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SOME RESULTS FOR ISOTONIC FUNCTIONALS VIA AN INEQUALITY DUE TO LIAO, WU AND ZHAO

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Abstract. In this paper we obtain some inequalities for isotonic functionals via a reverse of Young's inequality due to Liao, Wu and Zhao. Applications for integrals and n-tuples of real numbers are provided as well.

1. Introduction

The famous *Young inequality* for scalars says that, if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called ν-weighted arithmetic-geometric mean inequality.

Kittaneh and Manasrah [13], [14] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \leq R\left(\sqrt{a}-\sqrt{b}\right)^2 \tag{1.2}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. The case $\nu = \frac{1}{2}$ reduces (1.2) to an identity and is of no interest.

We recall that *Specht's ratio* is defined by [21]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function S is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's

$$S\left(\left(\frac{a}{b}\right)^r\right)a^{1-\nu}b^{\nu} \leq (1-\nu)a + \nu b \leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.4}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.4) is due to Tominaga [23] while the first one is due to Furuichi [12].

It is an open question for the author if in the right hand side of (1.4) we can replace $S\left(\frac{a}{b}\right)$ by $S\left(\left(\frac{a}{b}\right)^R\right)$ where $R = \max\{1-\nu, \nu\}$.

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We consider the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \ h > 0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^r\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \leq K^R\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{1.6}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [24] while the second by Liao et al. [15].

In [24] the authors also showed that

$$K^r(h) \geq S(h^r) \text{ for } h > 0 \text{ and } r \in \left[0, \frac{1}{2}\right]$$

implying that the lower bound in (1.6) is better than the lower bound from (1.4).

We can give a simple direct proof for (1.6) as follows.

Recall the following result obtained by the author in 2006 [8] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} & n \min_{j\in\{1,2,\dots,n\}} \{p_j\} \left[\frac{1}{n}\sum_{j=1}^{n} \Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)\right] \\ & \leq \frac{1}{P_n}\sum_{j=1}^{n} p_j \Phi(x_j) - \Phi\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) \\ & \leq n \max_{j\in\{1,2,\dots,n\}} \{p_j\} \left[\frac{1}{n}\sum_{j=1}^{n} \Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)\right], \end{aligned} \tag{1.7}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$.

For $n = 2$, we deduce from (1.7) that

$$\begin{aligned} & 2\min\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \\ & \leq \nu\Phi(x) + (1-\nu)\Phi(y) - \Phi[\nu x + (1-\nu) y] \\ & \leq 2\max\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{1.8}$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0,1]$.

Now, if we write the inequality (1.8) for the convex function $\Phi(x) = -\ln x$, and for the positive numbers a and b we get (1.6).

Let L be a *linear class* of real-valued functions $g : E \to \mathbb{R}$ having the properties:

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1$, $t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f+\beta g)=\alpha A(f)+\beta A(g)$ for all $f,g\in L$ and $\alpha,\beta\in\mathbb{R}$.
(A2) If $f\in L$ and $f\geq 0$, then $A(f)\geq 0$.
The mapping A is said to be *normalised if*
(A3) $A(\mathbf{1})=1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2], [17] and [18]). For other inequalities for isotonic functionals see [1], [4]-[16] and [19]-[22].

We note that common examples of such isotonic linear functionals A are given by

$$A(g)=\int_E g\,d\mu \text{ or } A(g)=\sum_{k\in E}p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second ($p_k\geq 0, k\in E$).

In this paper we obtain some inequalities for isotonic functionals via the reverse of Young's inequality (1.6). Applications for integrals and n-tuples of real numbers are also provided.

2. On Callebaut's Inequality

The functional version of *Callebaut's inequality* states that

$$A^2(fg)\leq A\left(f^{2-\nu}g^{\nu}\right)A\left(f^{\gamma}g^{2-\nu}\right)\leq A\left(f^2\right)A\left(g^2\right) \tag{2.1}$$

provided that f^2, g^2, $f^{2-\nu}g^{\nu}$, $f^{\nu}g^{2-\nu}$, $fg\in L$ for some $\nu\in[0,2]$. For the discrete and integral of one real variable versions see [3].

Let $a,b\in[m,M]\subset(0,\infty)$, then $\frac{m}{M}\leq\frac{a}{b}\leq\frac{M}{m}$ with $\frac{m}{M}<1<\frac{M}{m}$. If $\frac{a}{b}\in\left[\frac{m}{M},1\right)$ then $K\left(\frac{a}{b}\right)\leq K\left(\frac{m}{M}\right)=K\left(\frac{M}{m}\right)$. If $\frac{a}{b}\in\left(1,\frac{M}{m}\right]$ then also $K\left(\frac{a}{b}\right)\leq K\left(\frac{M}{m}\right)$. Therefore for any $a,b\in[m,M]$ we have from (1.6) that

$$(1-\nu)a+\nu b\leq K^R\left(\frac{M}{m}\right)a^{1-\nu}b^{\nu}, \tag{2.2}$$

where $R=\max\{1-\nu,\nu\}$.

We start with the following result:

Theorem 1. *Let $A,B:L\to\mathbb{R}$ be two normalised isotonic functionals. If $f,g:E\to\mathbb{R}$ are such that $f\geq 0$, $g>0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$ and*

$$0<m\leq\frac{f}{g}\leq M<\infty \tag{2.3}$$

for some constants m,M, then

$$\begin{aligned}&A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right)\\&\leq(1-\nu)A\left(f^2\right)B\left(g^2\right)+\nu A\left(g^2\right)B\left(f^2\right)\\&\leq K^R\left(\left(\frac{M}{m}\right)^2\right)A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right),\end{aligned} \tag{2.4}$$

where $R=\max\{1-\nu,\nu\}$.

Proof. For any $x,y\in E$ we have

$$m^2\leq\frac{f^2(x)}{g^2(x)},\frac{f^2(y)}{g^2(y)}\leq M^2.$$

If we use the inequalities (1.1) and (2.2) for

$$a=\frac{f^2(x)}{g^2(x)},\ b=\frac{f^2(y)}{g^2(y)},$$

then we get

$$\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}\leq(1-\nu)\frac{f^2(x)}{g^2(x)}+\nu\frac{f^2(y)}{g^2(y)}\leq K^R\left(\left(\frac{M}{m}\right)^2\right)\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu} \tag{2.5}$$

for any $x,y\in E$.

Now, if we multiply (2.5) by $g^2(x)g^2(y)>0$ then we get

$$\begin{aligned}&f^{2(1-\nu)}(x)\,g^{2\nu}(x)\,f^{2\nu}(y)\,g^{2(1-\nu)}(y)\\&\leq(1-\nu)f^2(x)\,g^2(y)+\nu g^2(x)\,f^2(y)\\&\leq K^R\left(\left(\frac{M}{m}\right)^2\right)f^{2(1-\nu)}(x)\,g^{2\nu}(x)\,f^{2\nu}(y)\,g^{2(1-\nu)}(y)\end{aligned} \tag{2.6}$$

for any $x,y\in E$.

Fix $y\in E$. Then by (2.6) we have in the order of L that

$$\begin{aligned}&f^{2\nu}(y)\,g^{2(1-\nu)}(y)\,f^{2(1-\nu)}g^{2\nu}\\&\leq(1-\nu)\,g^2(y)\,f^2+\nu f^2(y)\,g^2\\&\leq K^R\left(\left(\frac{M}{m}\right)^2\right)f^{2\nu}(y)\,g^{2(1-\nu)}(y)\,f^{2(1-\nu)}g^{2\nu}.\end{aligned} \tag{2.7}$$

If we take the functional A in (2.7) then we get

$$\begin{aligned}&f^{2\nu}(y)\,g^{2(1-\nu)}(y)\,A\left(f^{2(1-\nu)}g^{2\nu}\right)\\&\leq(1-\nu)\,g^2(y)\,A\left(f^2\right)+\nu f^2(y)\,A\left(g^2\right)\\&\leq K^R\left(\left(\frac{M}{m}\right)^2\right)f^{2\nu}(y)\,g^{2(1-\nu)}(y)\,A\left(f^{2(1-\nu)}g^{2\nu}\right),\end{aligned} \tag{2.8}$$

for any $y\in E$.

This inequality can be written in the order of L as

$$A\left(f^{2(1-\nu)}g^{2\nu}\right)f^{2\nu}g^{2(1-\nu)}\leq(1-\nu)\,A\left(f^2\right)g^2+\nu A\left(g^2\right)f^2\leq K^R\left(\left(\frac{M}{m}\right)^2\right)A\left(f^{2(1-\nu)}g^{2\nu}\right)f^{2\nu}g^{2(1-\nu)}. \tag{2.9}$$

Now, if we take the functional B in (2.9), then we get the desired result (2.4). □

The following reverse of two functional Cauchy-Bunyakovsky-Schwarz inequality for isotonic functionals holds:

Corollary 1. *Let $A, B : L \to \mathbb{R}$ be two normalised isotonic functionals. If $f, g : E \to \mathbb{R}$ are such that $f \geq 0$, $g > 0$, f^2, g^2, $fg \in L$ and the condition (2.3) holds true, then*

$$A(fg)B(fg) \leq \frac{1}{2}\left[A\left(f^2\right)B\left(g^2\right) + A\left(g^2\right)B\left(f^2\right)\right] \leq K^{1/2}\left(\left(\frac{M}{m}\right)^2\right)A(fg)B(fg) = \frac{m^2+M^2}{2mM}A(fg)B(fg). \tag{2.10}$$

The following reverse Callebaut type inequality holds:

Corollary 2. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional. If $f, g : E \to \mathbb{R}$ are such that $f \geq 0$, $g > 0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L$ for some $\nu \in [0,1]$ and the condition (2.3) is valid, then*

$$A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right) \leq A\left(f^2\right)A\left(g^2\right) \leq K^R\left(\left(\frac{M}{m}\right)^2\right)A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right), \tag{2.11}$$

where $R = \max\{1-\nu, \nu\}$.

Remark 1. *If we replace ν by $\frac{1}{2}(1-\nu)$ with $\nu \in [0,1]$ in (2.11), then we get*

$$A\left(f^{1+\nu}g^{1-\nu}\right)A\left(f^{1-\nu}g^{1+\nu}\right) \leq A\left(f^2\right)A\left(g^2\right) \leq K^{\frac{1+\nu}{2}}\left(\left(\frac{M}{m}\right)^2\right)A\left(f^{1+\nu}g^{1-\nu}\right)A\left(f^{1-\nu}g^{1+\nu}\right), \tag{2.12}$$

provided that $f \geq 0$, $g > 0$, f^2, g^2, $f^{1+\nu}g^{1-\nu}$, $f^{1-\nu}g^{1+\nu} \in L$ for some $\nu \in [0,1]$ and the condition (2.3) is valid.

Also, if we take $\nu = \frac{1}{2}\gamma$ with $\gamma \in [0,2]$, then we get from (2.11) that

$$A\left(f^{2-\gamma}g^{\gamma}\right)A\left(f^{\gamma}g^{2-\gamma}\right) \leq A\left(f^2\right)A\left(g^2\right) \leq K^T\left(\left(\frac{M}{m}\right)^2\right)A\left(f^{2-\gamma}g^{\gamma}\right)A\left(f^{\gamma}g^{2-\gamma}\right), \tag{2.13}$$

where $T = \max\left\{\frac{1}{2}\gamma, 1-\frac{1}{2}\gamma\right\}$, provided that $f \geq 0$, $g > 0$, f^2, g^2, $f^{2-\gamma}g^{\gamma}$, $f^{\gamma}g^{2-\gamma} \in L$ for some $\gamma \in [0,2]$ and the condition (2.3) is valid.

The inequality (2.13) is a reverse for the second inequality in the functional version of Callebaut inequality (2.1).

3. A Reverse of Hölder's and Related Inequalities

First, observe that if $a, b > 0$ and

$$0 < L^{-1} \leq \frac{a}{b} \leq L < \infty, \tag{3.1}$$

for some $L > 1$, then by (1.6) we have for every $\nu \in [0,1]$ that

$$(1-\nu)\,a + \nu b \le K^{R}(L)\,a^{1-\nu}b^{\nu}, \tag{3.2}$$

where $R = \max\{1-\nu, \nu\}$.

Theorem 2. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f,\ g : E \to \mathbb{R}$ are such that $fg,\ f^p,\ g^q \in L$ and*

$$0 < m_1 \le f \le M_1 < \infty,\ \ 0 < m_2 \le g \le M_2 < \infty, \tag{3.3}$$

for some constants $m_1,\ M_1,\ m_2,\ M_2$, then

$$[A(f^p)]^{1/p}\,[A(g^q)]^{1/q} \le K^{U}\left(\left(\frac{M_1}{m_1}\right)^{p}\left(\frac{M_2}{m_2}\right)^{q}\right)A(fg), \tag{3.4}$$

where $U = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Proof. Observe that, by (3.3) we have

$$m_1^p \le A(f^p) \le M_1^p \text{ and } m_2^q \le A(g^q) \le M_2^q.$$

Also

$$\left(\frac{m_1}{M_1}\right)^{p} \le \frac{f^p}{A(f^p)} \le \left(\frac{M_1}{m_1}\right)^{p}$$

and

$$\left(\frac{m_2}{M_2}\right)^{q} \le \frac{g^q}{A(g^q)} \le \left(\frac{M_2}{m_2}\right)^{q}$$

giving that

$$\left[\left(\frac{M_1}{m_1}\right)^{p}\left(\frac{M_2}{m_2}\right)^{q}\right]^{-1} \le \frac{\frac{f^p}{A(f^p)}}{\frac{g^q}{A(g^q)}} \le \left(\frac{M_1}{m_1}\right)^{p}\left(\frac{M_2}{m_2}\right)^{q}.$$

Using the inequality (3.2) for $\nu = \frac{1}{q}$, $a = \frac{f^p}{A(f^p)}$, $b = \frac{g^q}{A(g^q)}$ and $L = \left(\frac{M_1}{m_1}\right)^{p}\left(\frac{M_2}{m_2}\right)^{q}$, we get

$$\frac{1}{p}\frac{f^p}{A(f^p)} + \frac{1}{q}\frac{g^q}{A(g^q)} \le K^{U}\left(\left(\frac{M_1}{m_1}\right)^{p}\left(\frac{M_2}{m_2}\right)^{q}\right)\frac{fg}{[A(f^p)]^{1/p}\,[A(g^q)]^{1/q}}. \tag{3.5}$$

If we take the functional A in (3.5), then we get

$$1 = \frac{1}{p}\frac{A(f^p)}{A(f^p)} + \frac{1}{q}\frac{A(g^q)}{A(g^q)} \le K^{U}\left(\left(\frac{M_1}{m_1}\right)^{p}\left(\frac{M_2}{m_2}\right)^{q}\right)\frac{A(fg)}{[A(f^p)]^{1/p}\,[A(g^q)]^{1/q}},$$

which is equivalent with the desired result (3.4). □

Further, observe that if $a, b > 0$ and

$$0 < l^{-1} \le \frac{a}{b} \le L < \infty, \tag{3.6}$$

for some $L, l > 0$ with $Ll > 1$,then

$$K\left(\frac{a}{b}\right) \le \max\left\{K\left(l^{-1}\right), K(L)\right\} = \max\{K(l), K(L)\}$$

and by (1.6) we have for every $\nu \in [0,1]$ that

$$(1-\nu)\,a + \nu b \le \max\left\{K^{R}(l), K^{R}(L)\right\}a^{1-\nu}b^{\nu} \tag{3.7}$$

where $R = \max\{1-\nu, \nu\}$.

Theorem 3. *Let A, $B : L \to \mathbb{R}$ be two normalised isotonic functionals and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If f, g, $u, v : E \to \mathbb{R}$ are such that u, $v \geq 0$, u, v, uf, vg, uf^p, $vg^q \in L$ and the conditions (3.3) hold, then*

$$\begin{aligned} A(uf) B(vg) &\leq \frac{1}{p} A(uf^p) B(v) + \frac{1}{q} A(u) B(vg^q) \\ &\leq \max\left\{K^U\left(\frac{M_2^q}{m_1^p}\right), K^U\left(\frac{M_1^p}{m_2^q}\right)\right\} A(uf) B(vg), \end{aligned} \tag{3.8}$$

where $U = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

In particular,

$$\begin{aligned} A(uf) A(vg) &\leq \frac{1}{p} A(uf^p) A(v) + \frac{1}{q} A(u) A(vg^q) \\ &\leq \max\left\{K^U\left(\frac{M_2^q}{m_1^p}\right), K^U\left(\frac{M_1^p}{m_2^q}\right)\right\} A(uf) A(vg). \end{aligned} \tag{3.9}$$

Proof. Observe that, by (3.3) we have

$$\frac{m_1^p}{M_2^q} \leq \frac{f^p(x)}{g^q(y)} \leq \frac{M_1^p}{m_2^q}$$

for any $x, y \in E$.

Now, if we write the inequality (3.7) for $l = \frac{M_2^q}{m_1^p}$, $L = \frac{M_1^p}{m_2^q}$, $a = f^p(x)$, $b = g^q(y)$ and $\nu = \frac{1}{q}$, and use Young's inequality, then we get

$$\begin{aligned} f(x) g(y) &\leq \frac{1}{p} f^p(x) + \frac{1}{q} g^q(y) \\ &\leq \max\left\{K^U\left(\frac{M_2^q}{m_1^p}\right), K^U\left(\frac{M_1^p}{m_2^q}\right)\right\} f(x) g(y) \end{aligned} \tag{3.10}$$

for any $x, y \in E$.

If we multiply (3.10) by $u(x) v(y) \geq 0$ we get

$$\begin{aligned} v(y) g(y) fu &\leq \frac{1}{p} v(y) f^p u + \frac{1}{q} g^q(y) v(y) u \\ &\leq \max\left\{K^U\left(\frac{M_2^q}{m_1^p}\right), K^U\left(\frac{M_1^p}{m_2^q}\right)\right\} v(y) g(y) fu \end{aligned} \tag{3.11}$$

in the order of L, where $y \in E$.

If we take the functional A in (3.11), then we get

$$\begin{aligned} vgA(fu) &\leq \frac{1}{p} A(f^p u) v + \frac{1}{q} A(u) g^q v \\ &\leq \max\left\{K^U\left(\frac{M_2^q}{m_1^p}\right), K^U\left(\frac{M_1^p}{m_2^q}\right)\right\} A(fu) vg \end{aligned} \tag{3.12}$$

in the order of L.

Finally, if we take the functional B in (3.12) then we get the desired result (3.8). □

Corollary 3. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functionals and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let f, $g : E \to \mathbb{R}$ be such that the conditions (3.3) hold.*

(i) If f, g, f^2, g^2, f^{p+1}, $g^{q+1}\in L$, *then*

$$A\left(f^{2}\right)A\left(g^{2}\right)\leq\frac{1}{p}A\left(f^{p+1}\right)A\left(g\right)+\frac{1}{q}A\left(f\right)A\left(g^{q+1}\right)\leq\max\left\{K^{U}\left(\frac{M_{2}^{q}}{m_{1}^{p}}\right),K^{U}\left(\frac{M_{1}^{p}}{m_{2}^{q}}\right)\right\}A\left(f^{2}\right)A\left(g^{2}\right),\tag{3.13}$$

where $U=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

(ii) If f, g, fg, gf^p, $fg^q\in L$, *then*

$$A^{2}\left(fg\right)\leq\frac{1}{p}A\left(gf^{p}\right)A\left(f\right)+\frac{1}{q}A\left(g\right)A\left(fg^{q}\right)\leq\max\left\{K^{U}\left(\frac{M_{2}^{q}}{m_{1}^{p}}\right),K^{U}\left(\frac{M_{1}^{p}}{m_{2}^{q}}\right)\right\}A^{2}\left(fg\right).\tag{3.14}$$

The following result also holds:

Corollary 4. *Let* $A:L\to\mathbb{R}$ *be a normalised isotonic functionals and* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *Let* $\ell,h:E\to\mathbb{R}$, *with* $\ell\geq0$, $h>0$ *be such that the following condition holds*

$$0<m\leq\frac{\ell}{h}\leq M<\infty.\tag{3.15}$$

If h^2, $h\ell$, $h^{2-p}\ell^p$, $h^{2-q}\ell^q\in L$, *then we have*

$$A^{2}\left(h\ell\right)\leq\left[\frac{1}{p}A\left(h^{2-p}\ell^{p}\right)+\frac{1}{q}A\left(h^{2-q}\ell^{q}\right)\right]A\left(h^{2}\right)\leq\max\left\{K^{U}\left(\frac{M^{q}}{m^{p}}\right),K^{U}\left(\frac{M^{p}}{m^{q}}\right)\right\}A^{2}\left(h\ell\right),\tag{3.16}$$

where $U=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

Proof. Follows by Theorem 3 for $f=g=\frac{\ell}{h}$, $M_1=M_2=M$, $m_1=m_2=m$, and $u=v=h^2$. □

4. Some Integral Inequalities

Let $(\Omega,\mathcal{A},\mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq0$ for μ -a.e. (almost every) $x\in\Omega$, consider the Lebesgue space

$$L_{w}\left(\Omega,\mu\right):=\{f:\Omega\to\mathbb{R},\ f\text{ is }\mu\text{-measurable and }\int_{\Omega}\left|f\left(x\right)\right|w\left(x\right)d\mu\left(x\right)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega wd\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$. The same for other integrals involved below. We assume that $\int_\Omega wd\mu=1$.

Let f, g be μ-measurable functions with the property that there exists the constants M, $m>0$ such that

$$0<m\leq\frac{f}{g}\leq M<\infty\ \mu\text{-almost everywhere (a.e.) on }\Omega.$$

If $f^2,\ g^2 \in L_w(\Omega,\mu)$, then by (2.11) we have for any $s \in [0,1]$ that

$$\begin{aligned}(4.1)\qquad &\int_\Omega wf^{2(1-s)}g^{2s}d\mu \int_\Omega wf^{2s}g^{2(1-s)}d\mu \\ &\leq \int_\Omega wf^2d\mu \int_\Omega wg^2d\mu \\ &\leq K^R\left(\left(\frac{M}{m}\right)^2\right)\int_\Omega wf^{2(1-s)}g^{2s}d\mu \int_\Omega wf^{2s}g^{2(1-s)}d\mu,\end{aligned}$$

where $R = \max\{1-s, s\}$.

From (3.16) we also have

$$\begin{aligned}(4.2)\qquad \left(\int_\Omega wfgd\mu\right)^2 &\leq \left[\frac{1}{p}\int_\Omega wg^{2-p}f^pd\mu + \frac{1}{q}\int_\Omega wg^{2-q}f^qd\mu\right]\int_\Omega wg^2d\mu \\ &\leq \max\left\{K^U\left(\frac{M^q}{m^p}\right), K^U\left(\frac{M^p}{m^q}\right)\right\}\left(\int_\Omega wfgd\mu\right)^2,\end{aligned}$$

where $U = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, $\frac{1}{p}+\frac{1}{q} = 1$, $p, q > 1$.

Let f, g be μ-measurable functions with the property that there exist the constants m_1, M_1, m_2, M_2 such that

$$(4.3)\qquad 0 < m_1 \leq f \leq M_1 < \infty,\ 0 < m_2 \leq g \leq M_2 < \infty\ \mu\text{-a.e. on } \Omega.$$

Let $p,\ q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, then by (3.4) we have the following reverse of Hölder's inequality

$$(4.4)\qquad \left(\int_\Omega wf^pd\mu\right)^{1/p}\left(\int_\Omega wg^qd\mu\right)^{1/q} \leq K^U\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\int_\Omega wfgd\mu,$$

where $U = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

From (3.13) and (3.14) we also have

$$\begin{aligned}(4.5)\qquad \int_\Omega wf^2d\mu\int_\Omega wg^2d\mu &\leq \frac{1}{p}\int_\Omega wf^{p+1}d\mu\int_\Omega wgd\mu + \frac{1}{q}\int_\Omega wfd\mu\int_\Omega wg^{q+1}d\mu \\ &\leq \max\left\{K^U\left(\frac{M_2^q}{m_1^p}\right), K^U\left(\frac{M_1^p}{m_2^q}\right)\right\}\int_\Omega wf^2d\mu\int_\Omega wg^2d\mu\end{aligned}$$

and

$$\begin{aligned}(4.6)\qquad \left(\int_\Omega wfgd\mu\right)^2 &\leq \frac{1}{p}\int_\Omega wgf^pd\mu\int_\Omega wfd\mu + \frac{1}{q}\int_\Omega wgd\mu\int_\Omega wfg^qd\mu \\ &\leq \max\left\{K^U\left(\frac{M_2^q}{m_1^p}\right), K^U\left(\frac{M_1^p}{m_2^q}\right)\right\}\left(\int_\Omega wfgd\mu\right)^2,\end{aligned}$$

where $U = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

5. Inequalities for Real Numbers

We consider the n-tuples of positive numbers $a = (a_1,...,a_n)$, $b = (b_1,...,b_n)$ and the probability distribution $p = (p_1,...,p_n)$, i.e. $p_i \geq 0$ for any $i \in \{1,...,n\}$ with $\sum_{i=1}^n p_i = 1$.

If there exist the constants m, $M>0$ such that

$$0<m\leq\frac{a_i}{b_i}\leq M<\infty \text{ for any } i\in\{1,...,n\},$$

then by (4.1) for the counting discrete measure, we have for any $s\in[0,1]$ that

$$\begin{aligned}(5.1)\quad &\sum_{i=1}^{n}p_ia_i^{2(1-s)}b_i^{2s}\sum_{i=1}^{n}p_ia_i^{2s}b_i^{2(1-s)}\\ &\leq\sum_{i=1}^{n}p_ia_i^2\sum_{i=1}^{n}p_ib_i^2\\ &\leq K^R\left(\left(\frac{M}{m}\right)^2\right)\sum_{i=1}^{n}p_ia_i^{2(1-s)}b_i^{2s}\sum_{i=1}^{n}p_ia_i^{2s}b_i^{2(1-s)}\end{aligned}$$

where $R=\max\{1-s,s\}$.

From (4.2) we also have

$$\begin{aligned}(5.2)\quad \left(\sum_{i=1}^{n}p_ia_ib_i\right)^2 &\leq\left[\frac{1}{p}\sum_{i=1}^{n}p_ia_i^pb_i^{2-p}+\frac{1}{q}\sum_{i=1}^{n}p_ia_i^qb_i^{2-q}\right]\sum_{i=1}^{n}p_ib_i^2\\ &\leq\max\left\{K^U\left(\frac{M^q}{m^p}\right),K^U\left(\frac{M^p}{m^q}\right)\right\}\left(\sum_{i=1}^{n}p_ia_ib_i\right)^2,\end{aligned}$$

where $U=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$.

If there exist the constants m_1,M_1,m_2,M_2 such that

$$(5.3)\quad 0<m_1\leq a_i\leq M_1<\infty,\ 0<m_2\leq b_i\leq M_2<\infty \text{ for any } i\in\{1,...,n\}$$

and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by (4.4) we have the following reverse of Hölder's discrete inequality

$$(5.4)\quad \left(\sum_{i=1}^{n}p_ia_i^p\right)^{1/p}\left(\sum_{i=1}^{n}p_ib_i^q\right)^{1/q}\leq K^U\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\sum_{i=1}^{n}p_ia_ib_i.$$

From (4.5) and (4.6) we also have

$$\begin{aligned}(5.5)\quad \sum_{i=1}^{n}p_ia_i^2\sum_{i=1}^{n}p_ib_i^2 &\leq\frac{1}{p}\sum_{i=1}^{n}p_ia_i^{p+1}\sum_{i=1}^{n}p_ib_i+\frac{1}{q}\sum_{i=1}^{n}p_ia_i\sum_{i=1}^{n}p_ib_i^{q+1}\\ &\leq\max\left\{K^U\left(\frac{M_2^q}{m_1^p}\right),K^U\left(\frac{M_1^p}{m_2^q}\right)\right\}\sum_{i=1}^{n}p_ia_i^2\sum_{i=1}^{n}p_ib_i^2\end{aligned}$$

and

$$\begin{aligned}(5.6)\quad \left(\sum_{i=1}^{n}p_ia_ib_i\right)^2 &\leq\frac{1}{p}\sum_{i=1}^{n}p_ia_i\sum_{i=1}^{n}p_ia_i^pb_i+\frac{1}{q}\sum_{i=1}^{n}p_ib_i\sum_{i=1}^{n}p_ia_ib_i^q\\ &\leq\max\left\{K^U\left(\frac{M_2^q}{m_1^p}\right),K^U\left(\frac{M_1^p}{m_2^q}\right)\right\}\left(\sum_{i=1}^{n}p_ia_ib_i\right)^2,\end{aligned}$$

where $U=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$.

References

[1] D. Andrica and C. Badea, Grüss' inequality for positive linear functionals, *Periodica Math. Hung.,* **19** (1998), 155-167.
[2] P. R. Beesack and J. E. Pečarić, On Jessen's inequality for convex functions, *J. Math. Anal. Appl.,* **110** (1985), 536-552.
[3] D. K. Callebaut, Generalization of Cauchy-Schwarz inequality, *J. Math. Anal. Appl.* **12** (1965), 491-494.
[4] S. S. Dragomir, A refinement of Hadamard's inequality for isotonic linear functionals, *Tamkang J. Math* (Taiwan), **24** (1992), 101-106.
[5] S. S. Dragomir, On a reverse of Jessen's inequality for isotonic linear functionals, *J. Ineq. Pure & Appl. Math.*, **2**(3)(2001), Article 36.
[6] S. S. Dragomir, On the Jessen's inequality for isotonic linear functionals, *Nonlinear Analysis Forum,* **7**(2)(2002), 139-151.
[7] S. S. Dragomir, On the Lupaş-Beesack-Pečarić inequality for isotonic linear functionals, *Nonlinear Funct. Anal. & Appl.,* **7**(2)(2002), 285-298.
[8] S. S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), 417-478.
[9] S. S. Dragomir and N. M. Ionescu, On some inequalities for convex-dominated functions, *L'Anal. Num. Théor. L'Approx.,* **19** (1) (1990), 21-27.
[10] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications,* RGMIA Monographs, Victoria University, 2000. http://rgmia.vu.edu.au/monographs.html
[11] S. S. Dragomir, C. E. M. Pearce and J. E. Pečarić, On Jessen's and related inequalities for isotonic sublinear functionals, *Acta. Sci. Math.* (Szeged), **61** (1995), 373-382.
[12] S. Furuichi, Refined Young inequalities with Specht's ratio, *Journal of the Egyptian Mathematical Society* **20**(2012) , 46–49.
[13] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.,* **361** (2010), 262-269
[14] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra.,* **59** (2011), 1031-1037.
[15] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.
[16] A. Lupaş, A generalisation of Hadamard's inequalities for convex functions, *Univ. Beograd. Elek. Fak.,* 577-579 (1976), 115-121.
[17] J .E. Pečarić, On Jessen's inequality for convex functions (III), *J. Math. Anal. Appl.,* **156** (1991), 231-239.
[18] J. E. Pečarić and P. R. Beesack, On Jessen's inequality for convex functions (II), *J. Math. Anal. Appl.,* **156** (1991), 231-239.
[19] J. E. Pečarić and S. S. Dragomir, A generalisation of Hadamard's inequality for isotonic linear functionals, *Radovi Mat.* (Sarjevo), **7** (1991), 103-107.
[20] J. E. Pečarić and I. Raşa, On Jessen's inequality, *Acta. Sci. Math.* (Szeged), **56** (1992), 305-309.
[21] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.*, **74** (1960), pp. 91-98.
[22] G. Toader and S. S. Dragomir, Refinement of Jessen's inequality, *Demonstratio Mathematica,* **28** (1995), 329-334.
[23] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.
[24] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.

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