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# HERMITE-HADAMARD TYPE INEQUALITIES FOR $(n,m,h_1,h_2,\varphi)-$CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

ABDULLAH AKKURT AND HÜSEYIN YILDIRIM

Abstract. In this article we obtain new generalizations for Hermite-Hadamard Inequality by using Riemann-Liouville fractional integral and new type convex functions.

## 1. Introduction

**Definition 1.** ([7]) *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on $I$ if inequality*

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b), \tag{1.1}$$

*holds for all $a, b \in I$ and $t \in [0,1]$.*

The Hermite-Hadamard Inequlity: Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $a, b \in I$ with $a < b$. The following double inequality:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

is known in the literature as Hermite-Hadamard Inequlity for convex functions.

For several recent results concerning inequality (1.2), see ([6], [8]-[11], [13], [14], [16]-[18]) where further references are listed.

**Definition 2.** ([3]) *Let $s \in (0,1]$. A function $f : I \subseteq \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ is said to be $s-$convex in the second sense if*

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b), \tag{1.3}$$

*holds for all $a, b \in I$ and $t \in [0,1]$. This class of $s-$convex functions is usually denoted by $K_s^2$.*

**Definition 3.** ([12]) *Let $(0,1) \subseteq J \subseteq \mathbb{R}$, $I \subseteq \mathbb{R}$ be an interval, and $h : I \to \mathbb{R}_0$ is said to be $h-$convex if the inequality*

$$f(ta + (1-t)b) \leq h(t)\, f(a) + h(1-t)f(b). \tag{1.4}$$

**Definition 4.** ([1],[2],[5],[15]) *Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integral $J_{a^+}^{\alpha} f(x)$ and $J_{b^-}^{\alpha} f(x)$ of order $\alpha \geq 0$ are defined by*

$$J_{a^+}^{\alpha}[f(x)] = \tfrac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt \quad x > a \tag{1.5}$$

*and*

$$J_{b^-}^{\alpha}[f(x)] = \tfrac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt \quad x < b \tag{1.6}$$

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*respectively. Where* $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}du$ *is Gamma function and* $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

We give the following properties:

$$J^\alpha J^\beta [f(t)] = J^{\alpha+\beta} [f(t)], \quad \alpha \geq 0, \beta \geq 0, \tag{1.7}$$

$$J^\alpha J^\beta [f(t)] = J^\beta J^\alpha [f(t)], \quad \alpha \geq 0, \beta \geq 0. \tag{1.8}$$

**Definition 5.** *A function* $f$ *is said to be in the* $L_p(a,b)$ *space if*

$$L_p(a,b) = \left\{ f : \|f\|_{L_p} = \left( \int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty, \ 1 \leq p < \infty, \right\}. \tag{1.9}$$

and for the case $p = \infty$

$$||f||_\infty = ess \sup_{a \leq t \leq b} |f(t)|. \tag{1.10}$$

## 2. Preliminares

In order to established main results, we established the following generalized definition and lemma.

In paper ([4]), $(\alpha, \beta, a, b)-$convex functions are defined as solutions $f$ of the functional inequality

$$f(\alpha(t)x + \beta(t)y) \leq a(t) f(x) + b(t) f(y)$$

where $0 \neq T \subseteq [0,1]$ and $\alpha, \beta, a, b : T \to \mathbb{R}$ are given functions. We introduce a definition of $(n, m, h_1, h_2, \varphi)-$convex functions.

**Definition 6.** *Let* $\varphi : [a,b] \subset \mathbb{R} \to [a,b]$. *A function* $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$, $h_1, h_2 : [0,1] \to \mathbb{R}_0$, $m, n \in (0,1]$. *Then* $f$ *is said to be* $(n, m, h_1, h_2, \varphi)-$*convex if the inequality*

$$f(nt\varphi(a) + m(1-t)\varphi(b)) \leq nh_1(t) f(\varphi(a)) + mh_2(t) f(\varphi(b)). \tag{2.1}$$

*holds for all* $a, b \in I$ *and* $t \in [0,1]$. *If the inequality (2.1) reverses, then* $f$ *is said to be* $(n, m, h_1, h_2, \varphi)-$*concave on* $I$.

Taking $\varphi(x) = x$, $h_1(t) = t^\beta$ and $h_2(t) = 1 - t^\alpha$ in Definition 1, we obtain $(\beta, \alpha, n, m)-$convex functions in ([20]),

$$f(nta + m(1-t)b) \leq nt^\beta f(a) + m(1-t^\alpha) f(b).$$

**Lemma 1.** *Let* $f : I \to \mathbb{R}$ *be a differentiable mapping on* $(a,b)$ *with* $a < b$. *If* $f' \in L_1[\varphi(a), \varphi(b)]$ *for* $\varphi(a), \varphi(b) \in I$, *then the following equality for fractional integrals holds:*

$$\begin{aligned} &\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b) - \varphi(a))^\alpha} \left[ J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] \\ &= \frac{\varphi(b) - \varphi(a)}{2} \int_0^1 \left[ (1-t)^\alpha - t^\alpha \right] f'(t\varphi(a) + (1-t)\varphi(b)) dt. \end{aligned} \tag{2.2}$$

*Proof.* It suffices to note that

$$\begin{aligned} I &= \int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(t\varphi(a) + (1-t)\varphi(b))dt \\ &= \int_0^1 (1-t)^{\alpha} f'(t\varphi(a) + (1-t)\varphi(b))dt + \int_0^1 \left[-t^{\alpha}\right] f'(t\varphi(a) + (1-t)\varphi(b))dt \\ &= I_1 + I_2 \end{aligned}$$

By integration by parts, we get

$$\begin{aligned} I_1 &= \int_0^1 (1-t)^{\alpha} f'(t\varphi(a) + (1-t)\varphi(b))dt \\ &= (1-t)^{\alpha} \frac{f(t\varphi(a) + (1-t)\varphi(b))}{\varphi(a) - \varphi(b)}\Bigg|_0^1 + \frac{\alpha}{\varphi(a) - \varphi(b)} \int_0^1 (1-t)^{\alpha-1} f(t\varphi(a) + (1-t)\varphi(b))dt \\ &= \frac{f(\varphi(b))}{\varphi(b) - \varphi(a)} - \frac{\alpha}{\varphi(b) - \varphi(a)} \int_0^1 (1-t)^{\alpha-1} f(t\varphi(a) + (1-t)\varphi(b))dt \\ &= \frac{f(\varphi(b))}{\varphi(b) - \varphi(a)} - \frac{\alpha}{(\varphi(b) - \varphi(a))^{\alpha+1}} \int_{\varphi(a)}^{\varphi(b)} (u - \varphi(a))^{\alpha-1} f(u)du \\ &= \frac{f(\varphi(b))}{\varphi(b) - \varphi(a)} - \frac{\Gamma(\alpha+1)}{(\varphi(b) - \varphi(a))^{\alpha+1}} J^{\alpha}_{\varphi(b)-} f(\varphi(a)), \end{aligned}$$

and similarly,

$$\begin{aligned} I_2 &= \int_0^1 \left[-t^{\alpha}\right] f'(t\varphi(a) + (1-t)\varphi(b))dt \\ &= \left[-t^{\alpha}\right] \frac{f(t\varphi(a) + (1-t)\varphi(b))}{\varphi(a) - \varphi(b)}\Bigg|_0^1 + \frac{\alpha}{\varphi(a) - \varphi(b)} \int_0^1 t^{\alpha-1} f(t\varphi(a) + (1-t)\varphi(b))dt \\ &= \frac{f(\varphi(a))}{\varphi(b) - \varphi(a)} - \frac{\alpha}{\varphi(b) - \varphi(a)} \int_0^1 t^{\alpha-1} f(t\varphi(a) + (1-t)\varphi(b))dt \\ &= \frac{f(\varphi(a))}{\varphi(b) - \varphi(a)} - \frac{\alpha}{(\varphi(b) - \varphi(a))^{\alpha+1}} \int_{\varphi(a)}^{\varphi(b)} (\varphi(a) - u)^{\alpha-1} f(u)du \\ &= \frac{f(\varphi(a))}{\varphi(b) - \varphi(a)} - \frac{\Gamma(\alpha+1)}{(\varphi(b) - \varphi(a))^{\alpha+1}} J^{\alpha}_{\varphi(a)+} f(\varphi(b)). \end{aligned}$$

Thus can write,

$$I = I_1 + I_2 = \frac{f(\varphi(a)) + f(\varphi(b))}{\varphi(b) - \varphi(a)} - \frac{\Gamma(\alpha+1)}{(\varphi(b) - \varphi(a))^{\alpha+1}} \left[J^{\alpha}_{\varphi(a)+} f(\varphi(b)) + J^{\alpha}_{\varphi(b)-} f(\varphi(a))\right]$$

Multiplying the both sides by $\frac{\varphi(b) - \varphi(a)}{2}$, we obtain lemma which completes the proof. □

## 3. Main Results

**Theorem 1.** *Let $I$ be an interval $a, b \in I$ with $0 \leq a < b$ and $\varphi : I \to \mathbb{R}$ a continuous icreasing function. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^{\circ}$. If $f' \in L_1([\varphi(a), \varphi(b)])$ for $\varphi(a), \varphi(b) \in I$, $n, m \in (0, 1]$ and $h_1, h_2 : [0, 1] \to \mathbb{R}_0$, then*

(3.1)
$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J_{\varphi(a)+}^{\alpha}f(\varphi(b))+J_{\varphi(b)-}^{\alpha}f(\varphi(a))\right]\right|$$
$$\leq\frac{\varphi(b)-\varphi(a)}{2}\frac{2\left(1-2^{-\alpha}\right)}{\alpha+1}\left(n\left|f'\left(\frac{\varphi(a)}{n}\right)\right|\|h_1\|_{\infty}+m\left|f'\left(\frac{\varphi(b)}{m}\right)\right|\|h_2\|_{\infty}\right).$$

*Proof.* From Lemma 1 and $(n,m,h_1,h_2,\varphi)$ −convexity of $|f'|$, we obtain

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J_{\varphi(a)+}^{\alpha}f(\varphi(b))+J_{\varphi(b)-}^{\alpha}f(\varphi(a))\right]\right|$$
$$\leq\frac{\varphi(b)-\varphi(a)}{2}\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(t\varphi(a)+(1-t)\varphi(b))\right|dt$$
$$=\frac{\varphi(b)-\varphi(a)}{2}\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(nt\frac{\varphi(a)}{n}+m(1-t)\frac{\varphi(b)}{m})\right|dt$$
$$\leq\frac{\varphi(b)-\varphi(a)}{2}\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left(n\left|f'\left(\frac{\varphi(a)}{n}\right)\right|\|h_1\|_{\infty}+m\left|f'\left(\frac{\varphi(b)}{m}\right)\right|\|h_2\|_{\infty}\right)dt$$
$$=\frac{\varphi(b)-\varphi(a)}{2}\left\{\int_0^{1/2}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left(n\left|f'\left(\frac{\varphi(a)}{n}\right)\right|\|h_1\|_{\infty}+m\left|f'\left(\frac{\varphi(b)}{m}\right)\right|\|h_2\|_{\infty}\right)dt\right.$$
$$\left.+\int_{1/2}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]\left(n\left|f'\left(\frac{\varphi(a)}{n}\right)\right|\|h_1\|_{\infty}+m\left|f'\left(\frac{\varphi(b)}{m}\right)\right|\|h_2\|_{\infty}\right)dt\right\},$$

where

$$\int_0^{1/2}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt=\int_{1/2}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]dt=\frac{1-2^{-\alpha}}{\alpha+1},$$

which completes the proof. □

**Corollary 1.** *Under conditions of Theorem* 1, *if* $h_1(t)=h(t),\ h_2(t)=h(1-t)$, *then*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J_{\varphi(a)+}^{\alpha}f(\varphi(b))+J_{\varphi(b)-}^{\alpha}f(\varphi(a))\right]\right|$$
$$\leq\frac{\varphi(b)-\varphi(a)}{2}\frac{2-2^{1-\alpha}}{\alpha+1}\|h_1\|_{\infty}\left(n\left|f'\left(\frac{\varphi(a)}{n}\right)\right|+m\left|f'\left(\frac{\varphi(b)}{m}\right)\right|\right).$$

*Furthermore, if* $n=m=1$, *then*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J_{\varphi(a)+}^{\alpha}f(\varphi(b))+J_{\varphi(b)-}^{\alpha}f(\varphi(a))\right]\right|$$
$$\leq\frac{\varphi(b)-\varphi(a)}{2}\frac{2-2^{1-\alpha}}{\alpha+1}\|h_1\|_{\infty}\left(\left|f'(\varphi(a))\right|+\left|f'(\varphi(b))\right|\right).$$

**Corollary 2.** *Under the conditions of Crollary* 1, *if* $h_1(t)=h(t)=t^s,\ n=m=1$, *then*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J_{\varphi(a)+}^{\alpha}f(\varphi(b))+J_{\varphi(b)-}^{\alpha}f(\varphi(a))\right]\right|$$
$$\leq\frac{\varphi(b)-\varphi(a)}{2}\frac{2-2^{1-\alpha}}{\alpha+1}\frac{1}{s+1}\left(\left|f'(\varphi(a))\right|+\left|f'(\varphi(b))\right|\right).$$

*Specially,* $\alpha = s = n = m = 1$, *then*

$$\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int\limits_{\varphi(a)}^{\varphi(b)} f(x)dx \right| \leq \frac{\varphi(b) - \varphi(a)}{4} \left( |f'(\varphi(a))| + |f'(\varphi(b))| \right)$$

**Theorem 2.** *Let* $I$ *be an interval* $a, b \in I$ *with* $0 \leq a < b$ *and* $\varphi : I \to \mathbb{R}$ *a continuous icreasing function. Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* $I^{\circ}$. *If* $f' \in L_1([\varphi(a), \varphi(b)])$ *for* $\varphi(a), \varphi(b) \in I$, $n, m \in (0, 1]$ *and* $h_1, h_2 : [0, 1] \to \mathbb{R}_0$ , *then*

(3.2)
$$\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha + 1)}{2(\varphi(b) - \varphi(a))^{\alpha}} \left[ J^{\alpha}_{\varphi(a)^+} f(\varphi(b)) + J^{\alpha}_{\varphi(b)^-} f(\varphi(a)) \right] \right|$$
$$\leq \frac{\varphi(b) - \varphi(a)}{2} \left\{ \left\{ \frac{2}{\alpha p + 1} \left( 1 - \frac{1}{2^{\alpha p}} \right) \right\}^{1/p} \left[ n \, ||h_1||_q \left| f'(\frac{\varphi(a)}{n}) \right| + m \, ||h_2||_q \left| f'(\frac{\varphi(b)}{m}) \right| \right] \right\}.$$

*Proof.* From Lemma 1, Hölder inequality, and the $(n, m, h_1, h_2, \varphi)$ −convexity of $|f'|^q$, we obtain

$$\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha + 1)}{2(\varphi(b) - \varphi(a))^{\alpha}} \left[ J^{\alpha}_{\varphi(a)^+} f(\varphi(b)) + J^{\alpha}_{\varphi(b)^-} f(\varphi(a)) \right] \right|$$
$$\leq \frac{\varphi(b) - \varphi(a)}{2} \int\limits_0^1 |(1-t)^{\alpha} - t^{\alpha}| \, |f'(t\varphi(a) + (1-t)\varphi(b))| \, dt$$
$$\leq \frac{\varphi(b) - \varphi(a)}{2} \int\limits_0^1 |(1-t)^{\alpha} - t^{\alpha}| \left| f'(nt\frac{\varphi(a)}{n} + m(1-t)\frac{\varphi(b)}{m}) \right| dt$$
$$\leq \frac{\varphi(b) - \varphi(a)}{2} \int\limits_0^1 |(1-t)^{\alpha} - t^{\alpha}| \left[ n \, |h_1(t)| \left| f'(\frac{\varphi(a)}{n}) \right| + m \, |h_2(t)| \left| f'(\frac{\varphi(b)}{m}) \right| \right] dt$$
$$\leq \frac{\varphi(b) - \varphi(a)}{2} \left\{ \left( \int\limits_0^1 |(1-t)^{\alpha} - t^{\alpha}|^p \, dt \right)^{1/p} \left( \int\limits_0^1 n^q \, |h_1(t)|^q \left| f'(\frac{\varphi(a)}{n}) \right|^q dt \right)^{1/q} \right.$$
$$\left. + \left( \int\limits_0^1 |(1-t)^{\alpha} - t^{\alpha}|^p \, dt \right)^{1/p} \left( \int\limits_0^1 m^q \, |h_2(t)|^q \left| f'(\frac{\varphi(b)}{m}) \right|^q dt \right)^{1/q} \right\}$$
$$\leq \frac{\varphi(b) - \varphi(a)}{2} \left\{ \left( \int\limits_0^1 |(1-t)^{\alpha} - t^{\alpha}|^p \, dt \right)^{1/p} \left[ n \, ||h_1||_q \left| f'(\frac{\varphi(a)}{n}) \right| + m \, ||h_2||_q \left| f'(\frac{\varphi(b)}{m}) \right| \right] \right\}$$
$$\leq \frac{\varphi(b) - \varphi(a)}{2} \left\{ \left( \int\limits_0^{1/2} [(1-t)^{\alpha} - t^{\alpha}]^p \, dt + \int\limits_{1/2}^1 [t^{\alpha} - (1-t)^{\alpha}]^p \, dt \right)^{1/p} \right.$$
$$\left. \times \left[ n \, ||h_1||_q \left| f'(\frac{\varphi(a)}{n}) \right| + m \, ||h_2||_q \left| f'(\frac{\varphi(b)}{m}) \right| \right] \right\}$$
$$\leq \frac{\varphi(b) - \varphi(a)}{2} \left\{ \left( \int\limits_0^{1/2} [(1-t)^{\alpha p} - t^{\alpha p}] \, dt + \int\limits_{1/2}^1 [t^{\alpha p} - (1-t)^{\alpha p}] \, dt \right)^{1/p} \right.$$
$$\left. \times \left[ n \, ||h_1||_q \left| f'(\frac{\varphi(a)}{n}) \right| + m \, ||h_2||_q \left| f'(\frac{\varphi(b)}{m}) \right| \right] \right\}$$
$$\leq \frac{\varphi(b) - \varphi(a)}{2} \left\{ \left\{ \frac{2}{\alpha p + 1} \left( 1 - \frac{1}{2^{\alpha p}} \right) \right\}^{1/p} \left[ n \, ||h_1||_q \left| f'(\frac{\varphi(a)}{n}) \right| + m \, ||h_2||_q \left| f'(\frac{\varphi(b)}{m}) \right| \right] \right\}$$

where

$$\int_0^{1/2}\left[(1-t)^{\alpha p}-t^{\alpha p}\right]dt=\int_{1/2}^{1}\left[t^{\alpha p}-(1-t)^{\alpha p}\right]dt=\frac{1}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right),$$

which completes the proof. □

**Corollary 3.** *Under conditions of Theorem* 2, *if* $h_1(t)=h(t),\ h_2(t)=h(1-t)$, *then*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b))+J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]\right|$$
$$\leq\frac{\varphi(b)-\varphi(a)}{2}\,||h_1||_q\left\{\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right\}^{1/p}\left(n\left|f'(\frac{\varphi(a)}{n})\right|+m\left|f'(\frac{\varphi(b)}{m})\right|\right).$$

*Furthermore, if* $n=m=1$, *then*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b))+J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]\right|$$
$$\leq\frac{\varphi(b)-\varphi(a)}{2}\,||h_1||_q\left\{\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right\}^{1/p}\left(|f'(\varphi(a))|+|f'(\varphi(b))|\right).$$

**Corollary 4.** *Under the conditions of Crollary* 3, *if* $h_1(t)=h(t)=t^s,\ n=m=1$, *then*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b))+J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]\right|$$
$$\leq\frac{\varphi(b)-\varphi(a)}{2}\left(\frac{1}{sq+1}\right)^{1/q}\left\{\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right\}^{1/p}\left(|f'(\varphi(a))|+|f'(\varphi(b))|\right);$$

*Specially,* $\alpha=s=n=m=1$, *then*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{1}{(\varphi(b)-\varphi(a))}\int_{\varphi(a)}^{\varphi(b)}f(x)dx\right|$$
$$\leq\frac{(\varphi(b)-\varphi(a))}{2}\left(\frac{1}{q+1}\right)^{1/q}\left\{\frac{2}{p+1}\left(1-\frac{1}{2^{p}}\right)\right\}^{1/p}\left(|f'(\varphi(a))|+|f'(\varphi(b))|\right).$$

**Corollary 5.** *Under the conditions of Crollary* 4, *if* $h_1(t)=h(t)=t,\ n=m=1$ *and* $\varphi(x)=x$, *then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right]\right|$$
$$\leq\frac{b-a}{2}\left(\frac{1}{s+1}\right)^{1/q}\left\{\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right\}^{1/p}\left(|f'(a)|+|f'(b)|\right);$$

*Specially,* $\alpha=s=n=m=1$, *then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)dx\right|$$
$$\leq\frac{(b-a)}{2}\left(\frac{1}{q+1}\right)^{1/q}\left\{\frac{2}{p+1}\left(1-\frac{1}{2^{p}}\right)\right\}^{1/p}\left(|f'(\varphi(a))|+|f'(\varphi(b))|\right).$$

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[Department of Mathematics, Faculty of Science and Arts, University of Kahramanmaraş Sütçü İmam, 46000, Kahramanmaraş, Turkey

*E-mail address*: abdullahmat@gmail.com

[Department of Mathematics, Faculty of Science and Arts, University of Kahramanmaraş Sütçü İmam, 46000, Kahramanmaraş, Turkey

*E-mail address*: hyildir@ksu.edu.tr