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SOME EXTENSIONS OF HARDY'S INEQUALITY

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Abstract. In this paper, several extensions of the well-known Hardy's integral inequality using the method given by Pachpatte will be presented starting from a generalization of Holder's inequality given by Aldaz.

1. **Introduction**

We recall the classical integral inequality due to Hardy (see, [4], [5]) which states that for $f(x) \geq 0$ and $p > 1$

$$\int_0^{\infty} \{\frac{1}{x}F(x)\}^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f^p(x)dx,$$

where $F(x) = \int_0^x f(t)dt$.

If we suppose that all the integral exist on the respective domanis of their definition, then a generalization of Hardy's inequality which was given by Pachpatte in [10] is the following:

Theorem 1. *Let $p > 1$, $m > 1$ be constants. Let f be a nonnegative and integrable function on $(0, a)$, $0 < a < \infty$. If $F(x)$ is defined by*

$$F(x) = \int_{\frac{x}{2}}^{x} \frac{1}{t}\{\int_{\frac{t}{2}}^{t} \frac{f(s)}{s}ds\}dt, \quad x \in (0, a) \tag{1}$$

then

$$\int_0^a x^{-m}F^p(x)dx \leq \left(\frac{p}{m-1}\right)^{2p} \int_0^a x^{-m} \left|f(x) - f\left(\frac{x}{4}\right)\right|^p dx. \tag{2}$$

It also necessary to recall the C-P conditions and Theorem 2.3 which are given in [7].

We call the four binary functions f, h, w, and r satisfying the C-P condition on $(0, \infty) \times (0, \infty)$ if

(i) $f(x, y)$ is a nonnegative and integrable function on $(0, \infty) \times (0, \infty)$;

(ii) $h(x, y)$ is a positive continuous function on $(0, \infty) \times (0, \infty)$;

(iii) $w(x, y)$ and $r(x, y)$ are positive and absolutely continuous functions on $(0, \infty) \times (0, \infty)$;

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(iv) for almost all $(x,y) \in (0,\infty)\times(0,\infty)$, there is a positive constant such that

$$1-\frac{1}{|m-1|}\frac{H(x,y)}{h(x,y)}\frac{1}{w(x,y)}\frac{\partial w(x,y)}{\partial x}+\frac{p}{|m-1|}\frac{H(x,y)}{h(x,y)}\frac{1}{r(x,y)}\frac{\partial r(x,y)}{\partial x}\geq\frac{1}{\alpha},$$

where $H(x,y)=\int_0^x\int_0^y h(s,t)dsdt$.

Theorem 2. *Let $p>1$ and $m>1$ be constants, the four binary functions $f,\ h,\ w$ and r satisfy the C-P conditions on $(0,a)\times(0,b)$ and*

$$F^*(x,y)=\frac{1}{r(x,y)}\int_{\frac{x}{2}}^{x}\int_{\frac{y}{2}}^{y}r(s,t)h(s,t)f(s,t)dsdt$$

for $(x,y)\in(0,a)\times(0,b)$. Then,

$$\int_0^a\int_0^b w(x,y)H^{-m}(x,y)v(x,y)F^*(x,y)^p dxdy\leq$$

$$\leq\left(\alpha\frac{p}{m-1}\right)^p\int_0^a\int_0^b w(x,y)H^{p-m}(x,y)h^{1-p}(x,y)M^p(x,y)dxdy,$$

where

$$M(x,y)=\frac{1}{r(x,y)}\int_{\frac{y}{2}}^{y}\left|-\frac{1}{2}r\left(\frac{x}{2},t\right)h\left(\frac{x}{2},t\right)f\left(\frac{x}{2},t\right)+r(x,t)h(x,t)f(x,t)\right|dt.$$

This generalization of Holder's inequality was proved by Aldaz in [1] and will be used below.

Theorem 3. *Let $1<p<\infty$ and let $q=\frac{p}{p-1}$ be its conjugate exponent. If $f\in L^p,\ g\in L^q,\ \|f\|_p,\ \|g\|_q>0$, and $1<p\leq 2$ then*
(3),

$$\|f\|_p\|g\|_q\left(1-\frac{1}{p}\left\|\frac{|f|^{\frac{p}{2}}}{\|f\|_p^{\frac{p}{2}}}-\frac{|g|^{\frac{q}{2}}}{\|g\|_q^{\frac{q}{2}}}\right\|_2^2\right)_+\leq\|fg\|_1\leq\|f\|_p\|g\|_q\left(1-\frac{1}{q}\left\|\frac{|f|^{\frac{p}{2}}}{\|f\|_p^{\frac{p}{2}}}-\frac{|g|^{\frac{q}{2}}}{\|g\|_q^{\frac{q}{2}}}\right\|_2^2\right)$$

while if $2\leq p<\infty$, the terms $\frac{1}{p}$ and $\frac{1}{q}$ exchange their positions in the preceding inequalities.

Theorem 2.3 ([9]) *Let $f,\ g\geq 0,\ p>1,\ r>0$ and $\frac{1}{p}+\frac{1}{q}=1$. Also assume in addition that $k(x,y)$ is a non-negative homogeneous function of degree -2λ with $\lambda>1$ and that $\alpha_p=p(\lambda-1)+1$. If*

$$f_r(x)=\frac{1}{\Gamma(r)}\int_0^x(x-t)^{r-1}f(t)dt,\quad\text{and}\quad g_r(y)=\frac{1}{\Gamma(r)}\int_0^y(y-t)^{r-1}g(t)dt,$$

then

$$\int_0^\infty\int_0^\infty\left(\frac{f_r(x)}{x^{r-1}}\right)^{\frac{\alpha_p}{p}}\left(\frac{g_r(y)}{y^{r-1}}\right)^{\frac{\alpha_q}{q}}k(x,y)dxdy<$$

$$<C(p,q,\lambda)\Theta(p)\Theta(q)\|f\|_{\alpha_p}^{\frac{\alpha_p}{p}}\|g\|_{\alpha_q}^{\frac{\alpha_q}{q}},$$

where

$$C(p,q,\lambda)=\left(\int_0^\infty t^{\lambda-1}\cdot k(1,t)dt\right)^{\frac{1}{p}}\left(\int_0^\infty t^{\lambda-1}\cdot k(t,1)dt\right)^{\frac{1}{q}},$$

and

$$\Theta(s)=\left(\frac{\Gamma(1-\frac{1}{\alpha_s})}{\Gamma(r+1-\frac{1}{\alpha_s})}\right)^{\frac{\alpha_s}{s}}.$$

2. Some variants of Hardy-Pachpatte-Copson's inequalities

We suppose as in [10] that all the integral exist on the respective domanis of their definition. We also assumed that r_1 is $q=\frac{p}{p-1}$ if $1<p\leq 2$ or $r_1=p$ if $p>2$ in all the further theorems, without further mention, as in [1].

Proposition 1. *Let $p>1,\ m>1$ be constants. Let f be a nonnegative and integrable function on $(0,a),\ 0<a<\infty$. If $F(x)$ is defined by*

$$F(x)=\int_{\frac{x}{2}}^{x}\frac{1}{t}\{\int_{\frac{t}{2}}^{t}\frac{f(s)}{s}ds\}dt,\quad x\in(0,a)$$

then

$$\int_0^a x^{-m}F^p(x)dx\leq\left(\frac{p}{m-1}\right)^{2p}\int_0^a x^{-m}\left|f(x)-f\left(\frac{x}{4}\right)\right|^p dx \tag{4}$$

$$\times\left[1-\frac{2}{r_1}\left(1-\frac{\int_0^a x^{-m}F^{\frac{p}{2}}(x)\left(\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right)^{\frac{1}{2}}dx}{\left(\int_0^a x^{-m}F^p(x)dx\right)^{\frac{1}{2}}\left(\int_0^a x^{-m}\left(\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right)^p dx\right)^{\frac{1}{2}}}\right)\right]^p$$

$$\times\left[1-\frac{2}{r_1}\left(1-\frac{\int_0^a x^{-m}\left(\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\right)^{\frac{p}{2}}|f(x)-f\left(\frac{x}{4}\right)|^{\frac{p}{2}}dx}{\left(\int_0^a x^{-m}\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\}^p dx\right)^{\frac{1}{2}}\cdot\left(\int_0^a x^{-m}|f(x)-f\left(\frac{x}{4}\right)|^p dx\right)^{\frac{1}{2}}.}\right)\right]^p.$$

Proof. We use the same method as in [10], but the classical Holder's inequality will be use by an improvement given in [1]. First it is necessary to integrate the left side of inequality (4) by parts, obtaining:

$$\int_0^a x^{-m}F^p(x)dx=-\frac{a^{-m+1}}{m-1}F^p(a)+$$

$$+\frac{p}{m-1}\int_o^a x^{-m+1}F^{p-1}(x)\left[\frac{1}{x}\int_{\frac{x}{2}}^{x}\frac{f(s)}{s}ds-\frac{1}{2}\frac{2}{x}\int_{\frac{x}{4}}^{\frac{x}{2}}\frac{f(s)}{s}ds\right]dx$$

and from here

$$\int_0^a x^{-m}F^p(x)dx\leq\frac{p}{m-1}\int_0^a x^{-m}F^{p-1}(x)\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\}dx.$$

Using now the Aldaz's inequality with indices $\frac{p}{p-1}$, p like below, we observe that

$$\int_0^a x^{-m}F^p(x)dx\leq\frac{p}{m-1}\int_0^a x^{-m}F^{p-1}(x)\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\}dx=$$

$$=\frac{p}{m-1}\int_0^a\left[\{x^{-m}\}^{\frac{p-1}{p}}F^{p-1}(x)\right]\cdot\left[\{x^{-m}\}^{-\frac{p-1}{p}+1}\{\int_{\frac{x}{4}}^{x}\frac{|f(s)|}{s}ds\}\right]dx\leq$$

$$\leq \frac{p}{m-1}\left[\int_0^a x^{-m}F^p(x)dx\right]^{\frac{p-1}{p}}\cdot\left[\int_0^a x^{-m}\{\int_{\frac{x}{4}}^x \frac{|f(s)|}{s}ds\}^p dx\right]^{\frac{1}{p}}\times$$

$$\times\left[1-\frac{2}{r}\left(1-\frac{\int_0^a x^{-m}F^{\frac{p}{2}}(x)\left(\int_{\frac{x}{4}}^x\frac{|f(s)|}{s}ds\right)^{\frac{p}{2}}dx}{\left(\int_0^a x^{-m}F^p(x)dx\right)^{\frac{1}{2}}\cdot\left(\int_0^a x^{-m}\{\int_{\frac{x}{4}}^x\frac{|f(s)|}{s}ds\}^p\right)^{\frac{1}{2}}}\right)\right].$$

Like in [10], dividing in previous inequality by the first integral factor and then taking the pth power of both sides we have,

$$\int_0^a x^{-m}F^p(x)dx \leq \left(\frac{p}{m-1}\right)^p\int_0^a x^{-m}\{\int_{\frac{x}{4}}^x\frac{|f(s)|}{s}ds\}^p dx \tag{5}$$

$$\times\left[1-\frac{2}{r_1}\left(1-\frac{\int_0^a x^{-m}F^{\frac{p}{2}}(x)\left(\int_{\frac{x}{4}}^x\frac{|f(s)|}{s}ds\right)^{\frac{p}{2}}dx}{\left(\int_0^a x^{-m}F^p(x)dx\right)^{\frac{1}{2}}\cdot\left(\int_0^a x^{-m}\{\int_{\frac{x}{4}}^x\frac{|f(s)|}{s}ds\}^p\right)^{\frac{1}{2}}}\right)\right]^p.$$

Integrating by parts the expression

$$\int_0^a x^{-m}\{\int_{\frac{x}{4}}^x\frac{|f(s)|}{s}ds\}^p dx$$

and taking into account that the first term is negative, we will obtain,

$$\int_0^a x^{-m}\{\int_{\frac{x}{4}}^x\frac{|f(s)|}{s}ds\}^p dx \leq \frac{p}{m-1}\int_0^a x^{-m}\{\int_{\frac{x}{4}}^x\frac{|f(s)|}{s}ds\}^{p-1}\cdot\left[|f(x)|-|f\left(\frac{x}{4}\right)|\right]dx.$$

Now using again Aldaz's inequality with indices $\frac{p}{p-1}$ and p we get

$$\int_0^a x^{-m}\{\int_{\frac{x}{4}}^x\frac{|f(s)|}{s}ds\}^p dx \leq \frac{p}{m-1}\left[\int_0^a x^{-m}\{\int_{\frac{x}{4}}^x\frac{|f(s)|}{s}ds\}^p dx\right]^{\frac{p-1}{p}}\cdot$$

$$\cdot\left[\int_0^a x^{-m}|f(x)-f\left(\frac{x}{4}\right)|^p dx\right]^{\frac{1}{p}}\times$$

$$\times\left[1-\frac{2}{r_1}\left(1-\frac{\int_0^a x^{-m}\left(\int_{\frac{x}{4}}^x\frac{|f(s)|}{s}ds\right)^{\frac{p}{2}}|f(x)-f\left(\frac{x}{4}\right)|^{\frac{p}{2}}dx}{\left(\int_0^a x^{-m}\{\int_{\frac{x}{4}}^x\frac{|f(s)|}{s}ds\}^p dx\right)^{\frac{1}{2}}\cdot\left(\int_0^a x^{-m}|f(x)-f\left(\frac{x}{4}\right)|^p dx\right)^{\frac{1}{2}}}\right)\right].$$

Dividing both sides of last inequality by the first integral factor and then taking pth power we have,

$$\int_0^a x^{-m}\{\int_{\frac{x}{4}}^x\frac{|f(s)|}{s}ds\}^p dx \leq \frac{p}{m-1}\cdot\left[\int_0^a x^{-m}|f(x)-f\left(\frac{x}{4}\right)|^p dx\right]\times$$

$$\times\left[1-\frac{2}{r_1}\left(1-\frac{\int_0^a x^{-m}\left(\int_{\frac{x}{4}}^x\frac{|f(s)|}{s}ds\right)^{\frac{p}{2}}|f(x)-f\left(\frac{x}{4}\right)|^{\frac{p}{2}}dx}{\left(\int_0^a x^{-m}\{\int_{\frac{x}{4}}^x\frac{|f(s)|}{s}ds\}^p dx\right)^{\frac{1}{2}}\cdot\left(\int_0^a x^{-m}|f(x)-f\left(\frac{x}{4}\right)|^p dx\right)^{\frac{1}{2}}}\right)\right]^p.$$

Combining now last inequality with (5) we obtain the desired inequality.

∎

A slightly different version of Proposition 1 will be also stated below being an extension of Theorem 3 from[10].

Proposition 2. *Let p, m and f be as defined in Proposition 1. If*

$$F(x) = \int_{\frac{x}{2}}^{x} \frac{1}{t}\{\int_0^t \frac{f(s)}{s} ds\}dt, \quad x \in (0,a)$$

then

$$\int_0^a x^{-m} F^p(x)dx \le \left(\frac{p}{m-1}\right)^{2p} \int_0^a x^{-m}\left|f(x) - f\left(\frac{x}{2}\right)\right|^p dx \tag{6}$$

$$\times \left[1 - \frac{2}{r_1}\left(1 - \frac{\int_0^a x^{-m} F^{\frac{p}{2}}(x)\left(\int_{\frac{x}{2}}^{x} \frac{|f(s)|}{s} ds\right)^{\frac{1}{2}} dx}{\left(\int_0^a x^{-m}F^p(x)dx\right)^{\frac{1}{2}}\left(\int_0^a x^{-m}\left(\int_{\frac{x}{2}}^{x}\frac{|f(s)|}{s}ds\right)^p dx\right)^{\frac{1}{2}}}\right)\right]^p$$

$$\times \left[1 - \frac{2}{r_1}\left(1 - \frac{\int_0^a x^{-m}\left(\int_{\frac{x}{2}}^{x}\frac{|f(s)|}{s}ds\right)^{\frac{p}{2}} |f(x) - f\left(\frac{x}{2}\right)|^{\frac{p}{2}} dx}{\left(\int_0^a x^{-m}\left(\int_{\frac{x}{2}}^{x}\frac{|f(s)|}{s}ds\right)^p dx\right)^{\frac{1}{2}}\left(\int_0^a x^{-m}\left|f(x) - f\left(\frac{x}{2}\right)\right|^p dx\right)^{\frac{1}{2}}}\right)\right]^p.$$

.

Proof. The same reason like in Proposition 1 will be used. ∎

In all the further theorems it is assumed, without further mention, that the integrals exist on the respective domains of their definitions, like in [10], [11].

In **R** we have the following two theorems which are new improvements of the variant of the Hardy's inequality (1) given by Izumi and Izumi in [6], Theorem 2 and [11], Theorems 3 and 4. All these inequalities were obtained using the same reason as in the proof of Proposition 1.

Theorem 4. *Let $p > 1$ and $m > 1$ be constants. Let $f(x)$ be a nonnegatiue and integrable function on $(0,b)$, where $b > 0$ is a constant. Let $h(t)$ be a positive continuous function on $(0,b)$ and let $H(x) = \int_0^x h(t)dt$, for $x \in (0,b)$. Let also $w(x)$ and $r(x)$ be positive and absolutely continuous functions on $(0,b)$. If*

$$1 - \frac{1}{m-1}\frac{H(x)}{h(x)}\frac{w^{'}(x)}{w(x)} + \frac{p}{m-1}\frac{H(x)}{h(x)}\frac{r^{'}(x)}{r(x)} \ge \frac{1}{\gamma}$$

for almost all $x \in (0,b)$ and some positive constant γ and $G(x)$ is defined by

$$G(x) = \frac{1}{r(x)}\int_{\frac{x}{2}}^{x} r(t)h(t)f(t)dt,$$

for $x \in (0,b)$, then

$$\int_0^b w(x)H^{-m}(x)h(x)G^p(x)dx \le \left[\gamma\frac{p}{m-1}\right]^p \int_0^b w(x)H^{p-m}(x)h^{-(p-1)}(x)A^p(x)dx$$

$$\cdot\left[1 - \frac{2}{r_1}(1 - \frac{\int_0^b w(x)H^{\frac{p}{2}-m}(x)h^{1-\frac{p}{2}}(x)G^{\frac{p}{2}}(x)A^{\frac{p}{2}}(x)dx}{(\int_0^b w(x)H^{-m}(x)h(x)G^p(x)dx)^{\frac{1}{2}}(\int_0^b w(x)H^{p-m}(x)h^{-(p-1)}(x)A^p(x)dx)^{\frac{1}{2}}})\right]^p$$

where

$$A(x) = \frac{1}{r(x)}|r(x)h(x)f(x) - \frac{1}{2}r(\frac{x}{2})h(\frac{x}{2})f(\frac{x}{2})|.$$

Proof. We will use the same method like in the proof of Theorem 3 from [11] and Proposition 1.

∎

Theorem 5. *Let p, m, f, h, H, w and r be as in previous theorem and*

$$1-\frac{1}{m-1}\frac{H(x)}{h(x)}\frac{w^{'}(x)}{w(x)}-\frac{p}{m-1}\frac{H(x)}{h(x)}\frac{r^{'}(x)}{r(x)}\geq\frac{1}{\delta}$$

for almost all $x\in(0,b)$ and some positive constant δ and $G(x)$ is defined by

$$G(x)=r(x)\int_{\frac{x}{2}}^{x}\frac{h(t)f(t)}{r(t)}dt,$$

for $x\in(0,b)$, then

$$\int_0^b w(x)H^{-m}(x)h(x)G^p(x)dx\leq\left[\delta\frac{p}{m-1}\right]^p\int_0^b w(x)H^{p-m}(x)h^{-(p-1)}(x)B^p(x)dx$$

$$\cdot\left[1-\frac{2}{r_1}(1-\frac{\int_0^b w(x)H^{\frac{p}{2}-m}(x)h^{1-\frac{p}{2}}(x)G^{\frac{p}{2}}(x)B^{\frac{p}{2}}(x)dx}{(\int_0^b w(x)H^{-m}(x)h(x)G^p(x)dx)^{\frac{1}{2}}(\int_0^b w(x)H^{p-m}(x)h^{-(p-1)}(x)B^p(x)dx)^{\frac{1}{2}}})\right]^p$$

where

$$B(x)=r(x)|\frac{h(x)f(x)}{r(x)}-\frac{1}{2}\frac{h(\frac{x}{2})f(\frac{x}{2})}{r(\frac{x}{2})}|.$$

Proof. We will use the same method like in the proof of Theorem 4 from [11] and Proposition 1.

∎

The next two theorems are generalized forms of the slight variants of the Copson's inequalities given in [3], Theorems 5 and 6 and also the inequalities given by Love in [8], Theorems 6.1 and 6.3 and [11], Theorems 7 and 8.

Theorem 6. *Let $a<b<R$, $p>1$, $q<1$ and $\alpha>0$ be constants and $w(x),r(x)$ be two positive and locally absolutely continuous functions in (a,b). Let also $h(x)$ be a positive continuous function and $H(x)=\int_a^x h(t)dt$ for $x\in(a,b)$ and $f(x)$ be a nonnegative and measurable function on the interval (a,b).*

$$1-\frac{1}{1-q}\frac{H(x)}{h(x)}\frac{w^{'}(x)}{w(x)}\log\left(\frac{H(R)}{H(x)}\right)+\frac{p}{1-q}\frac{H(x)}{h(x)}\frac{r^{'}(x)}{r(x)}\log\left(\frac{H(R)}{H(x)}\right)\geq\frac{1}{\alpha}$$

for almost all $x\in(a,b)$ and $F(x)$ is given by

$$G(x)=\frac{1}{r(x)}\int_a^x r(t)h(t)f(t)dt,$$

for $x\in(a,b)$.

Then

$$\int_a^b w(x)H^{-1}(x)h(x)\left(\log\frac{H(R)}{H(x)}\right)^{-q}F^p(x)dx\leq$$

$$\leq\left[\alpha\frac{p}{1-q}\right]^p\int_a^b w(x)H^{p-1}(x)h(x)\left(\log\frac{H(R)}{H(x)}\right)^{p-q}f^p(x)dx\cdot[1-\frac{2}{r_1}\cdot(1-$$

$$-\frac{\int_a^b w(x)H^{\frac{p}{2}-1}(x)h(x)\left(\log\frac{H(R)}{H(x)}\right)^{\frac{p}{2}-q}F^{\frac{p}{2}}(x)f^{\frac{p}{2}}(x)dx}{[\int_a^b w(x)H^{-1}(x)h(x)\left(\log\frac{H(R)}{H(x)}\right)^{-q}F^p(x)dx\int_a^b w(x)H^{p-1}(x)h(x)\left(\log\frac{H(R)}{H(x)}\right)^{p-q}f^p(x)dx]^{\frac{1}{2}}})]^p.$$

Proof. We will apply the inequality of Aldaz and proceed as in like in the proof of Theorem 7 from [11].

∎

Theorem 7. *Let $a < b < R$, $p > 1, q > 1$, $\beta > 0$ be constants and w, r, h, H, f be as in previous theorem. If*

$$1+\frac{1}{q-1}\frac{H(x)}{h(x)}\frac{w'(x)}{w(x)}\left(\log\frac{H(R)}{H(x)}\right)-\frac{p}{q-1}\frac{H(x)}{h(x)}\frac{r'(x)}{r(x)}\left(\log\frac{H(R)}{H(x)}\right)\geq\frac{1}{\beta},$$

for almost all $x \in (a, b)$ and $F(x)$ is defined by

$$F(x)=\frac{1}{r(x)}\int_x^b h(t)f(t)r(t)dt,$$

for $x \in (a, b)$ then

$$\int_a^b w(x)H^{-1}(x)h(x)\left(\log\frac{H(R)}{H(x)}\right)^{-q}dx\leq$$

$$\leq\left[\beta\frac{p}{q-1}\right]^p\int_a^b w(x)H^{p-1}(x)h(x)\left(\log\frac{H(R)}{H(x)}\right)^{p-q}f^p(x)dx\cdot[1-\frac{2}{r_1}(1-$$

$$-\frac{\int_a^b w(x)H^{\frac{p}{2}-1}(x)h(x)\left(\log\frac{H(R)}{H(x)}\right)^{\frac{p}{2}-q}f^{\frac{p}{2}}(x)dx}{[\int_a^b w(x)H^{-1}(x)h(x)\left(\log\frac{H(R)}{H(x)}\right)^{-q}dx\int_a^b w(x)H^{p-1}(x)h(x)\left(\log\frac{H(R)}{H(x)}\right)^{p-q}f^p(x)dx]^{\frac{1}{2}}})]^p$$

.

Proof. We will apply the inequality of Aldaz and use the same method as in the proof of Theorem 8 from [11].

∎

Next inequality is an extension of an inequality given in [7]. The inequality given in [7] is a generalization of a Copson-Pachpatte (C-P) type inequality, which is a generalization of the Hardy integral inequality on binary functions.

Theorem 8. *Let p >1 and $m > 1$ be constants, the four binary functions f, h, w, and r satisfy the C-P condition on $(0, a) \times (0, b)$, and*

$$F^*(x,y)=\frac{1}{r(x,y)}\int_{\frac{x}{2}}^{x}\int_{\frac{y}{2}}^{y}r(s,t)h(s,t)f(s,t)dsdt$$

for $(x, y) \in (0, a) \times (0, b)$. Then we have

(7) $$\int_0^a\int_0^b w(x,y)H^{-m}(x,y)v(x,y)[F^*(x,y)]^p dxdy\leq$$

$$\leq\left(\alpha\frac{p}{m-1}\right)^p\int_0^a\int_0^b w(x,y)H^{p-m}(x,y)h^{1-p}(x,y)M^p(x,y)dxdy$$

$$\times[1-\frac{2}{r_1}(1-\frac{\int_0^a\int_0^b w(x,y))H^{\frac{p}{2}-m}(x,y)h^{1-\frac{p}{2}}(x,y)M^{\frac{p}{2}}(x,y)(F^*(x,y))^{\frac{p}{2}}dxdy}{(\int_0^a\int_0^b w(x,y)H^{p-m}(x,y)h^{1-p}(x,y)M^p(x,y)dxdy)^{\frac{1}{2}}}\cdot$$

$$\cdot\frac{1}{(\int_0^a\int_0^b w(x,y)H^{-m}(x,y)h(x,y)[F^*(x,y])^p dxdy)^{\frac{1}{2}}})]^p$$

where

$$M(x,y)=\frac{1}{r(x,y)}\int_{\frac{y}{2}}^{y}\left|-\frac{1}{2}r\left(\frac{x}{2},t\right)h\left(\frac{x}{2},t\right)f\left(\frac{x}{2},t\right)+r(x,t)h(x,t)f(x,t)\right|dt.$$

Proof. From the proof of Theorem 2.3, see [7] we find that for $(x,y)\in(0,a)\times(0,b)$ and $m>1$ we have

$$\int_0^a\int_0^b\left(1-\frac{1}{m-1}\frac{H(x,y)}{h(x,y)}\frac{\frac{\partial w(x,y)}{\partial x}}{w(x,y)}+\frac{p}{m-1}\frac{H(x,y)}{h(x,y)}\frac{\frac{\partial r(x,y)}{\partial x}}{r(x,y)}\right)\times$$

$$\times w(x,y)H^{-m}(x,y)v(x,y)[F^*(x,y)]^p dxdy\leq$$

$$\leq\frac{p}{m-1}\int_0^a\int_0^b w(x,y))H^{-m+1}(x,y)[F^*(x,y])^{p-1}M(x,y)dxdy$$

$$=\frac{p}{m-1}\int_0^a\int_0^b w^{\frac{1}{p}}(x,y)H^{\frac{p-m}{p}}(x,y)h^{-\frac{p-1}{p}}(x,y)M(x,y)\times$$

$$\times w^{\frac{p-1}{p}}(x,y)H^{\frac{-m(p-1)}{p}}(x,y)h^{\frac{p-1}{p}}(x,y)[F^*(x,y])^{p-1}dxdy.$$

Now applying the Aldaz's inequality with the indices p and $\frac{p}{p-1}$ on the right side of last inequality we will have,

$$\int_0^a\int_0^b w(x,y)H^{-m}(x,y)v(x,y)[F^*(x,y)]^p dxdy\leq$$

$$\leq\alpha\frac{p}{m-1}\left(\int_0^a\int_0^b w(x,y)H^{p-m}(x,y)h^{1-p}(x,y)M^p(x,y)\right)^{\frac{1}{p}}\cdot$$

$$\cdot\left(\int_0^a\int_0^b w(x,y)H^{-m}(x,y)h(x,y)[F^*(x,y)]^p\right)^{\frac{p-1}{p}}\times$$

$$\times[1-\frac{2}{r_1}(1-\frac{\int_0^a\int_0^b w(x,y))H^{\frac{p}{2}-m}(x,y)h^{1-\frac{p}{2}}(x,y)M^{\frac{p}{2}}(x,y)(F^*(x,y))^{\frac{p}{2}}dxdy}{(\int_0^a\int_0^b w(x,y)H^{p-m}(x,y)h^{1-p}(x,y)M^p(x,y)dxdy)^{\frac{1}{2}}}\cdot$$

$$\cdot\frac{1}{(\int_0^a\int_0^b w(x,y)H^{-m}(x,y)h(x,y)[F^*(x,y])^p dxdy)^{\frac{1}{2}}})].$$

Dividing both sides of last inequality by the second integral factor and then raising both sides to the pth power we obtain the inequality (7).

∎

Using the function f_r, the Riemann-Lioville integral of f of order r, we give an extension of Theorem 2.3 from [9] which proves an extension of Hardy's inequakity in two dimensions.

Theorem 9. *Let $f,\ g \geq 0,\ p > 1,\ r > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. We assume in addition that $k(x,y)$ is a non-negative homogeneous function of degree -2λ with $\lambda > 1$ and that $\alpha_p = p(\lambda - 1) + 1$. If*

$$f_r(x) = \frac{1}{\Gamma(r)} \int_0^x (x-t)^{r-1} f(t)dt, \quad \text{and} \quad g_r(y) = \frac{1}{\Gamma(r)} \int_0^y (y-t)^{r-1} g(t)dt,$$

then

$$\int_0^\infty \int_0^\infty \left(\frac{f_r(x)}{x^{r-1}}\right)^{\frac{\alpha_p}{p}} \left(\frac{g_r(y)}{y^{r-1}}\right)^{\frac{\alpha_q}{q}} k(x,y)dxdy <$$

$$< A^{\frac{1}{p}}(\lambda) B^{\frac{1}{q}}(\lambda)\Theta(p)\Theta(q) \left(\int_0^\infty f^{\alpha_p}(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^{\alpha_q}(y)dy\right)^{\frac{1}{q}} \times$$

$$\times[1 - \frac{2}{r_1}(1 - \frac{\int_0^\infty \int_0^\infty (f_r(x))^{\frac{\alpha_p}{2}} (g_r(y))^{\frac{\alpha_q}{2}} k(x,y) x^{\frac{(\lambda-1)(2-rp)-(r-1)}{2}} y^{\frac{(\lambda-1)(2-rq)-(r-1)}{2}} dxdy}{A^{\frac{1}{2}}(\lambda) B^{\frac{1}{2}}(\lambda) \left(\int_0^\infty \left(\frac{f_r(x)}{x^r}\right)^{\alpha_p} dx\right)^{\frac{1}{2}} \left(\int_0^\infty \left(\frac{g_r(y)}{y^r}\right)^{\alpha_q} dy\right)^{\frac{1}{2}}})],$$

where

$$A(\lambda) = \int_0^\infty t^{\lambda-1} \cdot k(1,t)dt, \quad B(\lambda) = \int_0^\infty t^{\lambda-1} \cdot k(t,1)dt$$

and

$$\Theta(s) = \left(\frac{\Gamma(1 - \frac{1}{\alpha_s})}{\Gamma(r + 1 - \frac{1}{\alpha_s})}\right)^{\frac{\alpha_s}{s}}.$$

Proof. We will use the same metdod as in [9], but we will take into account Aldaz's inequality, see [1], instead of Holder's inequality used in [9].

∎

Remark 1. *(a) We should mention that if we take $k(x,y) = \frac{1}{(x+y)^{2\lambda}}$ in previous theorem and $r = 1$, we also obtain a generalization of Theorem 1, see* [12] *as in* [9] *and therefore we have:*

$$\int_0^\infty \int_0^\infty \frac{F^{\frac{\alpha_p}{p}}(x) G^{\frac{\alpha_q}{q}}(y)}{(x+y)^{2\lambda}} dxdy <$$

$$< \beta(\lambda,\lambda) \left(\frac{\alpha_p}{\alpha_p - 1}\right)^{\frac{\alpha_p}{p}} \left(\frac{\alpha_q}{\alpha_q - 1}\right)^{\frac{\alpha_q}{q}} \left(\int_0^\infty f^{\alpha_p}(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^{\alpha_q}(y)dy\right)^{\frac{1}{q}} \times$$

$$\times \left[1 - \frac{2}{r_1}\left(1 - \frac{\int_0^\infty \int_0^\infty (F(x))^{\frac{\alpha_p}{2}} (G(y))^{\frac{\alpha_q}{2}} \frac{\left(x^{2-p} y^{2-q}\right)^{\frac{\lambda-1}{2}}}{(x+y)^{2\lambda}} dxdy}{\beta(\lambda,\lambda) \left(\int_0^\infty \left(\frac{F(x)}{x}\right)^{\alpha_p} dx \int_0^\infty \left(\frac{G(y)}{y}\right)^{\alpha_q} dy\right)^{\frac{1}{2}}}\right)\right]$$

(b) Assume that $A,\ B > 0$ and then taking $k(x,y) = \frac{1}{(Ax+By)^{2\lambda}}$, in previous theorem one may obtain a new generalization of Theorem 1, see [12]*. Thus we have:*

$$\int_0^\infty \int_0^\infty \frac{\left(\frac{f_r(x)}{x^{r-1}}\right)^{\frac{\alpha_p}{p}} \left(\frac{g_r(y)}{y^{r-1}}\right)^{\frac{\alpha_q}{q}}}{(Ax+By)^{2\lambda}} dxdy <$$

$$< \frac{\beta(\lambda,\lambda)}{(AB)^\lambda} \Theta(p)\Theta(q) \left(\int_0^\infty f^{\alpha_p}(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^{\alpha_q}(y)dy\right)^{\frac{1}{q}} \times$$

$$\times[1-\frac{2}{r_1}(1-\frac{A^\lambda B^\lambda \int_0^\infty \int_0^\infty \frac{(f_r(x))^{\frac{\alpha_p}{2}}(g_r(y))^{\frac{\alpha_q}{2}}}{(Ax+By)^{2\lambda}} x^{\frac{(\lambda-1)(2-rp)-(r-1)}{2}} y^{\frac{(\lambda-1)(2-rq)-(r-1)}{2}} dxdy}{\beta(\lambda,\lambda)\left(\int_0^\infty \left(\frac{f_r(x)}{x^r}\right)^{\alpha_p} dx\right)^{\frac{1}{2}} \left(\int_0^\infty \left(\frac{g_r(y)}{y^r}\right)^{\alpha_q} dy\right)^{\frac{1}{2}}})].$$

(c) As in [9] *we take* $k(x,y) = \frac{1}{x^{2\lambda}+y^{2\lambda}}$ *and obtain,*

$$\int_0^\infty \int_0^\infty \frac{\left(\frac{f_r(x)}{x^{r-1}}\right)^{\frac{\alpha_p}{p}} \left(\frac{g_r(y)}{y^{r-1}}\right)^{\frac{\alpha_q}{q}}}{x^{2\lambda}+y^{2\lambda}} dxdy <$$

$$< \frac{\pi\Theta(p)\Theta(q)}{2\lambda} \left(\int_0^\infty f^{\alpha_p}(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^{\alpha_q}(y)dy\right)^{\frac{1}{q}} \times$$

$$\times \left[1-\frac{2}{r_1}(1-\frac{2\lambda \int_0^\infty \int_0^\infty \frac{(f_r(x))^{\frac{\alpha_p}{2}}(g_r(y))^{\frac{\alpha_q}{2}}}{(x^{2\lambda}+y^{2\lambda})} x^{\frac{(\lambda-1)(2-rp)-(r-1)}{2}} y^{\frac{(\lambda-1)(2-rq)-(r-1)}{2}} dxdy}{\pi\left(\int_0^\infty \left(\frac{f_r(x)}{x^r}\right)^{\alpha_p} dx\right)^{\frac{1}{2}} \left(\int_0^\infty \left(\frac{g_r(y)}{y^r}\right)^{\alpha_q} dy\right)^{\frac{1}{2}}})\right].$$

As an extension of Theorem 2.10, see [9] we find the following:

Theorem 10. *Let* $f,\ g \geq 0,\ p > 1,\ r > 0$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *We assume that* $k(x,y)$ *is a non-negative homogeneous function of degree* -2λ *with* $\lambda \geq 2$ *and that* $\alpha_p = p(\lambda-1)$. *If*

$$f_r(x) = \frac{1}{\Gamma(r)}\int_0^x (x-t)^{r-1} f(t)dt, \quad \text{and} \quad g_r(y) = \frac{1}{\Gamma(r)}\int_0^y (y-t)^{r-1} g(t)dt,$$

then

$$\int_0^\infty \int_0^\infty x^{\frac{1}{p}} y^{\frac{1}{q}} \left(\frac{f_r(x)}{x^{r-1}}\right)^{\frac{\alpha_p}{p}} \left(\frac{g_r(y)}{y^{r-1}}\right)^{\frac{\alpha_q}{q}} k(x,y)dxdy <$$

$$< A_1^{\frac{1}{p}}(p) B_1^{\frac{1}{q}}(q)\Theta(p)\Theta(q) \left(\int_0^\infty f^{\alpha_p}(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^{\alpha_q}(y)dy\right)^{\frac{1}{q}} \times$$

$$\times \left[1-\frac{2}{r_1}(1-\frac{\int_0^\infty \int_0^\infty (f_r(x))^{\frac{\alpha_p}{2}} (g_r(y))^{\frac{\alpha_q}{2}} k^{\frac{p+q}{4}}(x,y) x^{\frac{1}{2}(\frac{q}{p}-\alpha_p(r-1))} y^{\frac{1}{2}(\frac{p}{q}-\alpha_q(r-1))} dxdy}{A_1^{\frac{1}{2}}(p) B_1^{\frac{1}{2}}(q) \left(\int_0^\infty \left(\frac{f_r(x)}{x^r}\right)^{\alpha_p} dx\right)^{\frac{1}{2}} \left(\int_0^\infty \left(\frac{g_r(y)}{y^r}\right)^{\alpha_q} dy\right)^{\frac{1}{2}}})\right],$$

where

$$A_1(p) = \int_0^\infty t^{p-1} \cdot k^{\frac{p}{2}}(1,t)dt, \quad B_1(q) = \int_0^\infty t^{q-1} \cdot k^{\frac{q}{2}}(t,1)dt$$

and

$$\Theta(s) = \left(\frac{\Gamma(1-\frac{1}{\alpha_s})}{\Gamma(r+1-\frac{1}{\alpha_s})}\right)^{\frac{\alpha_s}{s}}.$$

Proof. By the same metdod as in [9], but we will take into account Aldaz's inequality, see [1], instead of Holder's inequality we will prove the theorem.

∎

Remark 2. *We can take now, as in* [9], $r = 1$ *and* $k(x,y) = \frac{1}{(x+y)^4}$ *in previous theorem and we will find an improvement of Theorem 2 from* [13].

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