\right) A\left(f^{2\nu}g^{2(1-\nu)}\right). & \end{aligned} \tag{2.10}$$

Remark 1. *If we replace ν by $\frac{1}{2}(1-\nu)$ with $\nu \in [0,1]$ in (2.10), then we get*

$$\begin{aligned} A\left(f^{1+\nu}g^{1-\nu}\right) A\left(f^{1-\nu}g^{1+\nu}\right) &\leq A\left(f^2\right) A\left(g^2\right) \\ &\leq S\left(\left(\frac{M}{m}\right)^2\right) A\left(f^{1+\nu}g^{1-\nu}\right) A\left(f^{1-\nu}g^{1+\nu}\right), \end{aligned} \tag{2.11}$$

provided that $f \geq 0$, $g > 0$, f^2, g^2, $f^{1+\nu}g^{1-\nu}$, $f^{1-\nu}g^{1+\nu} \in L$ for some $\nu \in [0,1]$ and the condition (2.1) is valid.

Also, if we take $\nu = \frac{1}{2}\gamma$ with $\gamma \in [0,2]$, then we get

$$\begin{aligned} A\left(f^{2-\gamma}g^{\gamma}\right) A\left(f^{\gamma}g^{2-\gamma}\right) &\leq A\left(f^2\right) A\left(g^2\right) \\ &\leq S\left(\left(\frac{M}{m}\right)^2\right) A\left(f^{2-\gamma}g^{\gamma}\right) A\left(f^{\gamma}g^{2-\gamma}\right), \end{aligned} \tag{2.12}$$

provided that $f \geq 0$, $g > 0$, f^2, g^2, $f^{2-\gamma}g^{\gamma}$, $f^{\gamma}g^{2-\gamma} \in L$ for some $\nu \in [0,1]$ and the condition (2.1) is valid.

The inequality (2.12) is a reverse for the second inequality in the functional version of Callebaut inequality

$$A^2(fg) \leq A\left(f^{2-\gamma}g^{\gamma}\right) A\left(f^{\gamma}g^{2-\gamma}\right) \leq A\left(f^2\right) A\left(g^2\right) \tag{2.13}$$

provided that f^2, g^2, $f^{2-\gamma}g^{\gamma}$, $f^{\gamma}g^{2-\gamma}, fg \in L$ for some $\gamma \in [0,2]$. For the discrete and integral of one real variable versions see [3].

3. A Reverse of Hölder's and Related Inequalities

First, observe that if $a, b > 0$ and

$$0 < L^{-1} \leq \frac{a}{b} \leq L < \infty, \tag{3.1}$$

for some $L > 1$, then by (1.2) we have

$$(1-\nu)a + \nu b \leq S(L)\, a^{1-\nu}b^{\nu} \tag{3.2}$$

for every $\nu \in [0,1]$.

Theorem 2. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If f, $g : E \to \mathbb{R}$ are such that fg, f^p, $g^q \in L$ and*

$$0 < m_1 \leq f \leq M_1 < \infty,\ 0 < m_2 \leq g \leq M_2 < \infty, \tag{3.3}$$

then

$$\left[A\left(f^p\right)\right]^{1/p} \left[A\left(g^q\right)\right]^{1/q} \leq S\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right) A(fg). \tag{3.4}$$

Proof. Observe that, by (3.3) we have

$$m_1^p \le A\left(f^p\right) \le M_1^p \text{ and } m_2^q \le A\left(g^q\right) \le M_2^q.$$

Also

$$\left(\frac{m_1}{M_1}\right)^p \le \frac{f^p}{A\left(f^p\right)} \le \left(\frac{M_1}{m_1}\right)^p$$

and

$$\left(\frac{m_2}{M_2}\right)^q \le \frac{g^q}{A\left(g^q\right)} \le \left(\frac{M_2}{m_2}\right)^q$$

giving that

$$\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1} \le \frac{\frac{f^p}{A(f^p)}}{\frac{g^q}{A(g^q)}} \le \left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q.$$

Using the inequality (3.2) for $\nu = \frac{1}{q}$, $a = \frac{f^p}{A(f^p)}$, $b = \frac{g^q}{A(g^q)}$ and $L = \left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q$, we get

$$\frac{1}{p}\frac{f^p}{A\left(f^p\right)} + \frac{1}{q}\frac{g^q}{A\left(g^q\right)} \le S\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right) \frac{fg}{\left[A\left(f^p\right)\right]^{1/p}\left[A\left(g^q\right)\right]^{1/q}}. \tag{3.5}$$

If we take the functional A in (3.5) we get

$$1 = \frac{1}{p}\frac{A\left(f^p\right)}{A\left(f^p\right)} + \frac{1}{q}\frac{A\left(g^q\right)}{A\left(g^q\right)} \le S\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right) \frac{A\left(fg\right)}{\left[A\left(f^p\right)\right]^{1/p}\left[A\left(g^q\right)\right]^{1/q}},$$

which is equivalent with the desired result (3.4). □

The following reverse of Cauchy-Bunyakovsky-Schwarz inequality for isotonic functionals holds:

Corollary 3. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional, $f, g : E \to \mathbb{R}$ such that fg, f^2, $g^2 \in L$ and the condition (3.3) is valid, then*

$$\left[A\left(f^2\right)\right]^{1/2}\left[A\left(g^2\right)\right]^{1/2} \le S\left(\left(\frac{M_1}{m_1}\right)^2 \left(\frac{M_2}{m_2}\right)^2\right) A\left(fg\right). \tag{3.6}$$

Further, observe that if $a, b > 0$ and

$$0 < l^{-1} \le \frac{a}{b} \le L < \infty, \tag{3.7}$$

for some $L, l > 0$ with $Ll > 1$,then

$$S\left(\frac{a}{b}\right) \le \max\left\{S\left(l^{-1}\right), S\left(L\right)\right\} = \max\left\{S\left(l\right), S\left(L\right)\right\}$$

and by (1.2) we have

$$\left(1-\nu\right)a + \nu b \le \max\left\{S\left(l\right), S\left(L\right)\right\} a^{1-\nu} b^{\nu} \tag{3.8}$$

for every $\nu \in [0, 1]$.

Theorem 3. *Let A, $B : L \to \mathbb{R}$ be two normalised isotonic functionals and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If f, g, $u, v : E \to \mathbb{R}$ are such that u, $v \geq 0$, u, v, uf, vg, uf^p, $vg^q \in L$ and the conditions (3.3) hold, then*

$$A(uf)B(vg) \leq \frac{1}{p}A(uf^p)B(v) + \frac{1}{q}A(u)B(vg^q) \leq \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\}A(uf)B(vg). \tag{3.9}$$

In particular,

$$A(uf)A(vg) \leq \frac{1}{p}A(uf^p)A(v) + \frac{1}{q}A(u)A(vg^q) \leq \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\}A(uf)A(vg). \tag{3.10}$$

Proof. Observe that, by (3.3) we have

$$\frac{m_1^p}{M_2^q} \leq \frac{f^p(x)}{g^q(y)} \leq \frac{M_1^p}{m_2^q}$$

for any $x, y \in E$.

Now, if we write the inequality (3.8) for $l = \frac{M_2^q}{m_1^p}$, $L = \frac{M_1^p}{m_2^q}$, $a = f^p(x)$, $b = g^q(y)$ and $\nu = \frac{1}{q}$, and use Young's inequality, then we get

$$f(x)g(y) \leq \frac{1}{p}f^p(x) + \frac{1}{q}g^q(y) \leq \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\}f(x)g(y) \tag{3.11}$$

for any $x, y \in E$.

If we multiply (3.11) by $u(x)v(y) \geq 0$ we get

$$v(y)g(y)fu \leq \frac{1}{p}v(y)f^p u + \frac{1}{q}g^q(y)v(y)u \leq \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\}v(y)g(y)fu \tag{3.12}$$

in the order of L, where $y \in E$.

If we take the functional A in (3.12), then we get

$$vgA(fu) \leq \frac{1}{p}A(f^p u)v + \frac{1}{q}A(u)g^q v \leq \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\}A(fu)vg \tag{3.13}$$

in the order of L.

Finally, if we take the functional B in (3.13) then we get the desired result (3.9). □

Corollary 4. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functionals and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let f, $g : E \to \mathbb{R}$ be such that the conditions (3.3) hold.*

(i) If f, g, f^2, g^2, f^{p+1}, $g^{q+1} \in L$, then

$$A(f^2)A(g^2) \leq \frac{1}{p}A(f^{p+1})A(g) + \frac{1}{q}A(f)A(g^{q+1}) \leq \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\}A(f^2)A(g^2). \tag{3.14}$$

(ii) If f, g, fg, gf^p, $fg^q \in L$, then

$$A^2(fg) \le \frac{1}{p} A(gf^p) A(f) + \frac{1}{q} A(g) A(fg^q) \le \max\left\{ S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right) \right\} A^2(fg). \tag{3.15}$$

The following result also holds:

Corollary 5. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functionals and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $\ell, h : E \to \mathbb{R}$, with $\ell \ge 0$, $h > 0$ be such that the following condition holds*

$$0 < m \le \frac{\ell}{h} \le M < \infty. \tag{3.16}$$

If h^2, $h\ell$, $h^{2-p}\ell^p$, $h^{2-q}\ell^q \in L$, then we have

$$A^2(h\ell) \le \left[\frac{1}{p} A\left(h^{2-p}\ell^p\right) + \frac{1}{q} A\left(h^{2-q}\ell^q\right)\right] A\left(h^2\right) \le \max\left\{ S\left(\frac{M^q}{m^p}\right), S\left(\frac{M^p}{m^q}\right) \right\} A^2(h\ell). \tag{3.17}$$

Proof. Follows by Theorem 3 for $f = g = \frac{\ell}{h}$, $M_1 = M_2 = M$, $m_1 = m_2 = m$, and $u = v = h^2$. $\square$

We observe that for $p = q = 2$ we recapture from (3.17) the inequality (2.9).

4. Applications for Integrals

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \ge 0$ for μ -a.e. (almost every) $x \in \Omega$ and $p \ge 1$ consider the Lebesgue space

$$L_w^p(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|^p w(x)\, d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\, d\mu(x)$. The same for other integrals involved below. We assume that $\int_\Omega w d\mu = 1$.

Let f, g be μ-measurable functions with the property that there exists the constants M, $m > 0$ such that

$$0 < m \le \frac{f}{g} \le M < \infty \ \mu\text{-almost everywhere (a.e.) on } \Omega.$$

If f^2, $g^2 \in L_w(\Omega, \mu)$, then by (2.10) we have

$$\int_\Omega w f^{2(1-s)} g^{2s} d\mu \int_\Omega w f^{2s} g^{2(1-s)} d\mu \le \int_\Omega w f^2 d\mu \int_\Omega w g^2 d\mu \le S\left(\left(\frac{M}{m}\right)^2\right) \int_\Omega w f^{2(1-s)} g^{2s} d\mu \int_\Omega w f^{2s} g^{2(1-s)} d\mu \tag{4.1}$$

for any $s\in[0,1]$ and, in particular,

$$\left(\int_\Omega wfgd\mu\right)^2\le\int_\Omega wf^2d\mu\int_\Omega wg^2d\mu\le S\left(\left(\frac{M}{m}\right)^2\right)\left(\int_\Omega wfgd\mu\right)^2. \tag{4.2}$$

From (3.17) we also have

$$\begin{aligned}\left(\int_\Omega wfgd\mu\right)^2&\le\left[\frac{1}{p}\int_\Omega wg^{2-p}f^pd\mu+\frac{1}{q}\int_\Omega wg^{2-q}f^qd\mu\right]\int_\Omega wg^2d\mu\\&\le\max\left\{S\left(\frac{M^q}{m^p}\right),S\left(\frac{M^p}{m^q}\right)\right\}\left(\int_\Omega wfgd\mu\right)^2.\end{aligned} \tag{4.3}$$

Let f, g be μ-measurable functions with the property that there exists the constants m_1, M_1, m_2, M_2 such that

$$0<m_1\le f\le M_1<\infty,\ 0<m_2\le g\le M_2<\infty\ \mu\text{-a.e. on }\Omega. \tag{4.4}$$

Let p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by (3.4) we have the following reverse of Hölder's inequality

$$\left(\int_\Omega wf^pd\mu\right)^{1/p}\left(\int_\Omega wg^qd\mu\right)^{1/q}\le S\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\int_\Omega wfgd\mu \tag{4.5}$$

and, in particular, the reverse of Cauchy-Bunyakovsky-Schwarz inequality

$$\left(\int_\Omega wf^2d\mu\right)^{1/2}\left(\int_\Omega wg^2d\mu\right)^{1/2}\le S\left(\left(\frac{M_1}{m_1}\frac{M_2}{m_2}\right)^2\right)\int_\Omega wfgd\mu. \tag{4.6}$$

From (3.14) and (3.15) we also have

$$\begin{aligned}\int_\Omega wf^2d\mu\int_\Omega wg^2d\mu&\le\frac{1}{p}\int_\Omega wf^{p+1}d\mu\int_\Omega wgd\mu+\frac{1}{q}\int_\Omega wfd\mu\int_\Omega wg^{q+1}d\mu\\&\le\max\left\{S\left(\frac{M_2^q}{m_1^p}\right),S\left(\frac{M_1^p}{m_2^q}\right)\right\}\int_\Omega wf^2d\mu\int_\Omega wg^2d\mu\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\left(\int_\Omega wfgd\mu\right)^2&\le\frac{1}{p}\int_\Omega wgf^pd\mu\int_\Omega wfd\mu+\frac{1}{q}\int_\Omega wgd\mu\int_\Omega wfg^qd\mu\\&\le\max\left\{S\left(\frac{M_2^q}{m_1^p}\right),S\left(\frac{M_1^p}{m_2^q}\right)\right\}\left(\int_\Omega wfgd\mu\right)^2.\end{aligned} \tag{4.8}$$

5. Applications for Real Numbers

We consider the n-tuples of positive numbers $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ and the probability distribution $p=(p_1,...,p_n)$, i.e. $p_i\ge0$ for any $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$.

If there exist the constants m, $M>0$ such that

$$0<m\le\frac{a_i}{b_i}\le M<\infty\text{ for any }i\in\{1,...,n\},$$

then by (4.1) and (4.2) for the counting discrete measure, we have

(5.1)
$$\begin{aligned}\sum_{i=1}^{n} p_i a_i^{2(1-s)} b_i^{2s} \sum_{i=1}^{n} p_i a_i^{2s} b_i^{2(1-s)} &\leq \sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 \\ &\leq S\left(\left(\frac{M}{m}\right)^2\right) \sum_{i=1}^{n} p_i a_i^{2(1-s)} b_i^{2s} \sum_{i=1}^{n} p_i a_i^{2s} b_i^{2(1-s)}\end{aligned}$$

for any $s \in [0,1]$ and, in particular,

$$\left(\sum_{i=1}^{n} p_i a_i b_i\right)^2 \leq \sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 \leq S\left(\left(\frac{M}{m}\right)^2\right)\left(\sum_{i=1}^{n} p_i a_i b_i\right)^2. \tag{5.2}$$

From (4.3) we also have

$$\begin{aligned}\left(\sum_{i=1}^{n} p_i a_i b_i\right)^2 &\leq \left[\frac{1}{p}\sum_{i=1}^{n} p_i b_i^{2-p} a_i^p + \frac{1}{q}\sum_{i=1}^{n} p_i b_i^{2-q} a_i^q\right]\sum_{i=1}^{n} p_i b_i^2 \\ &\leq \max\left\{S\left(\frac{M^q}{m^p}\right), S\left(\frac{M^p}{m^q}\right)\right\}\left(\sum_{i=1}^{n} p_i a_i b_i\right)^2.\end{aligned} \tag{5.3}$$

If there exists the constants m_1, M_1, m_2, M_2 such that

$$0 < m_1 \leq a_i \leq M_1 < \infty,\ 0 < m_2 \leq b_i \leq M_2 < \infty \text{ for any } i \in \{1, ..., n\} \tag{5.4}$$

and $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (4.5) we have the following reverse of Hölder's discrete inequality

$$\left(\sum_{i=1}^{n} p_i a_i^p\right)^{1/p}\left(\sum_{i=1}^{n} p_i b_i^q\right)^{1/q} \leq S\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\sum_{i=1}^{n} p_i a_i b_i \tag{5.5}$$

and, in particular, the reverse of Cauchy-Bunyakovsky-Schwarz inequality

$$\left(\sum_{i=1}^{n} p_i a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i b_i^2\right)^{1/2} \leq S\left(\left(\frac{M_1}{m_1}\frac{M_2}{m_2}\right)^2\right)\sum_{i=1}^{n} p_i a_i b_i. \tag{5.6}$$

From (4.7) and (4.8) we also have

$$\begin{aligned}\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 &\leq \frac{1}{p}\sum_{i=1}^{n} p_i a_i^{p+1}\sum_{i=1}^{n} p_i b_i + \frac{1}{q}\sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i^{q+1} \\ &\leq \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\}\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}\left(\sum_{i=1}^{n} p_i a_i b_i\right)^2 &\leq \frac{1}{p}\sum_{i=1}^{n} p_i b_i a_i^p \sum_{i=1}^{n} p_i a_i + \frac{1}{q}\sum_{i=1}^{n} p_i b_i \sum_{i=1}^{n} p_i a_i b_i^q \\ &\leq \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\}\left(\sum_{i=1}^{n} p_i a_i b_i\right)^2.\end{aligned} \tag{5.8}$$

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