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SOME RESULTS FOR ISOTONIC FUNCTIONALS VIA AN INEQUALITY DUE TO KITTANEH AND MANASRAH

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Abstract. In this paper we obtain some inequalities for isotonic functionals via a reverse and refinement of Young's inequality due to Kittaneh and Manasrah.

1. Introduction

Let L be a *linear class* of real-valued functions $g : E \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1$, $t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalised if*
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2], [15] and [16]). For other inequalities for isotonic functionals see [1], [4]-[14] and [17]-[20].

We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second ($p_k \geq 0, k \in E$).

As is known to all, the famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0, 1]$, then

$$a^{1-\nu} b^{\nu} \leq (1 - \nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called ν-weighted arithmetic-geometric mean inequality.

Kittaneh and Manasrah [12], [13] provided a refinement and a reverse for Young inequality as follows:

$$r\left(\sqrt{a} - \sqrt{b}\right)^2 \leq (1 - \nu) a + \nu b - a^{1-\nu} b^{\nu} \leq R\left(\sqrt{a} - \sqrt{b}\right)^2 \tag{1.2}$$

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where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$. The case $\nu=\frac{1}{2}$ reduces (1.2) to an identity and is of no interest.

We observe that, if $a,b\in[m,M]\subset(0,\infty)$, then $\left|\sqrt{a}-\sqrt{b}\right|\le\sqrt{M}-\sqrt{m}$ and by (1.2) we obtain the following reverse of Young inequality

$$(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\le R\left(\sqrt{M}-\sqrt{m}\right)^2. \tag{1.3}$$

We can give a simple direct proof for (1.2) as follows.

Recall the following result obtained by Dragomir in 2006 [8] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned}(1.4)\qquad & n\min_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right]\\ &\le\frac{1}{P_n}\sum_{j=1}^{n}p_j\Phi(x_j)-\Phi\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\right)\\ &\le n\max_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right],\end{aligned}$$

where $\Phi:C\to\mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n=\sum_{j=1}^{n}p_j>0$.

For $n=2$, we deduce from (1.4) that

$$\begin{aligned}(1.5)\qquad & 2\min\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]\\ &\le\nu\Phi(x)+(1-\nu)\Phi(y)-\Phi[\nu x+(1-\nu)y]\\ &\le 2\max\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]\end{aligned}$$

for any $x,y\in\mathbb{R}$ and $\nu\in[0,1]$.

If we take $\Phi(x)=\exp(x)$, then we get from (1.5)

$$\begin{aligned}(1.6)\qquad & 2\min\{\nu,1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2}-\exp\left(\frac{x+y}{2}\right)\right]\\ &\le\nu\exp(x)+(1-\nu)\exp(y)-\exp[\nu x+(1-\nu)y]\\ &\le 2\max\{\nu,1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2}-\exp\left(\frac{x+y}{2}\right)\right]\end{aligned}$$

for any $x,y\in\mathbb{R}$ and $\nu\in[0,1]$. Further, denote $\exp(x)=a$, $\exp(y)=b$ with $a,b>0$, then from (1.6) we obtain the inequality (1.2).

In this paper we obtain some inequalities for isotonic functionals via the reverse and refinement of Young's inequality (1.2). Applications for integrals and n-tuples of real numbers are also provided.

2. On Callebaut's Inequality

The functional version of Callebaut inequality states that

$$A^2(fg)\le A\left(f^{2-\nu}g^{\nu}\right)A\left(f^{\gamma}g^{2-\nu}\right)\le A\left(f^2\right)A\left(g^2\right) \tag{2.1}$$

provided that f^2, g^2, $f^{2-\nu}g^{\nu}$, $f^{\nu}g^{2-\nu}$, $fg \in L$ for some $\nu \in [0,2]$. For the discrete and integral of one real variable versions see [3].

We have the following result that provides a refinement and reverse of Callebaut's inequality:

Theorem 1. *Let $A, B : L \to \mathbb{R}$ be two normalised isotonic functionals. If $f, g : E \to \mathbb{R}$ are such that, f^2, g^2, fg, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L$ for some $\nu \in [0,1]$, then*

$$\begin{aligned}(2.2)\qquad & r\left(A\left(f^2\right)B\left(g^2\right) - 2A\left(fg\right)B\left(fg\right) + A\left(g^2\right)B\left(f^2\right)\right) \\ & \leq \left(1-\nu\right)A\left(f^2\right)B\left(g^2\right) + \nu A\left(g^2\right)B\left(f^2\right) \\ & - A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right) \\ & \leq R\left(A\left(f^2\right)B\left(g^2\right) - 2A\left(fg\right)B\left(fg\right) + A\left(g^2\right)B\left(f^2\right)\right),\end{aligned}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

Proof. Let $x, y \in E$ such that $g(x), g(y) \neq 0$. If we use the inequalities (1.2) for

$$a = \frac{f^2(x)}{g^2(x)},\ b = \frac{f^2(y)}{g^2(y)},$$

then we get

$$\begin{aligned}(2.3)\qquad & r\left(\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)}\right)^2 \\ & \leq \left(1-\nu\right)\frac{f^2(x)}{g^2(x)} + \nu\frac{f^2(y)}{g^2(y)} - \left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu} \\ & \leq R\left(\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)}\right)^2\end{aligned}$$

where $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

Therefore

$$\begin{aligned}(2.4)\qquad & r\left(\frac{f^2(x)}{g^2(x)} - 2\frac{f(x)}{g(x)}\frac{f(y)}{g(y)} + \frac{f^2(y)}{g^2(y)}\right) \\ & \leq \left(1-\nu\right)\frac{f^2(x)}{g^2(x)} + \nu\frac{f^2(y)}{g^2(y)} - \left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu} \\ & \leq R\left(\frac{f^2(x)}{g^2(x)} - 2\frac{f(x)}{g(x)}\frac{f(y)}{g(y)} + \frac{f^2(y)}{g^2(y)}\right).\end{aligned}$$

If we multiply (2.4) by $g^2(x)\,g^2(y)$, then we get

$$\begin{aligned}(2.5)\qquad & r\left(f^2(x)\,g^2(y) - 2f(x)\,g(x)\,f(y)\,g(y) + f^2(y)\,g^2(x)\right) \\ & \leq \left(1-\nu\right)f^2(x)\,g^2(y) + \nu g^2(x)\,f^2(y) \\ & - f^{2(1-\nu)}(x)\,g^{2\nu}(x)\,f^{2\nu}(y)\,g^{2(1-\nu)}(y) \\ & \leq R\left(f^2(x)\,g^2(y) - 2f(x)\,g(x)\,f(y)\,g(y) + f^2(y)\,g^2(x)\right),\end{aligned}$$

which holds for any $x, y \in E$.

Fix $y \in E$. Then by (2.5) we have in the order of L that

$$\begin{aligned} r\left(g^2(y) f^2 - 2f(y) g(y) fg + f^2(y) g^2\right) & \\ \leq (1-\nu) g^2(y) f^2 + \nu f^2(y) g^2 - f^{2\nu}(y) g^{2(1-\nu)}(y) f^{2(1-\nu)} g^{2\nu} & \\ \leq R\left(g^2(y) f^2 - 2f(y) g(y) fg + f^2(y) g^2\right). & \end{aligned} \tag{2.6}$$

If we take the functional A in (2.6), then we get

$$\begin{aligned} & r\left(g^2(y) A\left(f^2\right) - 2f(y) g(y) A(fg) + f^2(y) A\left(g^2\right)\right) \\ & \leq (1-\nu) g^2(y) A\left(f^2\right) + \nu f^2(y) A\left(g^2\right) - f^{2\nu}(y) g^{2(1-\nu)}(y) A\left(f^{2(1-\nu)} g^{2\nu}\right) \\ & \leq R\left(g^2(y) A\left(f^2\right) - 2f(y) g(y) A(fg) + f^2(y) A\left(g^2\right)\right), \end{aligned}$$

for any $y \in E$.

If we write this inequality in the order of L, then we have

$$\begin{aligned} & r\left(A\left(f^2\right) g^2 - 2A(fg) fg + A\left(g^2\right) f^2\right) \\ & \leq (1-\nu) A\left(f^2\right) g^2 + \nu A\left(g^2\right) f^2 - A\left(f^{2(1-\nu)} g^{2\nu}\right) f^{2\nu} g^{2(1-\nu)} \\ & \leq Rr\left(A\left(f^2\right) g^2 - 2A(fg) fg + A\left(g^2\right) f^2\right), \end{aligned}$$

and by taking the functional B we deduce the desired result (2.2). □

Corollary 1. *Let $A : L \to \mathbb{R}$ be two normalised isotonic functionals. If $f, g : E \to \mathbb{R}$ are such that f^2, g^2, fg, $f^{2(1-\nu)} g^{2\nu}$, $f^{2\nu} g^{2(1-\nu)} \in L$ for some $\nu \in [0,1]$, then*

$$\begin{aligned} & 2r\left(A\left(f^2\right) A\left(g^2\right) - A^2(fg)\right) \\ & \leq A\left(f^2\right) A\left(g^2\right) - A\left(f^{2(1-\nu)} g^{2\nu}\right) A\left(f^{2\nu} g^{2(1-\nu)}\right) \\ & \leq 2R\left(A\left(f^2\right) A\left(g^2\right) - A^2(fg)\right). \end{aligned} \tag{2.7}$$

The following result also holds:

Theorem 2. *Let $A, B : L \to \mathbb{R}$ be two normalised isotonic functionals. If $f, g : E \to \mathbb{R}$ are such that $f \geq 0$, $g > 0$, f^2, g^2, $f^{2(1-\nu)} g^{2\nu}$, $f^{2\nu} g^{2(1-\nu)} \in L$ for some $\nu \in [0,1]$ and*

$$0 < m \leq \frac{f}{g} \leq M < \infty \tag{2.8}$$

for some constants m, M, then

$$\begin{aligned} & (1-\nu) A\left(f^2\right) B\left(g^2\right) + \nu A\left(g^2\right) B\left(f^2\right) - A\left(f^{2(1-\nu)} g^{2\nu}\right) B\left(f^{2\nu} g^{2(1-\nu)}\right) \\ & \leq R(M-m)^2 A\left(g^2\right) B\left(g^2\right). \end{aligned} \tag{2.9}$$

In particular, we have

$$A\left(f^2\right) A\left(g^2\right) - A\left(f^{2(1-\nu)} g^{2\nu}\right) A\left(f^{2\nu} g^{2(1-\nu)}\right) \leq R(M-m)^2 A^2\left(g^2\right). \tag{2.10}$$

Proof. For any $x, y \in E$ we have

$$m^2 \leq \frac{f^2(x)}{g^2(x)}, \frac{f^2(y)}{g^2(y)} \leq M^2.$$

If we use the inequality (1.3) for

$$a = \frac{f^2(x)}{g^2(x)},\ b = \frac{f^2(y)}{g^2(y)},$$

then we get

$$(1-\nu)\frac{f^2(x)}{g^2(x)}+\nu\frac{f^2(y)}{g^2(y)}-\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}\le R(M-m)^2 \tag{2.11}$$

for any $x,y\in E$.

Now, if we multiply (2.11) by $g^2(x)g^2(y)>0$ then we get

$$\begin{aligned}&(1-\nu)f^2(x)g^2(y)+\nu g^2(x)f^2(y)\\&-f^{2(1-\nu)}(x)g^{2\nu}(x)f^{2\nu}(y)g^{2(1-\nu)}(y)\\&\le R(M-m)^2g^2(x)g^2(y)\end{aligned}\tag{2.12}$$

for any $x,y\in E$.

Fix $y\in E$. Then by (2.12) we have in the order of L that

$$\begin{aligned}&(1-\nu)g^2(y)f^2+\nu f^2(y)g^2-f^{2\nu}(y)g^{2(1-\nu)}(y)f^{2(1-\nu)}g^{2\nu}\\&\le R(M-m)^2g^2(y)g^2.\end{aligned}\tag{2.13}$$

If we take the functional A in (2.13), then we get

$$\begin{aligned}&(1-\nu)g^2(y)A\left(f^2\right)+\nu f^2(y)A\left(g^2\right)-f^{2\nu}(y)g^{2(1-\nu)}(y)A\left(f^{2(1-\nu)}g^{2\nu}\right)\\&\le R(M-m)^2g^2(y)A\left(g^2\right),\end{aligned}\tag{2.14}$$

for any $y\in E$.

This inequality can be written in the order of L as

$$\begin{aligned}&(1-\nu)A\left(f^2\right)g^2+\nu A\left(g^2\right)f^2-A\left(f^{2(1-\nu)}g^{2\nu}\right)f^{2\nu}g^{2(1-\nu)}\\&\le R(M-m)^2A\left(g^2\right)g^2.\end{aligned}\tag{2.15}$$

Now, if we take the functional B in (2.15), then we get the desired result (2.9). □

Corollary 2. *Let $A,B:L\to\mathbb{R}$ be two normalised isotonic functionals. If $f,g:E\to\mathbb{R}$ are such that $f\ge 0$, $g>0$, f^2, g^2, $fg\in L$ and the condition (2.8) is valid, then*

$$\begin{aligned}&\frac{1}{2}\left[A\left(f^2\right)B\left(g^2\right)+A\left(g^2\right)B\left(f^2\right)\right]-A(fg)B(fg)\\&\le\frac{1}{2}(M-m)^2A\left(g^2\right)B\left(g^2\right).\end{aligned}\tag{2.16}$$

In particular, we have

$$A\left(f^2\right)A\left(g^2\right)-A^2(fg)\le\frac{1}{2}(M-m)^2A^2\left(g^2\right).\tag{2.17}$$

3. On Hölder's Inequality

We have:

Theorem 3. *Let $A:L\to\mathbb{R}$ be a normalised isotonic functional and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $f,g:E\to\mathbb{R}$ are such that $f,g\ge 0$ and fg, f^p, $g^q\in L$ then*

$$\begin{aligned}&2s\left(\sqrt{A(f^p)A(g^q)}-A\left(f^{\frac{p}{2}}g^{\frac{q}{2}}\right)\right)[A(f^p)]^{\frac{1}{p}-\frac{1}{2}}[A(g^q)]^{\frac{1}{q}-\frac{1}{2}}\\&\le[A(f^p)]^{1/p}[A(g^q)]^{1/q}-A(fg)\\&\le 2S\left(\sqrt{A(f^p)A(g^q)}-A\left(f^{\frac{p}{2}}g^{\frac{q}{2}}\right)\right)[A(f^p)]^{\frac{1}{p}-\frac{1}{2}}[A(g^q)]^{\frac{1}{q}-\frac{1}{2}},\end{aligned}\tag{3.1}$$

where $s=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ *and* $S=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

Proof. From (1.2) we have

$$s\left(a+b-2\sqrt{ab}\right)\leq\frac{1}{p}a+\frac{1}{q}b-a^{\frac{1}{p}}b^{\frac{1}{q}}\leq R\left(a+b-2\sqrt{ab}\right) \tag{3.2}$$

where $a,b\geq 0$.

If we choose in (3.2) $a=\frac{f^p}{A(f^p)}$, $b=\frac{g^q}{A(g^q)}$, then we get

$$\begin{aligned} &s\left(\frac{f^p}{A(f^p)}+\frac{g^q}{A(g^q)}-2\frac{f^{\frac{p}{2}}g^{\frac{q}{2}}}{\sqrt{A(f^p)A(g^q)}}\right) \\ &\leq\frac{1}{p}\frac{f^p}{A(f^p)}+\frac{1}{q}\frac{g^q}{A(g^q)}-\frac{fg}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \\ &\leq S\left(\frac{f^p}{A(f^p)}+\frac{g^q}{A(g^q)}-2\frac{f^{\frac{p}{2}}g^{\frac{q}{2}}}{\sqrt{A(f^p)A(g^q)}}\right) \end{aligned} \tag{3.3}$$

in the order of L.

If we take the functional A in (3.3), then we get

$$\begin{aligned} 2s\left(1-\frac{A\left(f^{\frac{p}{2}}g^{\frac{q}{2}}\right)}{\sqrt{A(f^p)A(g^q)}}\right)&\leq 1-\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \\ &\leq 2S\left(1-\frac{A\left(f^{\frac{p}{2}}g^{\frac{q}{2}}\right)}{\sqrt{A(f^p)A(g^q)}}\right), \end{aligned}$$

which is equivalent to the desired result (3.1). □

The following result also holds:

Theorem 4. *Let $A:L\to\mathbb{R}$ be a normalised isotonic functional and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $f,g:E\to\mathbb{R}$ are such that fg, f^p, $g^q\in L$ and*

$$0<m_1\leq f\leq M_1<\infty,\ 0<m_2\leq g\leq M_2<\infty, \tag{3.4}$$

for some constants m_1,m_2,M_1 and M_2, then

$$\begin{aligned} 0&\leq[A(f^p)]^{1/p}[A(g^q)]^{1/q}-A(fg) \\ &\leq S\left(\max\left\{\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}},\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right\}-\min\left\{\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}},\left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right\}\right)^2 \\ &\quad\times[A(f^p)]^{1/p}[A(g^q)]^{1/q}, \end{aligned} \tag{3.5}$$

where $S=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

Proof. Observe that, by (3.3) we have

$$m_1^p\leq A(f^p)\leq M_1^p \text{ and } m_2^q\leq A(g^q)\leq M_2^q.$$

Also

$$\left(\frac{m_1}{M_1}\right)^p\leq\frac{f^p}{A(f^p)}\leq\left(\frac{M_1}{m_1}\right)^p$$

and

$$\left(\frac{m_2}{M_2}\right)^q\leq\frac{g^q}{A(g^q)}\leq\left(\frac{M_2}{m_2}\right)^q.$$

Therefore

$$\min\left\{\left(\frac{m_1}{M_1}\right)^p,\left(\frac{m_2}{M_2}\right)^q\right\}\le\frac{f^p}{A(f^p)},\frac{g^q}{A(g^q)}\le\max\left\{\left(\frac{M_1}{m_1}\right)^p,\left(\frac{M_2}{m_2}\right)^q\right\}$$

and by (1.3) we have for $\nu=\frac{1}{q}$, $a=\frac{f^p}{A(f^p)}$, $b=\frac{g^q}{A(g^q)}$, $m=\min\left\{\left(\frac{m_1}{M_1}\right)^p,\left(\frac{m_2}{M_2}\right)^q\right\}$ and $M=\max\left\{\left(\frac{M_1}{m_1}\right)^p,\left(\frac{M_2}{m_2}\right)^q\right\}$ that

$$\begin{aligned}(3.6)\qquad &\frac{1}{p}\frac{f^p}{A(f^p)}+\frac{1}{q}\frac{g^q}{A(g^q)}-\frac{fg}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}\\ &\le S\left(\max\left\{\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}},\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right\}-\min\left\{\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}},\left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right\}\right)^2.\end{aligned}$$

Now, if we take the functional A in the inequality (3.6), then we get

$$\begin{aligned}&1-\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}\\ &\le S\left(\max\left\{\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}},\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right\}-\min\left\{\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}},\left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right\}\right)^2,\end{aligned}$$

which is equivalent to the desired result (3.5). □

Corollary 3. *Let $A:L\to\mathbb{R}$ be a normalised isotonic functional. If $f,g:E\to\mathbb{R}$ are such that fg, f^2, $g^2\in L$ and (3.4) is satisfied, then*

$$\begin{aligned}(3.7)\qquad 0&\le\left[A\left(f^2\right)\right]^{1/2}\left[A\left(g^2\right)\right]^{1/2}-A(fg)\\ &\le\frac{1}{2}\left(\max\left\{\frac{M_1}{m_1},\frac{M_2}{m_2}\right\}-\min\left\{\frac{m_1}{M_1},\frac{m_2}{M_2}\right\}\right)^2\\ &\quad\times\left[A\left(f^2\right)\right]^{1/2}\left[A\left(g^2\right)\right]^{1/2}.\end{aligned}$$

4. Applications for Integrals

Let $(\Omega,\mathcal{A},\mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\ge0$ for μ -a.e.(almost every) $x\in\Omega$ and $p\ge1$ consider the Lebesgue space

$$L_w^p(\Omega,\mu):=\{f:\Omega\to\mathbb{R},\ f\text{ is }\mu\text{-measurable and }\int_\Omega|f(x)|^p\,w(x)\,d\mu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega wd\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$. The same for other integrals involved below. We assume that $\int_\Omega wd\mu=1$.

Assume that f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L_w(\Omega,\mu)$ for some $\nu \in [0,1]$, then by (2.7) we have

$$
\begin{aligned}
(4.1)\qquad & 2r\left(\int_\Omega wf^2 d\mu \int_\Omega wg^2 d\mu - \left(\int_\Omega wfgd\mu\right)^2\right) \\
& \le \int_\Omega wf^2 d\mu \int_\Omega wg^2 d\mu - \int_\Omega wf^{2(1-\nu)}g^{2\nu} d\mu \int_\Omega wf^{2\nu}g^{2(1-\nu)} d\mu \\
& \le 2R\left(\int_\Omega wf^2 d\mu \int_\Omega wg^2 d\mu - \left(\int_\Omega wfgd\mu\right)^2\right).
\end{aligned}
$$

If $f \ge 0$, $g > 0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L_w(\Omega,\mu)$ for some $\nu \in [0,1]$ and

$$
(4.2)\qquad 0 < m \le \frac{f}{g} \le M < \infty
$$

for some constants m, M, then by (2.10)

$$
\begin{aligned}
(4.3)\qquad & \int_\Omega wf^2 d\mu \int_\Omega wg^2 d\mu - \int_\Omega wf^{2(1-\nu)}g^{2\nu} d\mu \int_\Omega wf^{2\nu}g^{2(1-\nu)} d\mu \\
& \le R(M-m)^2\left(\int_\Omega wg^2 d\mu\right)^2,
\end{aligned}
$$

where $R = \max\{1-\nu, \nu\}$.

If $f, g \ge 0$ and f^p, $g^q \in L_w(\Omega,\mu)$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (3.1)

$$
\begin{aligned}
(4.4)\qquad & 2s\left(\sqrt{\int_\Omega wf^p d\mu \int_\Omega wg^q d\mu} - \int_\Omega wf^{\frac{p}{2}}g^{\frac{q}{2}} d\mu\right) \\
& \times \left[\int_\Omega wf^p d\mu\right]^{\frac{1}{p}-\frac{1}{2}} \left[\int_\Omega wg^q d\mu\right]^{\frac{1}{q}-\frac{1}{2}} \\
& \le \left[\int_\Omega wf^p d\mu\right]^{1/p} \left[\int_\Omega wg^q d\mu\right]^{1/q} - \int_\Omega wfgd\mu \\
& \le 2S\left(\sqrt{\int_\Omega wf^p d\mu \int_\Omega wg^q d\mu} - \int_\Omega wf^{\frac{p}{2}}g^{\frac{q}{2}} d\mu\right) \\
& \times \left[\int_\Omega wf^p d\mu\right]^{\frac{1}{p}-\frac{1}{2}} \left[\int_\Omega wg^q d\mu\right]^{\frac{1}{q}-\frac{1}{2}},
\end{aligned}
$$

where $s = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and $S = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Let f, g be μ-measurable functions defined on Ω and

$$
(4.5)\qquad 0 < m_1 \le f \le M_1 < \infty,\ 0 < m_2 \le g \le M_2 < \infty \text{ a.e. on } \Omega
$$

for some constants m_1, m_2, M_1 and M_2, then by (3.5)

$$
\begin{aligned}
(4.6) \qquad 0 &\leq \left[\int_\Omega wf^p d\mu\right]^{1/p} \left[\int_\Omega wg^q d\mu\right]^{1/q} - \int_\Omega wfg d\mu \\
&\leq S\left(\max\left\{\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}, \left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right\} - \min\left\{\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}}, \left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right\}\right)^2 \\
&\times \left[\int_\Omega wf^p d\mu\right]^{1/p} \left[\int_\Omega wg^q d\mu\right]^{1/q}.
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
(4.7) \qquad 0 &\leq \left[\int_\Omega wf^2 d\mu\right]^{1/2} \left[\int_\Omega wg^2 d\mu\right]^{1/2} - \int_\Omega wfg d\mu \\
&\leq S\left(\max\left\{\frac{M_1}{m_1}, \frac{M_2}{m_2}\right\} - \min\left\{\frac{m_1}{M_1}, \frac{m_2}{M_2}\right\}\right)^2 \\
&\times \left[\int_\Omega wf^2 d\mu\right]^{1/2} \left[\int_\Omega wg^2 d\mu\right]^{1/2}.
\end{aligned}
$$

5. Applications for Real Numbers

We consider the n-tuples of positive numbers $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and the probability distribution $p = (p_1, ..., p_n)$, i.e. $p_i \geq 0$ for any $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$.

If we use the inequality (4.1) for the counting discrete measure, then we have

$$
\begin{aligned}
(5.1) \qquad & 2r\left(\sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \left(\sum_{i=1}^n p_i a_i b_i\right)^2\right) \\
&\leq \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \sum_{i=1}^n p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i=1}^n p_i a_i^{2\nu} b_i^{2(1-\nu)} \\
&\leq 2R\left(\sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \left(\sum_{i=1}^n p_i a_i b_i\right)^2\right),
\end{aligned}
$$

where $\nu \in [0, 1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

If there exists some constants m, M such that

$$
0 < m \leq \frac{a_i}{b_i} \leq M < \infty \text{ for any } i \in \{1, ..., n\}
$$

then by (4.3) we have that

$$
\begin{aligned}
(5.2) \qquad & \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \sum_{i=1}^n p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i=1}^n p_i a_i^{2\nu} b_i^{2(1-\nu)} \\
&\leq R(M-m)^2 \left(\sum_{i=1}^n p_i b_i^2\right)^2,
\end{aligned}
$$

where $R = \max\{1-\nu, \nu\}$.

If $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by (4.4) we have

$$2s\left(\sqrt{\sum_{i=1}^{n}p_ia_i^p\sum_{i=1}^{n}p_ib_i^q}-\sum_{i=1}^{n}p_ia_i^{\frac{p}{2}}b_i^{\frac{q}{2}}\right)\left[\sum_{i=1}^{n}p_ia_i^p\right]^{\frac{1}{p}-\frac{1}{2}}\left[\sum_{i=1}^{n}p_ib_i^q\right]^{\frac{1}{q}-\frac{1}{2}} \tag{5.3}$$
$$\leq\left[\sum_{i=1}^{n}p_ia_i^p\right]^{1/p}\left[\sum_{i=1}^{n}p_ib_i^q\right]^{1/q}-\sum_{i=1}^{n}p_ia_ib_i$$
$$\leq 2S\left(\sqrt{\sum_{i=1}^{n}p_ia_i^p\sum_{i=1}^{n}p_ib_i^q}-\sum_{i=1}^{n}p_ia_i^{\frac{p}{2}}b_i^{\frac{q}{2}}\right)\left[\sum_{i=1}^{n}p_ia_i^p\right]^{\frac{1}{p}-\frac{1}{2}}\left[\sum_{i=1}^{n}p_ib_i^q\right]^{\frac{1}{q}-\frac{1}{2}},$$

where $s=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ and $S=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

If for the constants m_1,m_2,M_1 and M_2 we have

$$0<m_1\leq a_i\leq M_1<\infty, 0<m_2\leq b_i\leq M_2<\infty,\ \text{for any } i\in\{1,...,n\}$$

then by (4.6) we have

$$0\leq\left[\sum_{i=1}^{n}p_ia_i^p\right]^{1/p}\left[\sum_{i=1}^{n}p_ib_i^q\right]^{1/q}-\sum_{i=1}^{n}p_ia_ib_i \tag{5.4}$$
$$\leq S\left(\max\left\{\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}},\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right\}-\min\left\{\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}},\left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right\}\right)^2$$
$$\times\left[\sum_{i=1}^{n}p_ia_i^p\right]^{1/p}\left[\sum_{i=1}^{n}p_ib_i^q\right]^{1/q}.$$

In particular, we have

$$0\leq\left[\sum_{i=1}^{n}p_ia_i^2\right]^{1/2}\left[\sum_{i=1}^{n}p_ib_i^2\right]^{1/2}-\sum_{i=1}^{n}p_ia_ib_i \tag{5.5}$$
$$\leq S\left(\max\left\{\frac{M_1}{m_1},\frac{M_2}{m_2}\right\}-\min\left\{\frac{m_1}{M_1},\frac{m_2}{M_2}\right\}\right)^2\left[\sum_{i=1}^{n}p_ia_i^2\right]^{1/2}\left[\sum_{i=1}^{n}p_ib_i^2\right]^{1/2}.$$

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