Received 13/08/15

ON SOME INTEGRAL INEQUALITIES FOR GENERALIZED FRACTIONAL INTEGRAL

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Abstract. In this article we obtain interal inequalities for generalized Riemann-Liouville fractional integrals and Chebyshev functional by using synchronous functions.

1. INTRODUCTION

Consider the following equality which is studied in [1]

$$T(f,g)=\frac{1}{b-a}\int_b^a f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \tag{1.1}$$

where f and g are two synchronous and integrable functions on $[a,b]$.

(1.1) Chebyshev functional is studied by many scientists, see [9], [10], [11], [12] for references.

In this paper we obtain some integral inequalities for (1.1) type functional for generalized fractional integrals.

2. Fractional Integrals

Now we will give fundamental definitions and notations for fractional integrals [3-7].

Definition 1. *Let $a,b\in\mathbb{R}$, $a<b$, and $\alpha>0$. For $f\in L_1(a,b)$*

$$\left(J_{a^+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ \alpha>0,\ x>a \tag{2.1}$$

and

$$\left(J_{b^-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ b>0,\ b>x. \tag{2.2}$$

These integrals are called right-sided Riemann-Liouville fractional integral and left-sided Riemann-Liouville fractional integral respectively [2]-[7].

Definition 2. *Let (a,b) be a finite interval of the real line $\mathbb{R}$ and $\Re(\alpha)>0$. Also let $h(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $h'(x)$ on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function h on $[a,b]$ are defined by [13]*

$$\left(J_{a^+,h}^{\alpha}f\right)(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left[h(x)-h(t)\right]^{\alpha-1}h'(t)f(t)\,dt,\ x\geq a,\ \Re(\alpha)>0 \tag{2.3}$$

Key words and phrases. Fractional Integral, Riemann-Liouville Fractional Integral, Chebyshev functional.

2010 Mathematics Subject Classification. 26A33, 26D10, 26D15,41A55.

and

$$\left(J_{b^-,h}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[h(t) - h(x)\right]^{\alpha-1} h'(t) f(t)\, dt,\ x \le b,\ \Re(\alpha) > 0. \tag{2.4}$$

For (2.3) and (2.4)

$$\left(J_{a^+,h}^{\alpha} f\right)(a) = \left(J_{b^-,h}^{\alpha} f\right)(b) = 0.$$

If we take $h(x) = x$ in (2.3) and (2.4) integral formulas, we will obtain

$$J_{a^+,h}^{\alpha} = J_{a^+}^{\alpha} \text{ and } J_{b^-,h}^{\alpha} = J_{b^-}^{\alpha}.$$

Also if we choose $h(x) = \dfrac{x^{k+1}}{k+1}$ for $k \ge 0$, then the equalities (2.3) and (2.4) will be

$$(J_{a^+,k}^{\alpha} f)(x) = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^{k+1} - t^{k+1})^{\alpha-1} t^k f(t) dt,\ x > a \tag{2.5}$$

and

$$(J_{b^-,k}^{\alpha} f)(x) = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \int_x^b (t^{k+1} - x^{k+1})^{\alpha-1} t^k f(t) dt,\ x < b \tag{2.6}$$

respectively. This kind of generalized fractional integrals are studied in [5],[7],[14] and [16].

For $a = 0$ in (2.3), we can write

$$\left(J_{0^+,h}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (h(x) - h(t))^{\alpha-1} h'(t) f(t) dt,\ x > 0 \tag{2.7}$$

$$\left(J_{0^+,h}^{0} f\right)(x) = f(x).$$

Semi group and commutative properties of (2.7) integral operator is the following

$$J_{a^+,h}^{\alpha} J_{a^+,h}^{\beta} f(x) = J_{a^+,h}^{\alpha+\beta} f(x),\ \alpha \ge 0,\ \beta \ge 0$$

and

$$J_{a^+,h}^{\alpha} J_{a^+,h}^{\beta} f(x) = J_{a^+,h}^{\beta} J_{a^+,h}^{\alpha} f(x).$$

To show the being unit operator property of (2.7) integral operator, we choose h function specially as $f(x) = h(x)$ we obtain the following equality

$$\begin{aligned} \left(J_{0^+,h}^{\alpha} h\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (h(x) - h(t))^{\alpha-1} h(t) h'(t)\, dt \\ &= \frac{(h(x) - h(0))^{\alpha}}{\Gamma(\alpha+2)} \left[h(x) + \alpha h(0)\right]. \end{aligned} \tag{2.8}$$

Let $\alpha = 0$ in (2.8), then

$$\left(J_{0^+,h}^{0} h\right)(x) = h(x).$$

In (2.7) let $f(x) = x^{\mu}$, $f(x) = 1$ and $h(x) = x^{k+1}$ for $\alpha > 0$; $k \ge 0$, $\mu > -1$, $t > 0$ then

$$\begin{aligned} J_{0^+,h}^{\alpha}(x^{\mu}) &= \frac{(k+1)^{-\alpha}\Gamma(\frac{k+\mu+1}{k+1})}{\Gamma(\alpha + \frac{k+\mu+1}{k+1})} t^{\alpha(k+1)+\mu} \\ J_{0^+,h}^{\alpha}(1) &= \frac{(k+1)^{-\alpha}}{\Gamma(\alpha+1)} t^{\alpha(k+1)}. \end{aligned} \tag{2.9}$$

3. Main Results

Theorem 1. *Let f and g are two synchronous functions on $[0,\infty]$. Then for $t>0$, $\alpha>0$;*

$$J_{a^+,h}^{\alpha}(fg)(t) \geq \frac{1}{J_{a^+,h}^{\alpha}(1)} J_{a^+,h}^{\alpha} f(t) J_{a^+,h}^{\alpha} g(t). \tag{3.1}$$

Proof. For f and g synchronous functions, we have

$$\left(f\left(\tau\right)-f(\rho)\right)\left(g\left(\tau\right)-g(\rho)\right) \geq 0. \tag{3.2}$$

From (3.2) it can be written as following

$$f(\tau)g(\tau)+f(\rho)g(\rho) \geq f(\tau)g(\rho)+f(\rho)g(\tau). \tag{3.3}$$

If we multiply two sides of the (3.3) with $\frac{\left(h\left(t\right)-h\left(\tau\right)\right)^{\alpha-1}}{\Gamma(\alpha)} h^{'}\left(\tau\right)$, $\tau \in (a,t)$, we obtain

$$\begin{aligned} &\frac{\left(h\left(t\right)-h\left(\tau\right)\right)^{\alpha-1}}{\Gamma(\alpha)} h^{'}\left(\tau\right) f(\tau)g(\tau)+\frac{\left(h\left(t\right)-h\left(\tau\right)\right)^{\alpha-1}}{\Gamma(\alpha)} h^{'}\left(\tau\right) f(\rho)g(\rho) \geq \\ &\frac{\left(h\left(t\right)-h\left(\tau\right)\right)^{\alpha-1}}{\Gamma(\alpha)} h^{'}\left(\tau\right) f(\tau)g(\rho)+\frac{\left(h\left(t\right)-h\left(\tau\right)\right)^{\alpha-1}}{\Gamma(\alpha)} h^{'}\left(\tau\right) f(\rho)g(\tau). \end{aligned} \tag{3.4}$$

Integrating (3.4) inequality on (a,t), then

$$\begin{aligned} &\tfrac{1}{\Gamma(\alpha)} \int_a^t \left(h\left(t\right)-h\left(\tau\right)\right)^{\alpha-1} h^{'}\left(\tau\right) f(\tau)g(\tau)d\tau+\tfrac{1}{\Gamma(\alpha)} \int_a^t \left(h\left(t\right)-h\left(\tau\right)\right)^{\alpha-1} h^{'}\left(\tau\right) f(\rho)g(\rho)d\tau \geq \\ &\tfrac{1}{\Gamma(\alpha)} \int_a^t \left(h\left(t\right)-h\left(\tau\right)\right)^{\alpha-1} h^{'}\left(\tau\right) f(\tau)g(\rho)d\tau+\tfrac{1}{\Gamma(\alpha)} \int_a^t \left(h\left(t\right)-h\left(\tau\right)\right)^{\alpha-1} h^{'}\left(\tau\right) f(\rho)g(\tau)d\tau. \end{aligned} \tag{3.5}$$

Therefore

$$\begin{aligned} &J_{a^+,h}^{\alpha}(fg)(t)+f(\rho)g(\rho)\tfrac{1}{\Gamma(\alpha)} \int_a^t \left(h\left(t\right)-h\left(\tau\right)\right)^{\alpha-1} h^{'}\left(\tau\right) d\tau \geq \\ &g(\rho)\tfrac{1}{\Gamma(\alpha)} \int_a^t \left(h\left(t\right)-h\left(\tau\right)\right)^{\alpha-1} h^{'}\left(\tau\right) f(\tau)d\tau+f(\rho)\frac{1}{\Gamma(\alpha)} \int_a^t \left(h\left(t\right)-h\left(\tau\right)\right)^{\alpha-1} h^{'}\left(\tau\right) g(\tau)d\tau \end{aligned} \tag{3.6}$$

and

$$J_{a^+,h}^{\alpha}(fg)(t)+f(\rho)g(\rho)J_{a^+,h}^{\alpha}(1) \geq g(\rho)J_{a^+,h}^{\alpha}(f)(t)+f(\rho)J_{a^+,h}^{\alpha}(g)(t). \tag{3.7}$$

Now multiplying two sides of (3.7) with $\frac{\left(h\left(t\right)-h\left(\rho\right)\right)^{\alpha-1}}{\Gamma(\alpha)} h^{'}\left(\rho\right)$, $\rho \in (a,t)$, we have

$$\begin{aligned} &\frac{\left(h\left(t\right)-h\left(\rho\right)\right)^{\alpha-1}}{\Gamma(\alpha)} h^{'}\left(\rho\right) J_{a^+,h}^{\alpha}(fg)(t)+\frac{\left(h\left(t\right)-h\left(\rho\right)\right)^{\alpha-1}}{\Gamma(\alpha)} h^{'}\left(\rho\right) f(\rho)g(\rho)J_{a^+,h}^{\alpha}(1) \geq \\ &\frac{\left(h\left(t\right)-h\left(\rho\right)\right)^{\alpha-1}}{\Gamma(\alpha)} h^{'}\left(\rho\right) g(\rho)J_{a^+,h}^{\alpha} f(t)+\frac{\left(h\left(t\right)-h\left(\rho\right)\right)^{\alpha-1}}{\Gamma(\alpha)} h^{'}\left(\rho\right) f(\rho)J_{a^+,h}^{\alpha}(g)(t). \end{aligned} \tag{3.8}$$

By integrating to (3.8) on (a,t), then

$$\begin{aligned} &J_{a^+,h}^{\alpha}(fg)(t) \int_a^t \frac{\left(h\left(t\right)-h\left(\rho\right)\right)^{\alpha-1}}{\Gamma(\alpha)} h^{'}\left(\rho\right) d\rho+\frac{J_{a^+,h}^{\alpha}(1)}{\Gamma(\alpha)} \int_a^t f(\rho)g(\rho)\left(h\left(t\right)-h\left(\rho\right)\right)^{\alpha-1} h^{'}\left(\rho\right) d\rho \geq \\ &\frac{J_{a^+,h}^{\alpha} f(t)}{\Gamma(\alpha)} \int_a^t \left(h\left(t\right)-h\left(\rho\right)\right)^{\alpha-1} h^{'}\left(\rho\right) g(\rho)d\rho+\frac{J_{a^+,h}^{\alpha} g(t)}{\Gamma(\alpha)} \int_a^t \left(h\left(t\right)-h\left(\rho\right)\right)^{\alpha-1} h^{'}\left(\rho\right) f(\rho)d\rho. \end{aligned} \tag{3.9}$$

This inequality is can be written as the following at the same time

$$J^{\alpha}_{a^+,h}(fg)(t) \geq \frac{1}{J^{\alpha}_{a^+,h}(1)} J^{\alpha}_{a^+,h}f(t) J^{\alpha}_{a^+,h}g(t). \tag{3.10}$$

So the proof is completed. □

Theorem 2. *Let f and g are two synchronous functions on $[a,b]$. Then for $t>a$, $\alpha>0$, and $\beta>0$,*

$$J^{\beta}_{a^+,h}(1) J^{\alpha}_{a^+,h}(fg)(t) + \frac{(h(t)-h(a))^{\alpha}}{\Gamma(\alpha+1)} J^{\beta}_{a^+,h}(fg)(t) \geq J^{\alpha}_{a^+,h}f(t) J^{\beta}_{a^+,h}g(t) + J^{\alpha}_{a^+,h}g(t) J^{\beta}_{a^+,h}f(t).$$

Proof. If we multiply two sides of (3.7) with $\frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho)$, then we obtain

$$\begin{aligned} &\frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) J^{\alpha}_{a^+,h}(fg)(t) + \frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) f(\rho) g(\rho) J^{\alpha}_{a^+,h}(1) \geq \\ &\frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) g(\rho) J^{\alpha}_{a^+,h}(f)(t) + \frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) f(\rho) J^{\alpha}_{a^+,h}(g)(t). \end{aligned} \tag{3.11}$$

Integrating to (3.11) on (a,t), then

$$\begin{aligned} &\int_a^t \frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) J^{\alpha}_{a^+,h}(fg)(t) dt + \int_a^t \frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) f(\rho) g(\rho) J^{\alpha}_{a^+,h}(1) dt \geq \\ &\int_a^t \frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) g(\rho) J^{\alpha}_{a^+,h}(f)(t) dt + \int_a^t \frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) f(\rho) J^{\alpha}_{a^+,h}(g)(t) dt. \end{aligned} \tag{3.12}$$

This is the proof of the theorem

$$J^{\beta}_{a^+,h}(1) J^{\alpha}_{a^+,h}(fg)(t) + J^{\alpha}_{a^+,h}(1) J^{\beta}_{a^+,h}(fg)(t) \geq J^{\alpha}_{a^+,h}f(t) J^{\beta}_{a^+,h}g(t) + J^{\alpha}_{a^+,h}g(t) J^{\beta}_{a^+,h}f(t). \tag{3.13}$$

□

Remark 1. *It is obvious that if we take $\alpha=\beta$ in this theorem we will obtain Theorem 1.*

Theorem 3. *Let $(f_i)_{i=1,\dots n}$ be increasing functions on $[0,\infty]$. Then for all $t>0$, $\alpha>0$,*

$$J^{\alpha}_{a^+,h}\left(\prod_{i=1}^{n} f_i\right)(t) \geq \left(J^{\alpha}_{a^+,h}(1)\right)^{1-n} \left(\prod_{i=1}^{n} J^{\alpha}_{a^+,h} f_i\right)(t) \tag{3.14}$$

Proof. We will prove this theorem by induction.It is clear that for $n=1$ and all $t>0$, $\alpha>0$, we have $J^{\alpha}_{a^+,h}(f_1)(t) \geq J^{\alpha}_{a^+,h}f_1(t)$. And for $n=2$, we obtain (3.1),

$$J^{\alpha}_{a^+,h}(f_1 f_2)(t) \geq \left(J^{\alpha}_{a^+,h}(1)\right)^{-1} \left(J^{\alpha}_{a^+,h} f_1\right)(t) \left(J^{\alpha}_{a^+,h} f_2\right)(t) \tag{3.15}$$

Now assume that(induction hypothesis)

$$J^{\alpha}_{a^+,h}\left(\prod_{i=1}^{n-1} f_i\right)(t) \geq \left(J^{\alpha}_{a^+,h}(1)\right)^{2-n} \left(\prod_{i=1}^{n-1} J^{\alpha}_{a^+,h} f_i\right)(t) \tag{3.16}$$

If $(f_i)_{i=1,\dots n}$ are positive increasing functions, then $\left(\prod_{i=1}^{n-1} f_i\right)(t)$ is an increasing function. So we can use Theorem 1 for functions $\prod_{i=1}^{n-1} f_i = g$, and $f_n = f$, therefore we obtain

(3.17)
$$J^{\alpha}_{a^+,h}(\prod_{i=1}^{n} f_i)(t) = J^{\alpha}_{a^+,h}(fg)(t) \geq \left(J^{\alpha}_{a^+,h}(1)\right)^{-1} \left(\prod_{i=1}^{n-1} J^{\alpha}_{a^+,h} f_i\right)(t) \left(J^{\alpha}_{a^+,h} f_n\right)(t).$$

By (3.16)

(3.18)
$$J^{\alpha}_{a^+,h}(\prod_{i=1}^{n} f_i)(t) \geq \left(J^{\alpha}_{a^+,h}(1)\right)^{-1} \left(J^{\alpha}_{a^+,h}(1)\right)^{2-n} \left(\prod_{i=1}^{n-1} J^{\alpha}_{a^+,h} f_i\right)(t) \left(J^{\alpha}_{a^+,h} f_n\right)(t).$$

This completes the proof. □

Theorem 4. *Let $h(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $h^{'}(x)$ on (a,b). If f is an increasing and g is a differentiable functions and there exist a real number $mh^{'}(t) := \inf_{t\geq 0} g^{'}(t)$ on $[0,+\infty]$. Then for all $t > 0$, $\alpha > 0$,*

(3.19)
$$J^{\alpha}_{a^+,h}(fg)(t) \geq \left(J^{\alpha}_{a^+,h}(1)\right)^{-1} J^{\alpha}_{a^+,h} f(t) J^{\alpha}_{a^+,h} g(t) - \frac{mh(t)}{\alpha+1} J^{\alpha}_{a^+,h} f(t) + m J^{\alpha}_{a^+,h}\left(h(t) f(t)\right).$$

Proof. Consider the given function $H(t) = g(t) - mh(t)$.It is clear that h is an increasing function and differentiable on $[0,+\infty]$. Then using Theorem 1 we obtain

(3.20)
$$\begin{aligned} J^{\alpha}_{a^+,h}(Hf)(t) &= J^{\alpha}_{a^+,h}((g(t) - mh(t)) f(t)) \\ &\geq \left(J^{\alpha}_{a^+,h}(1)\right)^{-1} J^{\alpha}_{a^+,h} f(t) \left(J^{\alpha}_{a^+,h} g(t) - m J^{\alpha}_{a^+,h} f(t)\right) \\ &\geq \left(J^{\alpha}_{a^+,h}(1)\right)^{-1} J^{\alpha}_{a^+,h} f(t) J^{\alpha}_{a^+,h} g(t) - \frac{m\left(J^{\alpha}_{a^+,h}(1)\right)^{-1}(h(t)-h(a))^{\alpha}(h(t)+\alpha h(a))}{\Gamma(\alpha+2)} J^{\alpha}_{a^+,h} f(t) \\ &\geq \left(J^{\alpha}_{a^+,h}(1)\right)^{-1} J^{\alpha}_{a^+,h} f(t) J^{\alpha}_{a^+,h} g(t) - \frac{m(h(t)+\alpha h(a))}{\alpha+1} J^{\alpha}_{a^+,h} f(t). \end{aligned}$$

Therefore

(3.21)
$$J^{\alpha}_{a^+,h}(fg)(t) \geq \left(J^{\alpha}_{a^+,h}(1)\right)^{-1} J^{\alpha}_{a^+,h} f(t) J^{\alpha}_{a^+,h} g(t) - \frac{m\left(h(t)+\alpha h(a)\right)}{\alpha+1} J^{\alpha}_{a^+,h} f(t) + m J^{\alpha}_{a^+,h}\left(h(t) f(t)\right).$$

This is the proof of theorem. □

Corollary 1. *Let f and g are functions on $[0,+\infty]$. Then*

I. If f is a decreasing and g is a differentiable functions and there exists a real number $Mh(t) := \sup_{t\geq 0} g^{'}(t)$. Then for all $t > 0$, $\alpha > 0$

(3.22)
$$J^{\alpha}_{a^+,h}(fg)(t) \geq \left(J^{\alpha}_{a^+,h}(1)\right)^{-1} J^{\alpha}_{a^+,h} f(t) J^{\alpha}_{a^+,h} g(t) - \frac{M\left(h(t)+\alpha h(a)\right)}{\alpha+1} J^{\alpha}_{a^+,h} f(t) + M J^{\alpha}_{a^+,h}\left(h(t) f(t)\right).$$

II. If f and g are differentiable functions and there exist real numbers $m_1 h(t) := \inf_{t\geq 0} f^{'}(x)$, and $m_2 h(t) := \inf_{t\geq 0} g^{'}(t)$. Then,

(3.23)
$$J_{a^+,h}^{\alpha}(fg)(t) - m_1 J_{a^+,h}^{\alpha} h(t)\, g(t) - m_2 J_{a^+,h}^{\alpha} h(t)\, f(t) + m_1 m_2 J_{a^+,h}^{\alpha} h(t)^2 \geq \left(J_{a^+,h}^{\alpha}(1)\right)^{-1} \Big[J_{a^+,h}^{\alpha} f(t) J_{a^+,h}^{\alpha} g(t) - m_1 J_{a^+,h}^{\alpha} h(t)\, J_{a^+,h}^{\alpha} g(t) - m_2 J_{a^+,h}^{\alpha} h(t)\, J_{a^+,h}^{\alpha} f(t) + m_1 m_2 \left(J_{a^+,h}^{\alpha} h(t)\right)^2 \Big].$$

III. If f and g are differentiable functions and there exist real numbers $M_1 h(t) := \sup_{t\geq 0} f^{'}(x)$, and $M_2 h(t) := \sup_{t\geq 0} g^{'}(t)$. Then

(3.24)
$$J_{a^+,h}^{\alpha}(fg)(t) - M_1 J_{a^+,h}^{\alpha} h(t)\, g(t) - M_2 J_{a^+,h}^{\alpha} h(t)\, f(t) + M_1 M_2 \left(J_{a^+,h}^{\alpha} h(t)\right)^2 \geq \left(J_{a^+,h}^{\alpha}(1)\right)^{-1} \Big[J_{a^+,h}^{\alpha} f(t) J_{a^+,h}^{\alpha} g(t) - M_1 J_{a^+,h}^{\alpha} t J_{a^+,h}^{\alpha} g(t) - M_2 J_{a^+,h}^{\alpha} h(t)\, J_{a^+,h}^{\alpha} f(t) + M_1 M_2 \left(J_{a^+,h}^{\alpha} h(t)\right)^2 \Big].$$

Proof. Applying Theorem1 for decreasing function F and G such that:
$$G(t) = g(t) - m_2 h(t).$$
we can prove (I).

Using the Theorem 3.1 to the increasing functions F and G, where:
$$F(t) = f(t) - m_1 h(t), \; G(t) = g(t) - m_2 h(t)$$
we can prove (II).

To prove (III), it suffice to consider the two decreasing functions
$$F(t) = f(t) - M_1 h(t), \; G(t) = g(t) - M_2 h(t).$$
□

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