Received 28/01/15

SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE EXPONENTIALS ARE CONVEX

S. S. DRAGOMIR[1,2] AND I. GOMM[1]

Abstract. Some inequalities of Hermite-Hadamard type for functions whose exponentials are convex are obtained.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the Hermite-Hadamard inequality.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the papers [1] – [58] and the references therein.

We denote by $\mathfrak{Expconv}(I)$ the class of all functions defined on the interval I of real numbers such that $\exp(f)$ is convex on I. If $\mathfrak{Conv}(I)$ is the class of convex functions defined on I then we have the following fact:

Proposition 1. *We have the strict inclusion*

$$\mathfrak{Conv}(I)\subsetneq\mathfrak{Expconv}(I).$$

Proof. Let f be convex on I. Then for any $x,y\in I$ and $\lambda\in[0,1]$ we have by the arithmetic mean - geometric mean inequality that

$$\begin{aligned}\exp f((1-\lambda)x+\lambda y) &\leq \exp[(1-\lambda)f(x)+\lambda f(y)]\\ &= [\exp f(x)]^{1-\lambda}[f(y)]^{\lambda}\\ &\leq (1-\lambda)\exp f(x)+\lambda\exp f(y),\end{aligned}$$

which shows that $\exp(f)$ is convex on I.

For $r\geq 1$ the function $f_r(x)=r\ln x,\ x>0$ is concave on $(0,\infty)$. We have $\exp(f_r(x))=x^r$ is a convex function, therefore $f_r\in\mathfrak{Expconv}(I)\setminus\mathfrak{Conv}(I)$. □

We observe that for twice differentiable functions g on $\mathring{I}$, the interior of I we have that

$$(\exp(g(x)))''=\left([g'(x)]^2+g''(x)\right)\exp g(x),\ x\in\mathring{I},$$

therefore $g\in\mathfrak{Expconv}(I)$ if and only if

$$[g'(x)]^2+g''(x)\geq 0 \text{ for any } x\in\mathring{I}.$$

1991 *Mathematics Subject Classification.* 26D15; 25D10.

Key words and phrases. Convex functions, Hermite-Hadamard inequality, Special means.

2. Some Hermite-Hadamard Type Inequalities

Now, if $g \in \mathfrak{Expconv}(I)$, then by the Hermite-Hadamard inequality for $\exp(g)$ we have for $a, b \in I$ with $a < b$ that

$$\exp g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \exp g(t)\,dt \le \frac{1}{2}\left[\exp g(a) + \exp g(b)\right]. \tag{2.1}$$

By Jensen's integral inequality for the exp function we also have for any integrable function $h : [a, b] \to \mathbb{R}$ that

$$\exp\left(\frac{1}{b-a}\int_a^b h(t)\,dt\right) \le \frac{1}{b-a}\int_a^b \exp h(t)\,dt. \tag{2.2}$$

We can improve the inequality (2.1) for convex functions as follows:

Theorem 1. *Let $f : I \to \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then we have for $f(b) \ne f(a)$ the inequalities*

$$\begin{aligned}\exp f\left(\frac{a+b}{2}\right) &\le \exp\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \le \frac{1}{b-a}\int_a^b \exp f(t)\,dt \\ &\le \frac{\exp f(b) - \exp f(a)}{f(b) - f(a)} \le \frac{1}{2}\left[\exp f(a) + \exp f(b)\right].\end{aligned} \tag{2.3}$$

Proof. The first inequality follows by Hermite-Hadamard inequality for the convex function f. The second inequality follows by (2.2).

By the convexity of f we have

$$\begin{aligned}\exp f(t) &= \exp f\left(\frac{b-t}{b-a}\cdot a + \frac{t-a}{b-a}\cdot b\right) \\ &\le \exp\left[\frac{b-t}{b-a}f(a) + \frac{t-a}{b-a}f(b)\right] \\ &= \exp\left[\frac{f(b)-f(a)}{b-a}t + \frac{bf(a)-af(b)}{b-a}\right]\end{aligned} \tag{2.4}$$

for any $t \in [a, b]$.

Integrating the inequality (2.4) on $[a, b]$ we get

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \exp f(t)\,dt \\ &\le \frac{1}{b-a}\int_a^b \exp\left[\frac{f(b)-f(a)}{b-a}t + \frac{bf(a)-af(b)}{b-a}\right]dt \\ &= \frac{1}{f(b)-f(a)}\left(\exp\left[\frac{f(b)-f(a)}{b-a}t + \frac{bf(a)-af(b)}{b-a}\right]\Bigg|_a^b\right) \\ &= \frac{1}{f(b)-f(a)}\left[\exp\left(\frac{f(b)-f(a)}{b-a}b + \frac{bf(a)-af(b)}{b-a}\right)\right. \\ &\left. - \exp\left(\frac{f(b)-f(a)}{b-a}a + \frac{bf(a)-af(b)}{b-a}\right)\right] \\ &= \frac{\exp f(b) - \exp f(a)}{f(b)-f(a)}\end{aligned}$$

and the third inequality in (2.3) is proved.

Utilising the Hermite-Hadamard inequality for the convex function exp we have

$$\frac{\exp d-\exp c}{d-c}=\frac{1}{d-c}\int_c^d \exp u du\leq \frac{\exp d+\exp c}{2}$$

for any $c,d\in\mathbb{R}$, $d\neq c$, therefore

$$\frac{\exp f(b)-\exp f(a)}{f(b)-f(a)}\leq \frac{1}{2}\left[\exp f(a)+\exp f(b)\right]$$

and the last inequality in (2.3) is proved. □

Consider the *identric mean* of two positive numbers

$$I=I(a,b):=\begin{cases} a & \text{if} \quad a=b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a\neq b, \end{cases} \quad a,b>0.$$

We observe that

$$\ln I(a,b)=\frac{1}{b-a}\int_a^b \ln u du$$

for $a,b>0$, $a\neq b$.

The following result holds:

Theorem 2. *Assume that $f\in\mathfrak{Expconv}(I)$ and $a,b\in I$ with $a<b$. Then we have*

$$\exp\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\leq I\left(\exp f(a),\exp f(b)\right) \tag{2.5}$$

and

$$\begin{aligned} &\exp f\left(\frac{a+b}{2}\right) \\ &\leq \exp\left(\frac{1}{b-a}\int_a^b \ln\left[\frac{\exp f(x)+\exp f(a+b-x)}{2}\right]dx\right) \\ &\leq \frac{1}{b-a}\int_a^b \exp f(x)\,dx. \end{aligned} \tag{2.6}$$

Proof. Since $f\in\mathfrak{Expconv}(I)$, then

$$\exp f((1-\lambda)a+\lambda b)\leq(1-\lambda)\exp f(a)+\lambda\exp f(b)$$

for any $\lambda\in[0,1]$, which is equivalent to

$$f((1-\lambda)a+\lambda b)\leq\ln\left[(1-\lambda)\exp f(a)+\lambda\exp f(b)\right] \tag{2.7}$$

for any $\lambda\in[0,1]$.

Integrating (2.7) on $[0,1]$ we get

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(t)\,dt &= \int_0^1 f((1-\lambda)a+\lambda b)\,d\lambda \\ &\leq \int_0^1 \ln\left[(1-\lambda)\exp f(a)+\lambda\exp f(b)\right]d\lambda \\ &= \frac{1}{\exp f(b)-\exp f(a)}\int_{\exp f(a)}^{\exp f(b)} \ln u du \\ &= \ln I\left(\exp f(a),\exp f(b)\right) \end{aligned} \tag{2.8}$$

and the inequality in (2.5) is proved.

From (2.7) we have

$$f\left(\frac{x+y}{2}\right)\leq \ln\left[\frac{\exp f(x)+\exp f(y)}{2}\right] \tag{2.9}$$

for any $x,y\in I$.

From (2.9) we have

$$f\left(\frac{a+b}{2}\right)\leq \ln\left[\frac{\exp f(x)+\exp f(a+b-x)}{2}\right] \tag{2.10}$$

for any $x\in[a,b]$.

Integrating the inequality (2.10) over x on $[a,b]$ we get the first inequality in (2.6).

By the Jensen inequality for the concave function ln we have

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \ln\left[\frac{\exp f(x)+\exp f(a+b-x)}{2}\right]dx \\ &\leq \ln\left(\frac{1}{b-a}\int_a^b \left[\frac{\exp f(x)+\exp f(a+b-x)}{2}\right]dx\right) \\ &= \ln\left(\frac{1}{2(b-a)}\int_a^b [\exp f(x)+\exp f(a+b-x)]\,dx\right) \\ &= \ln\left(\frac{1}{b-a}\int_a^b \exp f(x)\,dx\right) \end{aligned} \tag{2.11}$$

and the second inequality in (2.6) is proved. □

Remark 1. *If the function $g: I\to(0,\infty)$ is convex on I, then $f=\ln g\in \mathfrak{Expconv}(I)$ and for $a,b\in I$ with $a<b$ we have, by (2.5) and (2.6), the following inequalities*

$$\exp\left(\frac{1}{b-a}\int_a^b \ln g(t)\,dt\right)\leq I(g(a),g(b)) \tag{2.12}$$

and

$$\begin{aligned} g\left(\frac{a+b}{2}\right) &\leq \exp\left(\frac{1}{b-a}\int_a^b \ln\left[\frac{g(x)+g(a+b-x)}{2}\right]dx\right) \\ &\leq \frac{1}{b-a}\int_a^b g(x)\,dx. \end{aligned} \tag{2.13}$$

3. Related Results

The following related result also holds:

Theorem 3. *Assume that $f\in\mathfrak{Expconv}(I)$ and $a,b\in I$ with $a<b$. Then we have*

$$\begin{aligned} &\frac{f(a)(x-a)+f(b)(b-x)}{b-a}-\frac{1}{b-a}\int_a^b f(y)\,dy \\ &\geq \exp f(x)\left[\exp(-f(x))-\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy\right] \end{aligned} \tag{3.1}$$

for any $x \in [a,b]$.

In particular, we have

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(y)\,dy \geq \exp f\left(\frac{a+b}{2}\right)\left[\exp\left(-f\left(\frac{a+b}{2}\right)\right)-\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy\right]. \tag{3.2}$$

Proof. Since the function $\exp(f)$ is convex, it has lateral derivatives in each point of (a,b) and $f=\ln(\exp f)$ does the same. Then for any $x,y\in(a,b)$ we have

$$\exp f(x)-\exp f(y)\geq f'_-(y)(x-y)\exp f(y)$$

and dividing by $\exp f(y)>0$ we get

$$\exp f(x)\exp[-f(y)]-1\geq f'_-(y)(x-y) \tag{3.3}$$

for any $x,y\in(a,b)$.

Integrating (3.3) over y on $[a,b]$ and dividing by $b-a$ we get

$$\begin{aligned}\exp f(x)\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy-1 &\geq \frac{1}{b-a}\int_a^b f'_-(y)(x-y)\,dy\\ &=\frac{1}{b-a}\left[f(y)(x-y)\big|_a^b+\int_a^b f(y)\,dy\right]\\ &=\frac{1}{b-a}\left[\int_a^b f(y)\,dy-f(a)(x-a)-f(b)(b-x)\right]\end{aligned} \tag{3.4}$$

for any $x\in[a,b]$, which is equivalent to the desired inequality (3.1). □

Corollary 1. *With the assumptions of Theorem 3 we have*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(y)\,dy \geq \frac{\exp f(a)+\exp f(b)}{2}\left[1-\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy\right]. \tag{3.5}$$

Proof. If we take $x=a$ and $x=b$ in (3.4) we get

$$\exp f(a)\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy-1\geq\frac{1}{b-a}\int_a^b f(y)\,dy-f(b)$$

and

$$\exp f(b)\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy-1\geq\frac{1}{b-a}\int_a^b f(y)\,dy-f(a).$$

Adding these inequalities and dividing by two we get

$$\frac{\exp f(a)+\exp f(b)}{2}\left[\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy-1\right]\geq\frac{1}{b-a}\int_a^b f(y)\,dy-\frac{f(a)+f(b)}{2},$$

which is equivalent to the desired inequality (3.5). □

Corollary 2. *With the assumptions of Theorem 3 and if*

$$x_0 := \frac{f(b)\,b - f(a)\,a - \int_a^b f(y)\,dy}{f(b) - f(a)} \in [a,b], \tag{3.6}$$

where $f(b) \neq f(a)$, *then we have*

$$\frac{1}{b-a}\int_a^b \exp[-f(y)]\,dy \geq \exp\left(-f\left(\frac{f(b)\,b - f(a)\,a - \int_a^b f(y)\,dy}{f(b) - f(a)}\right)\right). \tag{3.7}$$

Proof. Follows by (3.1) by taking $x = x_0$ defined in (3.6). □

Remark 2. *Since*

$$x_0 = \frac{\int_a^b f'(y)\,y\,dy}{\int_a^b f'(y)\,dy},$$

then a sufficient condition for (3.6) to hold is that f *is monotonic nondecreasing or nonincreasing on the whole interval* $[a,b]$.

Remark 3. *If the function* $g : I \to (0,\infty)$ *is convex on* I, *then* $f = \ln g \in \mathfrak{Expconv}(I)$ *and for* $a, b \in I$ *with* $a < b$ *we have, by (3.1), (3.2) and (3.5), the following inequalities*

$$\ln\left([g(a)]^{\frac{x-a}{b-a}}\,[g(b)]^{\frac{b-x}{b-a}}\right) - \frac{1}{b-a}\int_a^b \ln g(y)\,dy \geq g(x)\left[\frac{1}{g(x)} - \frac{1}{b-a}\int_a^b \frac{1}{g(y)}dy\right], \tag{3.8}$$

$$\ln\left(\sqrt{g(a)\,g(b)}\right) - \frac{1}{b-a}\int_a^b \ln g(y)\,dy \geq g\left(\frac{a+b}{2}\right)\left[\frac{1}{g\left(\frac{a+b}{2}\right)} - \frac{1}{b-a}\int_a^b \frac{1}{g(y)}dy\right], \tag{3.9}$$

and

$$\ln\left(\sqrt{g(a)\,g(b)}\right) - \frac{1}{b-a}\int_a^b \ln g(y)\,dy \geq \frac{g(a)+g(b)}{2}\left[1 - \frac{1}{b-a}\int_a^b \frac{1}{g(y)}dy\right]. \tag{3.10}$$

If

$$x_0 := \frac{\ln\left(\frac{[g(b)]^b}{[g(a)]^a}\right) - \int_a^b \ln g(y)\,dy}{\ln\left(\frac{g(b)}{g(a)}\right)} \in [a,b], \tag{3.11}$$

where $g(b) \neq g(a)$, *then we have*

$$\frac{1}{b-a}\int_a^b \frac{1}{g(y)}dy \geq \frac{1}{g\left(\frac{\ln\left(\frac{[g(b)]^b}{[g(a)]^a}\right) - \int_a^b \ln g(y)dy}{\ln\left(\frac{g(b)}{g(a)}\right)}\right)}. \tag{3.12}$$

References

[1] Allasia, G., Giordano, C. & Pečarić, J. (1999). Hadamard-type Inequalities for $(2r)$-convex Functions with Applications, *Atti Acad. Sci. Torino-Cl. Sc. Fis.*, 133, 1-14.

[2] Alzer, H. (1989). A Note on Hadamard's Inequalities, *C.R. Math. Rep. Acad. Sci. Canada,* 11, 255-258.

[3] Alzer, H. (1989). On an Integral Inequality, *Math. Rev. Anal. Numer. Theor. Approx.,* 18, 101-103.

[4] Azpeitia, A.G. (1994). Convex Functions and the Hadamard Inequality, *Rev.-Colombiana-Mat.*, 28(1), 7–12.

[5] Barbu, D., Dragomir, S. S. & Buşe, C. (1993). A Probabilistic Argument for the Convergence of Some Sequences Associated to Hadamard's Inequality, *Studia Univ. Babeş-Bolyai, Math.,* 38 (1), 29-33.

[6] Buşe, C., Dragomir, S .S. & Barbu, D. (1996). The Convergence of Some Sequences Connected to Hadamard's Inequality, *Demostratio Math.,* 29(1) (1996), 53-59.

[7] Dragomir, S. S. (1991). A Mapping in Connection to Hadamard's Inequalities, *An. Öster. Akad. Wiss. Math.-Natur.,* (Wien), 128, 17-20.

[8] Dragomir, S. S. (1993). A Refinement of Hadamard's Inequality for Isotonic Linear Functionals, *Tamkang J. of Math.* (Taiwan), 24, 101-106.

[9] Dragomir, S. S. (1992). On Hadamard's Inequalities for Convex Functions, *Mat. Balkanica,* 6, 215-222.

[10] Dragomir, S. S. (2000). On Hadamard's Inequality for the Convex Mappings Defined on a Ball in the Space and Applications, *Math. Ineq. & Appl.*, 3(2) , 177-187.

[11] Dragomir, S. S. (2000). On Hadamard's Inequality on a Disk, *Journal of Ineq. in Pure & Appl. Math.,* 1(1), Article 2, 1-11.

[12] Dragomir, S. S. (1992). Some Integral Inequalities for Differentiable Convex Functions, *Contributions, Macedonian Acad. of Sci. and Arts*, 13(1) , 13-17.

[13] Dragomir, S. S. (1992). Two mappings in Connection to Hadamard's Inequalities, *J. Math. Anal. Appl.,* **167**, 49-56.

[14] Dragomir, S. S. (1994). Some Remarks on Hadamard's Inequalities for Convex Functions, *Extracta Math.*, 9 (2) , 88-94.

[15] Dragomir, S. S. & Agarwal, R. P. (1998). Two New Mappings Associated with Hadamard's Inequalities for Convex Functions, *Appl. Math. Lett.,* 11 (3), 33-38.

[16] Dragomir, S. S. & Buşe, C. (1995). Refinements of Hadamard's Inequality for Multiple Integrals, *Utilitas Math.* (Canada), 47, 193-195.

[17] Dragomir, S. S., Cho, Y. J. & Kim, S. S. (2000). Inequalities of Hadamard's Type for Lipschitzian Mappings and their Applications, *J. of Math. Anal. Appl.,* 245 (2) , 489-501.

[18] Dragomir, S. S. & Gomm, I. (2011). Bounds for Two Mappings Associated to the Hermite-Hadamard Inequality, *Aust. J. Math. Anal. Appl.*, 8, Art. 5,1-9.

[19] Dragomir, S. S. & Gomm, I. (2012). Some New Bounds for Two Mappings Related to the Hermite-Hadamard Inequality for Convex Functions, *Num. Alg. Cont. & Opt.* 2(2), 271-278.

[20] Dragomir, S. S. & Fitzpatrick, S. (1998). The Hadamard's Inequality for $s-$convex Functions in the First Sense, *Demonstratio Math.,* 31(3) , 633-642.

[21] Dragomir, S. S. & Fitzpatrick, S. (1999). The Hadamard's Inequality for $s-$convex Functions in the Second Sense, *Demonstratio Math.*, 32 (4) , 687-696.

[22] Dragomir, S. S. & Ionescu, N. M. (1990). On Some Inequalities for Convex-dominated Functions, A*nal. Num. Theor. Approx.*, 19, 21-28.

[23] Dragomir, S. S., Milošević, D. S. & Sándor, J. (1993). On Some Refinements of Hadamard's Inequalities and Applications, *Univ. Belgrad, Publ. Elek. Fak. Sci. Math.,* 4, 21-24.

[24] Dragomir, S. S. & Mond, B. (1997). On Hadamard's Inequality for a Class of Functions of Godunova and Levin, *Indian J. Math.*, 39(1), 1–9.

[25] Dragomir, S. S. & Pearce, C. E. M. (1998). Quasi-convex Functions and Hadamard's Inequality, *Bull. Austral. Math. Soc.,* 57, 377-385.

[26] Dragomir, S. S. & Pearce, C. E. M. (2000). *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, [Online http://rgmia.org/monographs/hermite_hadamard.html].

[27] Dragomir, S. S., Pearce, C. E. M. & Pečarić, J. (1995). On Jessen's and Related Inequalities for Isotonic Sublinear Functionals, *Acta Math. Sci.* (Szeged), 61, 373-382.

[28] Dragomir, S. S., Pečarić, J. & Persson, L. E. (1995). Some Inequalities of Hadamard type, *Soochow J. of Math.* (Taiwan), 21, 335-341.

[29] Dragomir, S. S., Pečarić, J. & Sándor, J. (1990). A Note on the Jensen-Hadamard Inequality, *Anal. Num. Theor. Approx.,* 19, 21-28.

[30] Dragomir, S. S. & Toader, G. H. (1993). Some Inequalities for $m-$convex Functions, *Studia Univ. Babeş-Bolyai, Math.*, 38(1) , 21-28.

[31] Fink, A. M. (1998). A Best Possible Hadamard Inequality, *Math. Ineq. & Appl.,* 2, 223-230.

[32] Fink, A. M. (1994). Toward a Theory of Best Possible Inequalities, *Nieuw Archief von Wiskunde,* 12, 19-29.

[33] Fink, A. M. (1995). Two Inequalities, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat.,* 6, 48-49.

[34] Gavrea, B. (2000). On Hadamard's Inequality for the Convex Mappings Defined on a Convex Domain in the Space, *Journal of Ineq. in Pure & Appl. Math.,* 1(1), Article 9, 1-6.

[35] Gill, P. M., Pearce, C. E. M. & Pečarić, J. (1997). Hadamard's Inequality for $r-$convex Functions, *J. of Math. Anal. and Appl.,* 215, 461-470.

[36] Hardy, G. H., Littlewood, E. J. & Pólya, G. (1952). *Inequalities*, 2nd Ed., Cambridge University Press.

[37] Lee, K.-C. & Tseng, K.-L. (2000). On a Weighted Generalisation of Hadamard's Inequality for G-convex Functions, *Tamsui Oxford Journal of Math. Sci.*, 16(1) , 91-104.

[38] Lupaş, A. (1997). The Jensen-Hadamard Inequality for Convex Functions of Higher Order, *Octogon Math. Mag.*, 5(2), 8-9.

[39] Lupaş, A. (1976). A Generalisation of Hadamard's Inequality for Convex Functions, *Univ. Beograd. Publ. Elek. Fak. Ser. Mat. Fiz.,* No. 544-576, 115-121.

[40] Maksimović, D. M. (1979). A Short Proof of Generalized Hadamard's Inequalities, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No. 634-677, 126–128.

[41] Mitrinović. D. S. & Lacković, I. (1985). Hermite and Convexity, *Aequat. Math.*, 28, 229–232.

[42] Mitrinović. D. S, Pečarić, J. & Fink, A. M. (1993) *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London.

[43] Neuman, E. (1986). Inequalities Involving Generalised Symmetric Means, *J. Math. Anal. Appl.,* 120, 315-320.

[44] Neuman, E. & Pečarić, J. (1989). Inequalities involving multivariate convex functions, *J. Math. Anal. Appl.*, 137, 514-549.

[45] Neuman, E. (1990). Inequalities Involving Multivariate Convex Functions II, *Proc. Amer. Math. Soc.*, 109, 965-974.

[46] Niculescu, C. P. (2000). A Note on the Dual Hermite-Hadamard Inequality, *The Math. Gazette*, (Romania), July, 1-3.

[47] Niculescu, C. P. (2000). Convexity According to the Geometric Mean, *Math. Ineq. & Appl.*, 3(2) , 155-167.

[48] Pearce, C. E. M. Pečarić, J. & Šimić, V. (1998). Stolarsky Means and Hadamard's Inequality, *J. Math. Anal. Appl.,* 220, 99-109.

[49] Pearce, C. E. M. & Rubinov, A. M. (1999). $P-$functions, Quasi-convex Functions and Hadamard-type Inequalities, *J. Math. Anal. Appl.,* 240(1), 92-104.

[50] Pečarić, J. (1992). Remarks on Two Interpolations of Hadamard's Inequalities, *Contributions, Macedonian Acad. of Sci. and Arts, Sect. of Math. and Technical Sciences,* (Scopje), 13, 9-12.

[51] Pečarić, J. & Dragomir, S. S.(1991). A Generalization of Hadamard's Integral Inequality for Isotonic Linear Functionals, *Rudovi Mat.* (Sarajevo), 7 , 103-107.

[52] Pečarić, J., Proschan, F. & Tong, Y. L. (1992) *Convex Functions, Partial Orderings and Statistical Applications,* Academic Press, Inc..

[53] Sándor, J. (1990). A Note on the Jensen-Hadamard Inequality, *Anal. Numer. Theor. Approx.*, 19(1), 29-34.

[54] Sándor, J. (1990). An Application of the Jensen-Hadamard Inequality, *Nieuw-Arch.-Wisk.*, 8(1), 63-66.

[55] Sándor, J. (1991). On the Jensen-Hadamard inequality, *Studia Univ. Babes-Bolyai, Math.,* 36(1), 9-15.

[56] Vasić, P. M., Lacković, I. B. & Maksimović, D. M. (1980). Note on Convex Functions IV: On Hadamard's Inequality for Weighted Arithmetic Means, *Univ. Beograd Publ. Elek. Fak., Ser. Mat. Fiz.,* No. 678-715, 199-205.

[57] Yang, G. S. & Hong, M. C. (1997). A Note on Hadamard's Inequality, *Tamkang J. Math.,* 28(1) , 33-37.

[58] Yang, G. S. & Tseng, K. L. (1999). On Certain Integral Inequalities Related to Hermite-Hadamard Inequalities, *J. Math. Anal. Appl.,* 239, 180-187.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au; ian.gomm@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa.