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CLASSIFICATION, NORM INEQUALITIES AND APPLICATIONS FOR BUSCHMAN–ERDELYI TRANSMUTATIONS

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This survey paper contains brief historical information, main known facts and original author's results on the theory of transmutations and some applications. Operators of Buschman–Erdelyi type were first studied by E.T. Copson, R.G. Buschman and A. Erdelyi as integral operators. In 1990's the author was first to prove transmutational nature of these operators and published papers with detailed study of their properties. This class include as special cases such famous objects as Sonine–Poisson–Delsarte transmutations and fractional Riemann–Lioville integrals. In this paper Buschman–Erdelyi transmutations are fully classified as operators of the first kind with special case of zero order smoothness operators, second kind and third kind with special case of unitary Sonine–Katrakhov and Poisson–Katrakhov transmutations. We study such properties as transmutational conditions, factorizations, norm estimates, connections with classical integral transforms. Applications are considered to singular partial differential equations, embedding theorems with sharp constants in Kipriyanov spaces, Euler–Poisson–Darboux equation including Copson lemma, generalized translations, Dunkl operators, Radon transform, generalized harmonics theory, Hardy operators, V. Katrakhov's results on pseudodifferential operators and problems of new kind for equations with solutions of arbitrary growth at singularity.

KEY WORDS: Transmutation operators, Buschman–Erdelyi transmutations, Sonine–Poisson–Delsarte transmutations, Sonine–Katrakhov and Poisson–Katrakhov transmutations, norm estimates, Hardy operator, Kipriyanov space

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1 Introduction: an idea of transmutations, historical information and applications.

1.1 Transmutation operators.

Transmutation theory is an essential generalization of matrix similarity theory. Let start with the main definition.

Definition 1. For a given pair of operators (A, B) an operator T is called transmutation (or intertwining) operator if on elements of some functional spaces the next property is valid

$$TA = BT. \tag{1}$$

It is obvious that the notion of transmutation is direct and far going generalization of similarity notion from linear algebra. But transmutations do not reduce to similar operators because intertwining operators often are not bounded in classical spaces and the inverse operator may be not exist or bounded in the same space. As a consequence spectra of intertwining operators are not the same as a rule. Moreover transmutations may be unbounded. It is the case for Darboux transformations which are defined for a pair of differential operators and are differential operators themselves, in this case all three operators are unbounded in classical spaces. But the theory of Darboux transformations is included in transmutation theory too. Also a pair of intertwining operators may not be differential ones. In transmutation theory there are problems for next varied types of operators: integral, integro–differential, difference–differential (e.g. the Dunkl operator), differential or integro–differential of infinite order (e.g. in connection with Schur's lemma), general linear operators in functional spaces, pseudodifferential and abstract differential operators.

All classical integral transforms due to the definition 1 are also special cases of transmutations, they include Fourier, Petzval (Laplace), Mellin, Hankel, Weierstrass, Kontorovich–Lebedev, Meyer, Stankovic, finite Grinberg and other transforms.

In quantum physics in study of Shrödinger equation and inverse scattering theory underlying transmutations are called wave operators.

Commuting operators are also a special class of transmutations. The most important class consists of operators commuting with derivatives. In this case transmutations as commutants are usually in the form of formal series, pseudodifferential or infinite order differential operators. Finding of commutants is directly connected with finding all transmutations in the given functional space. For these problems works a theory of operator convolutions, including Berg–Dimovski convolutions. Also more and more applications are developed connected with transmutation theory for commuting differential operators, such problems are based on classical results of J.L. Burchnall, T.W. Chaundy. Transmutations are also connected with factorization problems for integral and differential operators. Special class of transmutations are the so called Dirichlet–to–Neumann and Neumann–to–Dirichlet operators which link together solutions of the same equation but with different kinds of boundary conditions.

And how transmutations usually works? Suppose we study properties for a rather complicated operator A. But suppose also that we know corresponding properties for a model more simple operator B and transmutation (1) readily exists. Then we usually may copy results for the model operator B to a more complicated operator A. This is shortly the main idea of transmutations.

Let us for example consider an equation $Au = f$, then applying to it a transmutation with property (1) we consider a new equation $Bv = g$, with $v = Tu, g = Tf$. So if we can solve the simpler equation $Bv = g$ then the initial one is also solved and has solution $u = T^{-1}v$. Of course it is supposed that the inverse operator exist and its explicit form is known. This is a simple application of transmutation technique for proving formulas for solutions of ordinary and partial differential equations.

The next monographs [1]-[6] are completely devoted to the transmutation theory and its applications. Moreover essential parts of monographs [9]-[21] include material on transmutations, the complete list of books which consider some transmutational problems is now near of 100 items.

We specially distinguish the book [4]. In it the most difficult problems of transmutation theory were solved. Among them an existence of transmutations was proved for high order differential equations with variable coefficients including correcting errors of previous papers of Delsarte and Lions, the complete theory of Bianchi equation, extension of V. Marchenko theory of operator–analytic functions, results on operators commuting in spaces of analytic functions.

We use the term "transmutation" due to [3]: "Such operators are often called transformation operators by the Russian school (Levitan, Naimark, Marchenko et. al.), but transformation seems too broad a term, and, since some of the machinery seems "magical" at times, we have followed Lions and Delsarte in using the word "transmutation".

Now the transmutation theory is a completely formed part of mathematical world in which methods and ideas from different areas are used: differential and integral equations, functional analysis, function theory, complex analysis, special functions, fractional integrodifferentiation.

Transmutation theory is deeply connected with many applications in different fields of mathematics. Transmutations are applied in inverse problems via the generalized Fourier transform, spectral function and famous Levitan equation; in scattering theory the Marchenko equation is formulated in terms of transmutations; in spectral theory transmutations help to prove trace formulas and asymptotics for spectral function; estimates for transmutational kernels control stability in inverse and scattering problems; for nonlinear equations via Lax method transmutations for Sturm–Lioville problems lead to proving existence and explicit formulas for soliton solutions. Special kinds of transmutations are generalized analytic functions, generalized translations and convolutions, Darboux transformations. In the theory of partial differential equations transmutations works for proving explicit correspondence formulas among solutions of perturbed and non–perturbed equations, for singular and degenerate equations, pseudodifferential operators, problems with essential singularities at inner or corner points, estimates of solution decay for elliptic and ultraelliptic equations. In function theory transmutations are applied to embedding theorems and generalizations of Hardy operators, Paley–Wiener theory, generalizations of harmonic analysis based on generalized translations. Methods of transmutations are used in many applied problems: investigation of Jost solutions in scattering theory, inverse problems, Dirac and other matrix systems of differential equations, integral equations with special function kernels, probability theory and random processes, stochastic random equations, linear stochastic estimation, inverse problems of geophysics and transsound gas dynamics. Also a number of applications of transmutations to nonlinear equations is permanently increased.

In fact the modern transmutation theory originated from two basic examples [7]. The first is transmutations T for Sturm–Lioville problems with some potential $q(x)$ and natural boundary conditions

$$T(D^2 y(x) + q(x)y(x)) = D^2 (Ty(x)), D^2 y(x) = y''(x), \tag{2}$$

The second example is a problem of studying transmutations intertwining the Bessel operator B_ν and the second derivative:

$$T\left(B_\nu\right)f=\left(D^2\right)Tf,\quad B_\nu=D^2+\frac{2\nu+1}{x}D,\quad D^2=\frac{d^2}{dx^2},\quad \nu\in\mathbb{C}. \tag{3}$$

This class of transmutations includes Sonine–Poisson–Delsarte, Buschman–Erdelyi operators and generalizations. Such transmutations found many applications for a special class of partial differential equations with singular coefficients. A typical equation of this class is the B–elliptic equation with the Bessel operator in some variables of the form

$$\sum_{k=1}^{n} B_{\nu,x_k}u(x_1,\dots,x_n)=f. \tag{4}$$

Analogously B–hyperbolic and B–parabolic equations are considered, this terminology was proposed by I. Kipriyanov. This class of equations was first studied by Euler, Poisson, Darboux and continued in Weinstein's theory of generalized axially symmetric potential (GASPT). These problems were further investigated by Zhitomirslii, Kudryavtsev, Lizorkin, Matiychuk, Mikhailov, Olevskii, Smirnov, Tersenov, He Kan Cher, Yanushauskas, Egorov and others.

In the most detailed and complete way equations with Bessel operators were studied by the Voronezh mathematician I.A. Kipriyanov and his disciples Ivanov, Ryzhkov, Katrakhov, Arhipov, Baidakov, Bogachov, Brodskii, Vinogradova, Zaitsev, Zasorin, Kagan, Katrakhova, Kipriyanova, Kononenko, Kluchantsev, Kulikov, Larin, Leizin, Lyakhov, Muravnik, Polovinkin, Sazonov, Sitnik, Shatskii, Yaroslavtseva. The essence of Kipriyanov's school results was published in [15]. For classes of equations with Bessel operators I. Kipriyanov introduced special functional spaces which were named after him [51]. In this field interesting results were investigated by Katrakhov and his disciples, now these problems are considered by Gadjiev, Guliev, Glushak, Lyakhov with their coauthors and students. Abstract equations of the form (4) originated from the monograph [9] were considered by Egorov, Repnikov, Kononenko, Glushak, Shmulevich and others. And transmutations are one of basic tools for equations with Bessel operators, they are applied to construction of solutions, fundamental solutions, study of singularities, new boundary–value and other problems.

Some words about the structure of this publication. This is a survey article on transmutations of special classes. But the main result on norm estimates and unitarity of Buschman–Erdelyi transmutations is completely proved (theorem 5) as many other facts are consequences of this theorem. In the first section historical and priority information is provided. An author's classification of different classes of Buschman–Erdelyi transmutations is introduced. Based on this classification Buschman–Erdelyi transmutations of the first kind and zero order operators are studied in the second section. Buschman–Erdelyi transmutations of the second kind are considered in the third section. In the fourth section Buschman–Erdelyi transmutations of the third kind and also Sonine–Katrakhov and Poisson–Katrakhov unitary transmutations are considered. In the final fifth section different applications of Buschman–Erdelyi transmutations are listed, mostly inevitably briefly. They include embedding theorems for Kipriyanov spaces, solution representations for partial differential equations with Bessel operators, Euler–Poisson–Darboux equations and Copson's lemma for them, generalized translations, Dunkl operators, Radon transform, generalized spherical harmonics and B–harmonic polynomials, unitarity for some generalizations of Hardy operators. In the final part of this section some results of V. Katrakhov is mentioned on a new class of pseudodifferential operators and remarkable problems introduced by him with K—trace for solutions with infinite order singularities.

Also we must note that the term "operator" is used in this paper for brevity in the broad and sometimes not exact meaning, so appropriate domains and function classes are not always specified. It is easy to complete and make strict for every special result.

1.2 Buschman–Erdelyi transmutations.

The term "Buschman–Erdelyi transmutations" was introduced by the author and is now accepted. Integral equations with these operators were studied in mid–1950th. The author was first to prove the transmutational nature of these operators. The classical Sonine and Poisson operators are special cases of Buschman–Erdelyi transmutations and Sonine–Dimovski and Poisson–Dimovski transmutations are their generalizations for hyper–Bessel equations and functions.

Buschman–Erdelyi transmutations have many modifications. The author introduced convenient classification of them. Due to this classification we introduce Buschman–Erdelyi transmutations of the first kind, their kernels are expressed in terms of Legendre functions of the first kind. In the limiting case we define Buschman–Erdelyi transmutations of zero order smoothness being important in applications. Kernels of Buschman–Erdelyi transmutations of the second kind are expressed in terms of Legendre functions of the second kind. Some combination of operators of the first kind and the second kind leads to operators of the third kind. For the special choice of parameters they are unitary operators in the standard Lebesgue space. The author proposed terms "Sonine–Katrakhov" and "Poisson–Katrakhov" transmutations in honor of V. Katrakhov who introduced and studied these operators.

The study of integral equations and invertibility for Buschman–Erdelyi operators was started in 1960-th by P.Buschman and A.Erdelyi [22]–[25]. These operators also were investigated by Higgins, Ta Li, Love, Habibullah, K.N. Srivastava, Ding Hoang An, Smirnov, Virchenko, Fedotova, Kilbas, Skoromnik and others. During this period for this class of operators were considered only problems of solving integral equations, factorization and invertibility, cf. [26].

The most detailed study of Buschman–Erdelyi transmutations was taken by the author in 1980–1990th [29]–[32] and continued in [27]–[44] and some other papers. Interesting results were proved by N.Virchenko and A.Kilbas and their disciples [45]–[46], [47].

2 Buschman–Erdelyi transmutations of the first kind.

2.1 Sonine–Poisson–Delsarte transmutations.

Let us first consider the most well–known transmutations for the Bessel operator and the second derivative:

$$T(B_\nu)f = \left(D^2\right)Tf, B_\nu = D^2 + \frac{2\nu+1}{x}D, D^2 = \frac{d^2}{dx^2}, \nu \in \mathbb{C}. \tag{4}$$

Definition 2. The Poisson transmutation is defined by

$$P_\nu f = \frac{1}{\Gamma(\nu+1)2^\nu x^{2\nu}} \int_0^x \left(x^2-t^2\right)^{\nu-\frac{1}{2}} f(t)\,dt, \mathrm{Re}\,\nu > -\frac{1}{2}. \tag{5}$$

The Sonine transmutation is defined by

$$S_\nu f = \frac{2^{\nu+\frac{1}{2}}}{\Gamma(\frac{1}{2}-\nu)} \frac{d}{dx} \int_0^x \left(x^2-t^2\right)^{-\nu-\frac{1}{2}} t^{2\nu+1} f(t)\,dt, \mathrm{Re}\,\nu < \frac{1}{2}. \tag{6}$$

Operators (5)–(6) intertwine by formulas

$$S_\nu B_\nu = D^2 S_\nu,\ P_\nu D^2 = B_\nu P_\nu. \tag{7}$$

The definition may be extended to $\nu \in \mathbb{C}$. We will use more historically exact term Sonine–Poisson–Delsarte transmutations [8].

An important generalization for Sonine–Poisson–Delsarte are transmutations for hyper–Bessel functions. Such functions were first considered by Kummer and Delerue. The detailed study was done by Dimovski and his coauthors [10]. These transmutations are called Sonine–Dimovski and Poisson–Dimovski by Kiryakova [11]. In hyper–Bessel functions theory the leading role is for Obreshkoff integral transform [11]. It is a transform with Mayer's G–function kernel which generalize Laplace, Mellin, sine and cosine Fourier, Hankel, Mayer and other classical transforms. Different results on hyper–Bessel functions, connected equations and transformed were many times reopened. The same is true for the Obreshkoff integral transform. It my opinion the Obreshkoff transform together with Fourier, Mellin, Laplace, Stankovic transforms are basic elements from which many other transforms are constructed with corresponding applications.

2.2 Definition and main properties of Buschman–Erdelyi transmutations of the first kind.

Let define and study main properties of Buschman–Erdelyi transmutations of the first kind. This class of transmutations for some choice of parameters generalizes Sonine–Poisson–Delsart transmutations, Riemann–Liouville and Erdelyi–Kober fractional integrals, Mehler–Fock transform.

Definition 3. Define Buschman–Erdelyi operators of the first kind by

$$B_{0+}^{\nu,\mu}f=\int\limits_0^x \left(x^2-t^2\right)^{-\frac{\mu}{2}} P_\nu^\mu\left(\frac{x}{t}\right)f(t)dt, \tag{8}$$

$$E_{0+}^{\nu,\mu}f=\int\limits_0^x \left(x^2-t^2\right)^{-\frac{\mu}{2}} \mathbb{P}_\nu^\mu\left(\frac{t}{x}\right)f(t)dt, \tag{9}$$

$$B_{-}^{\nu,\mu}f=\int\limits_x^\infty \left(t^2-x^2\right)^{-\frac{\mu}{2}} P_\nu^\mu\left(\frac{t}{x}\right)f(t)dt, \tag{10}$$

$$E_{-}^{\nu,\mu}f=\int\limits_x^\infty \left(t^2-x^2\right)^{-\frac{\mu}{2}} \mathbb{P}_\nu^\mu\left(\frac{x}{t}\right)f(t)dt. \tag{11}$$

here $P_\nu^\mu(z)$ is the Legendre function of the first kind , $\mathbb{P}_\nu^\mu(z)$ is this function on the cut $-1\leqslant t\leqslant 1$, $f(x)$ is a locally summable function with some growth conditions at $x\to 0, x\to\infty$. Parameters $\mu,\nu\in\mathbb{C}$, $\operatorname{Re}\mu<1$, $\operatorname{Re}\nu\geqslant -1/2$.

Now consider main properties for this class of transmutations following essentially [29], [32], and also [7], [27]. All functions further are defined on positive semiaxis. So we use notations L_2 for the functional space $L_2(0,\infty)$ and $L_{2,k}$ for power weighted space $L_{2,k}(0,\infty)$ equipped with norm

$$\int\limits_0^\infty |f(x)|^2 x^{2k+1}\,dx. \tag{12}$$

$\mathbb{N}$ denote set of natural, $\mathbb{N}_0$–positive integer, $\mathbb{Z}$–integer and $\mathbb{R}$–real numbers.

First add to definition 3 a case of parameter $\mu=1$.

Definition 4. Define for $\mu=1$ Buschman–Erdelyi operators of zero order smoothness by

$$B_{0+}^{\nu,1}f=\frac{d}{dx}\int\limits_0^x P_\nu\left(\frac{x}{t}\right)f(t)\,dt, \tag{13}$$

$$E_{0+}^{\nu,1}f=\int\limits_0^x P_\nu\left(\frac{t}{x}\right)\frac{df(t)}{dt}\,dt, \tag{14}$$

$$B_{-}^{\nu,1}f=\int\limits_x^\infty P_\nu\left(\frac{t}{x}\right)(-\frac{df(t)}{dt})\,dt, \tag{15}$$

$$E_{-}^{\nu,1}f=(-\frac{d}{dx})\int\limits_x^\infty P_\nu\left(\frac{x}{t}\right)f(t)\,dt, \tag{16}$$

here $P_\nu(z)=P_\nu^0(z)$ is the Legendre function.

Theorem 1. The next formulas hold true for factorizations of Buschman–Erdelyi transmutations for suitable functions via Riemann–Liouville fractional integrals and Buschman–Erdelyi operators of zero order smoothness:

$$B_{0+}^{\nu,\mu} f = I_{0+}^{1-\mu}\ {}_1S_{0+}^{\nu} f,\ B_{-}^{\nu,\mu} f = {}_1P_{-}^{\nu}\ I_{-}^{1-\mu} f, \tag{17}$$

$$E_{0+}^{\nu,\mu} f = {}_1P_{0+}^{\nu}\ I_{0+}^{1-\mu} f,\ E_{-}^{\nu,\mu} f = I_{-}^{1-\mu}\ {}_1S_{-}^{\nu} f. \tag{18}$$

These formulas allow to separate parameters ν and μ. We will prove soon that operators (13)–(16) are isomorphisms of $L_2(0,\infty)$ except for some special parameters. So operators (8)–(11) roughly speaking are of the same smoothness in L_2 as integrodifferentiations $I^{1-\mu}$ and they coincide with them for $\nu = 0$. It is also possible to define Buschman–Erdelyi operators for all $\mu \in \mathbb{C}$.

Definition 5. Define the number $\rho = 1 - Re\,\mu$ as smoothness order for Buschman–Erdelyi operators (8)–(11).

So for $\rho > 0$ (otherwise for $Re\,\mu > 1$) Buschman–Erdelyi operators are smoothing and for $\rho < 0$ (otherwise for $Re\,\mu < 1$) they decrease smoothness in L_2 spaces. Operators (13)–(16) for which $\rho = 0$ due to definition 5 are of zero smoothness order.

For some special parameters ν, μ Buschman–Erdelyi operators of the first kind are reduced to other known operators. So for $\mu = -\nu$ or $\mu = \nu + 2$ they reduce to Erdelyi–Kober operators, for $\nu = 0$ they reduce to fractional integrodifferentiation $I_{0+}^{1-\mu}$ or $I_{-}^{1-\mu}$, for $\nu = -\frac{1}{2}$, $\mu = 0$ or $\mu = 1$ kernels reduce to elliptic integrals, for $\mu = = 0$, $x = 1$, $v = it - \frac{1}{2}$ the operator $B_{-}^{\nu,0}$ differs only by a constant from Mehler–Fock transform.

As a pair for the Bessel operator consider a connected one

$$L_\nu = D^2 - \frac{\nu(\nu+1)}{x^2} = \left(\frac{d}{dx} - \frac{\nu}{x}\right)\left(\frac{d}{dx} + \frac{\nu}{x}\right), \tag{19}$$

which for $\nu \in \mathbb{N}$ is an angular momentum operator from quantum physics. Their transmutational relations are established in the next theorem.

Theorem 2. For a given pair of transmutations X_ν, Y_ν

$$X_\nu L_\nu = D^2 X_\nu, Y_\nu D^2 = L_\nu Y_\nu \tag{20}$$

define the new pair of transmutations by formulas

$$S_\nu = X_{\nu-1/2} x^{\nu+1/2}, P_\nu = x^{-(\nu+1/2)} Y_{\nu-1/2}. \tag{21}$$

Then for the new pair S_ν, P_ν the next formulas are valid:

$$S_\nu B_\nu = D^2 S_\nu, P_\nu D^2 = B_\nu P_\nu. \tag{22}$$

Theorem 3. Let $Re\,\mu \leqslant 1$. Then an operator $B_{0+}^{\nu,\mu}$ is a Sonine type transmutation and (20) is valid.

The same result holds true for other Buschman–Erdelyi operators, $E_{-}^{\nu,\mu}$ is Sonine type and $E_{0+}^{\nu,\mu}$, $B_{-}^{\nu,\mu}$ are Poisson type transmutations.

From these transmutational connections we conclude that Buschman–Erdelyi operators link corresponding eigenfunctions for two operators. They lead to formulas for Bessel functions via exponents and trigonometric functions and vice versa which generalize classical Sonine and Poisson formulas.

Now consider factorizations of Buschman–Erdelyi operators. First let list main forms of fractional integrodifferentiations: Riemann–Liouville, Erdelyi–Kober, fractional integral by function $g(x)$, cf. [26].

$$I_{0+,x}^{\alpha}f=\frac{1}{\Gamma(\alpha)}\int\limits_0^x (x-t)^{\alpha-1}f(t)dt, \qquad (23)$$

$$I_{-,x}^{\alpha}f=\frac{1}{\Gamma(\alpha)}\int\limits_x^\infty (t-x)^{\alpha-1}f(t)dt,$$

$$I_{0+,2,\eta}^{\alpha}f=\frac{2x^{-2(\alpha+\eta)}}{\Gamma(\alpha)}\int\limits_0^x \left(x^2-t^2\right)^{\alpha-1}t^{2\eta+1}f(t)dt, \qquad (24)$$

$$I_{-,2,\eta}^{\alpha}f=\frac{2x^{2\eta}}{\Gamma(\alpha)}\int\limits_x^\infty \left(t^2-x^2\right)^{\alpha-1}t^{1-2(\alpha+\eta)}f(t)dt,$$

$$I_{0+,g}^{\alpha}f=\frac{1}{\Gamma(\alpha)}\int\limits_0^x (g(x)-g(t))^{\alpha-1}g'(t)f(t)dt, \qquad (25)$$

$$I_{-,g}^{\alpha}f=\frac{1}{\Gamma(\alpha)}\int\limits_x^\infty (g(t)-g(x))^{\alpha-1}g'(t)f(t)dt,$$

in all cases $\mathrm{Re}\,\alpha>0$ and operators may be further defined for all α [26]. In case of $g(x)=x$ (25) reduces to Riemann–Liouville, in case of $g(x)=x^2$ (25) reduces to Erdelyi–Kober and in case of $g(x)=\ln x$ to Hadamard fractional integrals.

Theorem 4. The next factorization formulas are valid for Buschman–Erdelyi operators of the first kind via Riemann–Liouville and Erdelyi–Kober fractional integrals

$$B_{0+}^{\nu,\mu} = I_{0+}^{\nu+1-\mu} I_{0+;\,2,\,\nu+\frac{1}{2}}^{-(\nu+1)} \left(\frac{2}{x}\right)^{\nu+1}, \tag{26}$$

$$E_{0+}^{\nu,\mu} = \left(\frac{x}{2}\right)^{\nu+1} I_{0+;\,2,\,-\frac{1}{2}}^{\nu+1} I_{0+}^{-(\nu+\mu)}, \tag{27}$$

$$B_{-}^{\nu,\mu} = \left(\frac{2}{x}\right)^{\nu+1} I_{-;\,2,\,\nu+1}^{-(\nu+1)} I_{-}^{\nu-\mu+2}, \tag{28}$$

$$E_{-}^{\nu,\mu} = I_{-}^{-(\nu+\mu)} I_{-;\,2,\,0}^{\nu+1} \left(\frac{x}{2}\right)^{\nu+1}. \tag{29}$$

Sonine–Poisson–Delsarte transmutations also are special cases for this class of operators.

Now let study properties of Buschman–Erdelyi operators of zero order smoothness defined by (13). A similar operator was introduced by Katrakhov by multiplying the Sonine operator with fractional integral, his aim was to work with transmutation obeying good estimates in $L_2(0,\infty)$.

We use the Mellin transform defined by [48]

$$g(s) = Mf(s) = \int_0^\infty x^{s-1} f(x)\, dx. \tag{30}$$

The Mellin convolution is defined by

$$(f_1 * f_2)(x) = \int_0^\infty f_1\left(\frac{x}{y}\right) f_2(y)\, \frac{dy}{y}, \tag{31}$$

so the convolution operator with kernel K acts under Mellin transform as multiplication on multiplicator

$$M[Af](s) = M\ [\int_0^\infty K\left(\frac{x}{y}\right) f(y)\, \frac{dy}{y}](s) = M[K*f](s) = m_A(s) Mf(s), \tag{32}$$
$$m_A(s) = M[K](s).$$

We observe that Mellin transform is a generalized Fourier transform on semiaxis with Haar measure $\frac{dy}{y}$ [56]. It plays important role for special functions, for example the gamma function is a Mellin transform of the exponential. With Mellin transform the important breakthrough in evaluating integrals was done in 1970th when mainly by O.Marichev the famous Slater's theorem was adapted for calculations. The Slater's theorem taking the Mellin transform as input give the function itself as output via hypergeometric functions [48]. This theorem occurred to be the milestone of powerful computer method for calculating integrals for many problems in differential and integral equations. The package MATHEMATICA of Wolfram Research is based on this theorem in calculating integrals.

Theorem 5. Buschman–Erdelyi operator of zero order smoothness $B_{0+}^{\nu,1}$ defined by (13) acts under the Mellin transform as convolution (32) with multiplicator

$$m(s)=\frac{\Gamma(-s/2+\frac{\nu}{2}+1)\Gamma(-s/2-\frac{\nu}{2}+1/2)}{\Gamma(1/2-\frac{s}{2})\Gamma(1-\frac{s}{2})} \tag{33}$$

for $\mathrm{Re}\, s<\min(2+\mathrm{Re}\,\nu, 1-\mathrm{Re}\,\nu)$. Its norm is a periodic in ν and equals

$$\|B_{0+}^{\nu,1}\|_{L_2}=\frac{1}{\min(1,\sqrt{1-\sin\pi\nu})}. \tag{34}$$

This operator is bounded in $L_2(0,\infty)$ if $\nu\neq 2k+1/2, k\in\mathbb{Z}$ and unbounded if $\nu=$ $=2k+1/2, k\in\mathbb{Z}$.

This theorem is the most important result of this article so we give a complete proof.

1. First let us prove the formula (33) with a proper multiplicator. For it using consequently formulas (7), p. 130, (2) p. 129, (4) p. 130 from [48] we evaluate

$$M(B_{0+}^{\nu,1})(s)=\frac{\Gamma(2-s)}{\Gamma(1-s)}M\left[\int_0^\infty\left\{H(\frac{x}{y}-1)P_\nu(\frac{x}{y})\right\}\{yf(y)\}\,\frac{dy}{y}\right](s-1)=$$

$$=\frac{\Gamma(2-s)}{\Gamma(1-s)}M\left[(x^2-1)_+{}^0P_\nu^0(x)\right](s-1)M\left[f\right](s),$$

we use notations from [48] for Heaviside and cutting power functions

$$x_+{}^\alpha=\begin{cases}x^\alpha, \text{если } x\geqslant 0\\ 0, \text{если } x<0\end{cases},\quad H(x)=x_+{}^0=\begin{cases}1, \text{если } x\geqslant 0\\ 0, \text{если } x<0.\end{cases}$$

Further using formulas 14(1) p. 234 и 4 p. 130 from [48] we evaluate

$$M\left[(x-1)_+{}^0P_\nu^0(\sqrt{x})\right](s)=\frac{\Gamma(\frac{1}{2}+\frac{\nu}{2}-s)\Gamma(-\frac{\nu}{2}-s)}{\Gamma(1-s)\Gamma(\frac{1}{2}-s)},$$

$$M\left[(x^2-1)_+{}^0P_\nu^0(x)\right](s-1)=\frac{1}{2}\cdot\frac{\Gamma(\frac{1}{2}+\frac{\nu}{2}-\frac{s-1}{2})\Gamma(-\frac{\nu}{2}-\frac{s-1}{2})}{\Gamma(1-\frac{s-1}{2})\Gamma(\frac{1}{2}-\frac{s-1}{2})}=$$

$$=\frac{1}{2}\cdot\frac{\Gamma(-\frac{s}{2}+\frac{\nu}{2}+1)\Gamma(-\frac{s}{2}-\frac{\nu}{2}+\frac{1}{2})}{\Gamma(-\frac{s}{2}+\frac{3}{2})\Gamma(-\frac{s}{2}+1)}$$

under conditions $\mathrm{Re}\, s<\min(2+\mathrm{Re}\,\nu, 1-\mathrm{Re}\,\nu)$. Now evaluate formula for

$$M(B_{0+}^{\nu,1})(s)=\frac{1}{2}\cdot\frac{\Gamma(2-s)}{\Gamma(1-s)}\cdot\Gamma(-\frac{s}{2}+\frac{3}{2})\Gamma(-\frac{s}{2}+1).$$

Applying to $\Gamma(2-s)$ the Legendre duplication formula (cf. [49]) we evaluate

$$M(B_{0+}^{\nu,1})(s)=\frac{2^{-s}}{\sqrt{\pi}}\cdot\frac{\Gamma(-\frac{s}{2}+\frac{\nu}{2}+1)\Gamma(-\frac{s}{2}-\frac{\nu}{2}+\frac{1}{2})}{\Gamma(1-s)}.$$

Apply the Legendre duplication formula once more to $\Gamma(1-s)$ and the formula for the multiplicator (33) is proved. In the paper [29] it was shown that restrictions may be reduced to $0<\operatorname{Re} s<1$ for proper ν. These restrictions may be weakened because they were derived for the class of all hypergeometric functions but we need just one special case of the Legendre function for which specified restrictions may be easily verified directly.

2. Now prove the formula (34) for a norm. From the multiplicator value we just found and theorem 4.7 from [7] on the line $\operatorname{Re} s=1/2, s=iu+1/2$ it follows

$$|M(B_{0+}^{\nu,1})(iu+1/2)|=\frac{1}{\sqrt{2\pi}}\left|\frac{\Gamma(-i\frac{u}{2}-\frac{\nu}{2}+\frac{1}{4})\Gamma(-i\frac{u}{2}+\frac{\nu}{2}+\frac{3}{4})}{\Gamma(\frac{1}{2}-iu)}\right|.$$

Below operator symbol in multiplicator will be omitted. Use formulas for modulus $|z|=\sqrt{z\bar{z}}$ and gamma–function $\overline{\Gamma(z)}=\Gamma(\bar{z})$ following from its definition as integral. The last property is true in general for the class of real–analytic functions. So we derive

$$|M(B_{0+}^{\nu,1})(iu+1/2)|=$$

$$=\frac{1}{\sqrt{2\pi}}\left|\frac{\Gamma(-i\frac{u}{2}-\frac{\nu}{2}+\frac{1}{4})\Gamma(i\frac{u}{2}-\frac{\nu}{2}+\frac{1}{4})\Gamma(-i\frac{u}{2}+\frac{\nu}{2}+\frac{3}{4})\Gamma(i\frac{u}{2}+\frac{\nu}{2}+\frac{3}{4})}{\Gamma(\frac{1}{2}-iu)\Gamma(\frac{1}{2}+iu)}\right|.$$

In the numerator combine outer and inner terms and transform three pair of gamma-functions by the formula (см. [49])

$$\Gamma(\frac{1}{2}+z)\ \Gamma(\frac{1}{2}-z)=\frac{\pi}{\cos\pi z}.$$

As a result we evaluate

$$|M(B_{0+}^{\nu,1})(iu+1/2)|=\sqrt{\frac{\cos(\pi iu)}{2\cos\pi(\frac{\nu}{2}+\frac{1}{4}+i\frac{u}{2})\cos\pi(\frac{\nu}{2}+\frac{1}{4}-i\frac{u}{2})}}=$$

$$=\sqrt{\frac{\operatorname{ch}(\pi iu)}{\operatorname{ch}\pi u-\sin\pi\nu}}$$

Further denote as $t=\operatorname{ch}\pi u, 1\leqslant t<\infty$. So derive once more applying theorem 4.7 from [7]

$$\sup_{u\in\mathbb{R}}|m(iu+\frac{1}{2})|=\sup_{1\leqslant t<\infty}\sqrt{\frac{t}{t-\sin\pi\nu}}.$$

So if $\sin\pi\nu\geqslant 0$ then supremum achieved at $t=1$ and for the norm the formula (34) is valid

$$\|B_{0+}^{\nu,1}\|_{L_2}=\frac{1}{\sqrt{1-\sin\pi\nu}}.$$

Otherwise if $\sin\pi\nu\leqslant 0$ then supremum achieved at $t\to\infty$ and the next formula is valid

$$\|B_{0+}^{\nu,1}\|_{L_2}=1.$$

This part of the theorem is proved.

3. From the explicit values for norms and above cited theorem 4.7 from [7] follow conditions of boundedness or unboundedness and periodicity. The theorem is completely proved.

Now proceed to finding multiplicators for all Buschman–Erdelyi operator of zero order smoothness.

Theorem 6. Buschman–Erdelyi operator of zero order smoothness acts under the Mellin transform as convolutions (32). For their multiplicators the next formulas are valid

$$m_{{}_1S^{\nu}_{0+}}(s)=\frac{\Gamma(-\frac{s}{2}+\frac{\nu}{2}+1)\Gamma(-\frac{s}{2}-\frac{\nu}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{s}{2})\Gamma(1-\frac{s}{2})}=; \tag{35}$$

$$=\frac{2^{-s}}{\sqrt{\pi}}\frac{\Gamma(-\frac{s}{2}-\frac{\nu}{2}+\frac{1}{2})\Gamma(-\frac{s}{2}+\frac{\nu}{2}+1)}{\Gamma(1-s)}, Re\, s<\min(2+Re\,\nu, 1-Re\,\nu);$$

$$m_{{}_1P^{\nu}_{0+}}(s)=\frac{\Gamma(\frac{1}{2}-\frac{s}{2})\Gamma(1-\frac{s}{2})}{\Gamma(-\frac{s}{2}+\frac{\nu}{2}+1)\Gamma(-\frac{s}{2}-\frac{\nu}{2}+\frac{1}{2})},\ Re\, s<1; \tag{36}$$

$$m_{{}_1P^{\nu}_{-}}(s)=\frac{\Gamma(\frac{s}{2}+\frac{\nu}{2}+1)\Gamma(\frac{s}{2}-\frac{\nu}{2})}{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+\frac{1}{2})}, Re\, s>\max(Re\,\nu, -1-Re\,\nu); \tag{37}$$

$$m_{{}_1S^{\nu}_{-}}(s)=\frac{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+\frac{1}{2})}{\Gamma(\frac{s}{2}+\frac{\nu}{2}+\frac{1}{2})\Gamma(\frac{s}{2}-\frac{\nu}{2})}, Re\, s>0 \tag{38}$$

The next formulas are valid for norms of Buschman–Erdelyi operator of zero order smoothness in L_2:

$$\|{}_1S^{\nu}_{0+}\|=\|{}_1P^{\nu}_{-}\|=1/\min(1,\sqrt{1-\sin\pi\nu}), \tag{39}$$

$$\|{}_1P^{\nu}_{0+}\|=\|{}_1S^{\nu}_{-}\|=\max(1,\sqrt{1-\sin\pi\nu}). \tag{40}$$

Similar results are proved in [27]–[29] for power weight spaces.

Corollary 1. Norms of operators (13) – (16) are periodic in ν with period 2 $\|X^{\nu}\|=\|X^{\nu+2}\|$, X^{ν} is any of operators (13) – (16).

Corollary 2. Norms of operators ${}_1S^{\nu}_{0+}$, ${}_1P^{\nu}_{-}$ are not bounded in total, every norm is greater or equals to 1. Norms are equal to 1 if $\sin\pi\nu\leqslant 0$. Operators ${}_1S^{\nu}_{0+}$, ${}_1P^{\nu}_{-}$ are unbounded in L_2 if and only if $\sin\pi\nu=1$ (or $\nu=(2k)+1/2,\ k\in\mathbb{Z}$).

Corollary 3. Norms of operators ${}_1P^{\nu}_{0+}$, ${}_1S^{\nu}_{-}$ are all bounded in ν, every norm is not greater then $\sqrt{2}$. Norms are equal to 1 if $\sin\pi\nu\geqslant 0$. Operators ${}_1P^{\nu}_{0+}$, ${}_1S^{\nu}_{-}$ are bounded in L_2 for all ν. Maximum of norm equals $\sqrt{2}$ is achieved if and only if $\sin\pi\nu=-1$ (или $\nu=-1/2+(2k),\ k\in\mathbb{Z}$).

The most important property of Buschman–Erdelyi operators of zero order smoothness is unitarity for integer ν. It is just the case if interpret for these parameters the operator L_{ν} as angular momentum operator in quantum mechanics.

Theorem 7. The operators (13) – (16) are unitary in L_2 if and only if the parameter ν is an integer. In this case pairs of operators $({}_1S^{\nu}_{0+},\ {}_1P^{\nu}_{-})$ and $({}_1S^{\nu}_{-},\ {}_1P^{\nu}_{0+})$ are mutually inverse.

To formulate an interesting special case let us propose that operators (13) – (16) act on functions permitting outer or inner differentiation in integrals, it is enough to suppose that $xf(x) \to 0$ for $x \to 0$. Then for $\nu = 1$

$$ {}_1P^1_{0+}f = (I - H_1)f,\ {}_1S^1_{-}f = (I - H_2)f, \tag{41} $$

and $H_1,\ H_2$ are famous Hardy operators,

$$ H_1 f = \frac{1}{x}\int_0^x f(y)dy,\ H_2 f = \int_x^\infty \frac{f(y)}{y}dy, \tag{42} $$

I is the identic operator.

Corollary 4. Operators (41) are unitary in L_2 and mutually inverse. They are transmutations for a pair of differential operators d^2/dx^2 и $d^2/dx^2 - 2/x^2$.

Unitarity of shifted Hardy operators (41) in L_2 is a known fact [50]. Below in application section we introduce a new class of generalizations for classical Hardy operators.

Now we list some properties of operators acting as convolutions by the formula (32) and with some multiplicator under the Mellin transform and being transmutations for the second derivative and angular momentum operator in quantum mechanics.

Theorem 8. Let an operator S_ν acts by formulas (32) and (20). Then

a) its multiplicator satisfy a functional equation

$$ m(s) = m(s-2)\frac{(s-1)(s-2)}{(s-1)(s-2) - \nu(\nu+1)}; \tag{43} $$

б) if any function $p(s)$ is periodic with period 2 ($p(s) = p(s-2)$) then a function $p(s)m(s)$ is a multiplicator for a new transmutation operator S_2^ν also acting by the rule (20).

This theorem confirms the importance of studying transmutations in terms of the Mellin transform and multiplicator functions.

Define the Stieltjes transform by (cf. [26])

$$ (Sf)(x) = \int_0^\infty \frac{f(t)}{x+t}dt. $$

This operator also acts by the formula (32) with multiplicator $p(s) = \pi/sin(\pi s)$, it is bounded in L_2. Obviously $p(s) = p(s-2)$. So from the theorem 8 it follows that a convolution of the Stieltjes transform with bounded transmutations (13)–(16) are also transmutations of the same class bounded in L_2.

On this way many new classes of transmutations were introduced with special function kernels.

3 Buschman–Erdelyi transmutations of the second kind.

Now we consider Buschman–Erdelyi transmutations of the second kind.

Definition 5. Define a new pair of Buschman–Erdelyi transmutations of the second kind with Legendre functions of the second kind in kernels

$$ {}_2S^\nu f = \frac{2}{\pi}\left(-\int_0^x (x^2-y^2)^{-\frac{1}{2}} Q_\nu^1\left(\frac{x}{y}\right) f(y)dy + \int_x^\infty (y^2-x^2)^{-\frac{1}{2}} \mathbb{Q}_\nu^1\left(\frac{x}{y}\right) f(y)dy\right), \quad (42) $$

$$ {}_2P^\nu f = \frac{2}{\pi}\left(-\int_0^x (x^2-y^2)^{-\frac{1}{2}} \mathbb{Q}_\nu^1\left(\frac{y}{x}\right) f(y)dy - \int_x^\infty (y^2-x^2)^{-\frac{1}{2}} Q_\nu^1\left(\frac{y}{x}\right) f(y)dy\right). \quad (43) $$

These operators are analogues of Buschman–Erdelyi transmutations of zero order smoothness. If $y \to x \pm 0$ then integrals are defined by principal values. It is proved that they are transmutations of Sonine type for (42) and of Poisson type for (43).

Theorem 9. Operators (42) – (43) are of the form (32) with multiplicators

$$ m_{{}_2S^\nu}(s) = p(s)\ m_{{}_1S_-^\nu}(s), \quad (44) $$

$$ m_{{}_2P^\nu}(s) = \frac{1}{p(s)}\ m_{{}_1P_-^\nu}(s), \quad (45) $$

with multiplicators of operators ${}_1S_-{}^\nu$, ${}_1P_-{}^\nu$ defined by (37) – (38), with period 2 function $p(s)$ equals

$$ p(s) = \frac{\sin\pi\nu + \cos\pi s}{\sin\pi\nu - \sin\pi s}. \quad (46) $$

Theorem 10. The next formulas for norms are valid

$$ \|{}_2S^\nu\|_{L_2} = \max(1, \sqrt{1+\sin\pi\nu}), \quad (47) $$

$$ \|{}_2P^\nu\|_{L_2} = 1/\min(1, \sqrt{1+\sin\pi\nu}). \quad (48) $$

Corollary. Operator ${}_2S^\nu$ is bounded for all ν. Operator ${}_2P^\nu$ is not bounded if and only if then $\sin\pi\nu = -1$.

Theorem 11. Operators ${}_2S^\nu$ and ${}_2P^\nu$ are unitary in L_2 if and only if $\nu \in \mathbb{Z}$.

Theorem 12. Let $\nu = i\beta + 1/2,\ \beta \in \mathbb{R}$. Then

$$ \|{}_2S^\nu\|_{L_2} = \sqrt{1+\operatorname{ch}\pi\beta},\ \|{}_2P^\nu\|_{L_2} = 1. \quad (49) $$

Theorem 13. The next formulas are valid

$$ {}_2S^0 f = \frac{2}{\pi} \int\limits_0^\infty \frac{y}{x^2 - y^2} f(y)\, dy, \tag{50} $$

$$ {}_2S^{-1} f = \frac{2}{\pi} \int\limits_0^\infty \frac{x}{x^2 - y^2} f(y)\, dy. \tag{51} $$

So in this case the operator ${}_2S^\nu$ reduce to a pair of semiaxis Hilbert transforms [26].

For operators of the second kind also introduce more general ones with two parameters analogically to Buschman–Erdelyi transmutations of the first kind by formulas

$$ {}_2S^{\nu,\mu} f = \frac{2}{\pi} \left(\int\limits_0^x (x^2 + y^2)^{-\frac{\mu}{2}} e^{-\mu\pi i} Q_\nu^\mu(\frac{x}{y}) f(y)\, dy + \right. \tag{52} $$

$$ \left. + \int\limits_x^\infty (y^2 + x^2)^{-\frac{\mu}{2}} \mathbb{Q}_\nu^\mu(\frac{x}{y}) f(y)\, dy \right), $$

here $Q_\nu^\mu(z)$ is the Legendre function of the second kind, $\mathbb{Q}_\nu^\mu(z)$ is this function on the cut [49], $Re\,\nu < 1$. The second operator may be defined as formally conjugate in $L_2(0,\infty)$ to (52).

Theorem 14. The operator (52) on $C_0^\infty(0,\infty)$ is well defined and acts by

$$ M[{}_2S^\nu](s) = m(s) \cdot M[x^{1-\mu} f](s), $$

$$ m(s) = 2^{\mu-1} \left(\frac{\cos \pi(\mu - s) - \cos \pi\nu}{\sin \pi(\mu - s) - \sin \pi\nu} \right) \cdot \tag{53} $$

$$ \cdot \left(\frac{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2} + \frac{1}{2}))}{\Gamma(\frac{s}{2} + \frac{1-\nu-\mu}{2})\Gamma(\frac{s}{2} + 1 + \frac{\nu-\mu}{2})} \right). $$

4 Buschman–Erdelyi transmutations of the third kind.

4.1 Sonine–Katrakhov and Poisson–Katrakhov transmutations.

Now we construct transmutations which are unitary for all ν. They are defined by formulas

$$ S_U^\nu f = -\sin\frac{\pi\nu}{2}\ {}_2S^\nu f + \cos\frac{\pi\nu}{2}\ {}_1S_-{}^\nu f, \tag{54} $$

$$ P_U^\nu f = -\sin\frac{\pi\nu}{2}\ {}_2P^\nu f + \cos\frac{\pi\nu}{2}\ {}_1P_-{}^\nu f. \tag{55} $$

For all values $\nu \in \mathbb{R}$ they are linear combinations of Buschman–Erdelyi transmutations of the first and second kinds of zero order smoothness. Also they are in the defined below class of Buschman–Erdelyi transmutations of the third kind. Integral representations are valid

$$S_U^\nu f = \cos\frac{\pi\nu}{2}\left(-\frac{d}{dx}\right)\int_x^\infty P_\nu\left(\frac{x}{y}\right)f(y)\,dy + \tag{56}$$

$$+\frac{2}{\pi}\sin\frac{\pi\nu}{2}\left(\int_0^x (x^2-y^2)^{-\frac{1}{2}}Q_\nu^1\left(\frac{x}{y}\right)f(y)\,dy - \int_x^\infty (y^2-x^2)^{-\frac{1}{2}}\mathbb{Q}_\nu^1\left(\frac{x}{y}\right)f(y)\,dy\right),$$

$$P_U^\nu f = \cos\frac{\pi\nu}{2}\int_0^x P_\nu\left(\frac{y}{x}\right)\left(\frac{d}{dy}\right)f(y)\,dy - \tag{57}$$

$$-\frac{2}{\pi}\sin\frac{\pi\nu}{2}\left(-\int_0^x (x^2-y^2)^{-\frac{1}{2}}\mathbb{Q}_\nu^1\left(\frac{y}{x}\right)f(y)\,dy - \int_x^\infty (y^2-x^2)^{-\frac{1}{2}}Q_\nu^1\left(\frac{y}{x}\right)f(y)\,dy\right).$$

Theorem 15. Operators (54)– (55), (56)– (57) for all $\nu \in \mathbb{R}$ are unitary, mutually inverse and conjugate in L_2. They are transmutations acting by (19). S_U^ν is a Sonine type transmutation and P_U^ν is a Poisson type one.

Transmutations like (56)–(57) but with kernels into more complicated form with hypergeometric functions were first introduced by Katrakhov in 1980. Due to it the author propose terms for this class of operators as Sonine–Katrakhov and Poisson–Katrakhov. In author's papers these operators were reduced to more simple form of Buschman–Erdelyi ones. It made possible to include this class of operators in general composition (or factorization) method [54], [33], [35].

4.2 Buschman–Erdelyi transmutations of the third kind with arbitrary weight function.

Define sine and cosine Fourier transforms with inverses

$$F_c f = \sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\cos(ty)\,dy, \quad F_c^{-1} = F_c, \tag{58}$$

$$F_s f = \sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\sin(ty)\,dy, \quad F_s^{-1} = F_s. \tag{59}$$

Define Hankel (Fourier–Bessel) transform and its inverse by

$$F_\nu f = \frac{1}{2^\nu \Gamma(\nu+1)} \int_0^\infty f(y)\, j_\nu(ty)\, y^{2\nu+1}\, dy =$$

$$= \int_0^\infty f(y) \frac{J_\nu(ty)}{(ty)^\nu}\, y^{2\nu+1}\, dy = \frac{1}{t^\nu} \int_0^\infty f(y) J_\nu(ty)\, y^{\nu+1}\, dy, \tag{60}$$

$$F_\nu^{-1} f = \frac{1}{(y)^\nu} \int_0^\infty f(t) J_\nu(yt)\, t^{\nu+1}\, dt. \tag{61}$$

Here $J_\nu(\cdot)$ is the Bessel function [49], $j_\nu(\cdot)$ is normalized Bessel function [15]. Operators (58)- (59) are unitary self–conjugate in $L_2(0,\infty)$. Operators (60)– (61) are unitary self–conjugate in power weighted space $L_{2,\nu}(0,\infty)$.

Now define on proper functions the first pair of Buschman–Erdelyi transmutations of the third kind

$$S_{\nu,c}^{(\varphi)} = F_c^{-1} \left(\frac{1}{\varphi(t)} F_\nu \right), \tag{62}$$

$$P_{\nu,c}^{(\varphi)} = F_\nu^{-1} \left(\varphi(t) F_c \right), \tag{63}$$

and the second pair by

$$S_{\nu,s}^{(\varphi)} = F_s^{-1} \left(\frac{1}{\varphi(t)} F_\nu \right), \tag{64}$$

$$P_{\nu,s}^{(\varphi)} = F_\nu^{-1} \left(\varphi(t) F_s \right).,\tag{65}$$

with $\varphi(t)$ being an arbitrary weight function.

The operators defined on proper functions are transmutations for B_ν and D^2. They may be expressed in the integral form.

Theorem 16. Define transmutations for B_ν and D^2 by formulas

$$S^{(\varphi)}_{\nu,\left\{\begin{smallmatrix} s \\ c \end{smallmatrix}\right\}} = F^{-1}_{\left\{\begin{smallmatrix} s \\ c \end{smallmatrix}\right\}} \left(\frac{1}{\varphi(t)} F_\nu \right),$$

$$P^{(\varphi)}_{\nu,\left\{\begin{smallmatrix} s \\ c \end{smallmatrix}\right\}} = F_\nu^{-1} \left(\varphi(t) F_{\left\{\begin{smallmatrix} s \\ c \end{smallmatrix}\right\}} \right).$$

Then for the Sonine type transmutation an integral form is valid

$$\left(S^{(\varphi)}_{\nu,\left\{ s \atop c \right\}} f \right)(x) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty K(x,y) f(y)\, dy, \tag{66}$$

где

$$K(x,y) = y^{\nu+1} \int\limits_0^\infty \frac{\left\{ \sin(xt) \atop \cos(xt) \right\}}{\varphi(t)\, t^\nu} J_\nu(yt) dt.$$

For the Poisson type transmutation an integral form is valid

$$\left(P^{(\varphi)}_{\nu,\left\{ s \atop c \right\}} f \right)(x) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty G(x,y) f(y)\, dy, \tag{67}$$

где

$$G(x,y) = \frac{1}{x^\nu} \int\limits_0^\infty \varphi(t)\, t^{\nu+1} \left\{ \sin(yt) \atop \cos(yt) \right\} J_\nu(xt) dt.$$

Introduced before unitary transmutations of Sonine–Katrakhov and Poisson–Katrakhov are special cases of this class operators. For this case we must choice a weight function $\varphi(t)$ as a power function depending on the parameter ν. The author plans to publish a special paper on Buschman–Erdelyi transmutations of the third kind with arbitrary weight function.

5 Some Applications of Buschman–Erdelyi transmutations.

In this section we gather some applications of Buschman–Erdelyi operators (but not all). Due to the article restrictions most items are only briefly mentioned with most informative facts and instructive references. Some applications only mentioned as problems for future investigations.

5.1 Norm estimates and embedding theorems in Kipriyanov spaces.

Consider a set of functions $\mathbb{D}(0,\infty)$. If $f(x) \in \mathbb{D}(0,\infty)$ then $f(x) \in \in C^\infty(0,\infty)$, $f(x)$ is zero at infinity. On this set define seminorms

$$\|f\|_{h_2^\alpha} = \|D_-^\alpha f\|_{L_2(0,\infty)} \tag{58}$$

$$\|f\|_{\widehat{h}_2^\alpha} = \|x^\alpha (-\frac{1}{x}\frac{d}{dx})^\alpha f\|_{L_2(0,\infty)} \tag{59}$$

here $D_-{}^\alpha$ is the Riemann–Liouville fractional integrodifferentiation, operator in (59) is defined by

$$(-\frac{1}{x}\frac{d}{dx})^\beta = 2^\beta I_{-;2,0}^{-\beta} x^{-2\beta}, \tag{60}$$

$I_{-;2,0}^{-\beta}$ is Erdelyi–Kober operator, $\alpha \in \mathbb{R}$. For $\beta = n \in \mathbb{N}_0$ expression (60) reduces to classical derivatives.

Theorem 17. Let $f(x)\in\mathbb{D}(0,\infty)$. Then the next formulas are valid:

$$D_-^\alpha f = {}_1S_-^{\alpha-1}x^\alpha(-\frac{1}{x}\frac{d}{dx})^\alpha f, \tag{61}$$

$$x^\alpha(-\frac{1}{x}\frac{d}{dx})^\alpha f = {}_1P_-^{\alpha-1}D_-^\alpha f. \tag{62}$$

So Buschman–Erdelyi transmutations of zero order smoothness for $\alpha\in\mathbb{N}$ links differential operators in seminorms definitions (58) and (59).

Theorem 18. Let $f(x)\in\mathbb{D}(0,\infty)$. Then the next inequalities hold true for seminorms

$$\|f\|_{h_2^\alpha}\leqslant \max(1,\sqrt{1+\sin\pi\alpha})\|f\|_{\widehat{h}_2^\alpha}, \tag{63}$$

$$\|f\|_{\widehat{h}_2^\alpha}\leqslant \frac{1}{\min(1,\sqrt{1+\sin\pi\alpha})}\|f\|_{h_2^\alpha}, \tag{64}$$

here α is any real number except $\alpha\neq-\frac{1}{2}+2k,\ k\in\mathbb{Z}$.

Constants in inequalities (63)–(64) are not greater than 1, it will be used below. If $\sin\pi\alpha=-1$ or $\alpha=-\frac{1}{2}+2k,\ k\in\mathbb{Z}$ then the estimate (64) is not valid.

Define on $\mathbb{D}(0,\infty)$ the Sobolev norm

$$\|f\|_{W_2^\alpha}=\|f\|_{L_2(0,\infty)}+\|f\|_{h_2^\alpha}. \tag{65}$$

Define one more norm

$$\|f\|_{\widehat{W}_2^\alpha}=\|f\|_{L_2(0,\infty)}+\|f\|_{\widehat{h}_2^\alpha} \tag{66}$$

Define spaces W_2^α, $\widehat{W}_2^\alpha$ as closures of $D(0,\infty)$ in (65) or (66) respectively.

Theorem 19. a) For all $\alpha\in\mathbb{R}$ the space $\widehat{W}_2^\alpha$ is continuously imbedded in W_2^α, moreover

$$\|f\|_{W_2^\alpha}\leqslant A_1\|f\|_{\widehat{W}_2^\alpha}, \tag{67}$$

with $A_1=\max(1,\sqrt{1+\sin\pi\alpha})$.

б) Let $\sin\pi\alpha\neq-1$ or $\alpha\neq-\frac{1}{2}+2k,\ k\in\mathbb{Z}$. Then the inverse embedding of W_2^α in $\widehat{W}_2^\alpha$ is valid, moreover

$$\|f\|_{\widehat{W}_2^\alpha}\leqslant A_2\|f\|_{W_2^\alpha}, \tag{68}$$

with $A_2=1/\min(1,\sqrt{1+\sin\pi\alpha})$.

в) Let $\sin\pi\alpha\neq-1$, then spaces W_2^α and $\widehat{W}_2^\alpha$ are isomorphic with equivalent norms.

г) Constants in embedding inequalities (67)–(68) are sharp.

In fact this theorem is a direct corollary of results on boundedness and norm estimates in L_2 of Buschman–Erdelyi transmutations of zero order smoothness. At the same manner from unitarity of these operators follows the next

Theorem 20. Norms

$$\|f\|_{W_2^\alpha} = \sum_{j=0}^{s} \|D_-^j f\|_{L_2}, \tag{69}$$

$$\|f\|_{\widehat{W}_2^\alpha} = \sum_{j=0}^{s} \|x^j(-\frac{1}{x}\frac{d}{dx})^j f\|_{L_2} \tag{70}$$

are equivalent for integer $s \in \mathbb{Z}$. Moreover every term in (69) equals to appropriate term in (70) of the same index j.

I. Kipriyanov introduced in [51] function spaces which essentially influenced the theory of partial differential equations with Bessel operators and in more general sense theory of singular and degenerate equations. These spaces are defined by the next way. First consider subset of even functions in $\mathbb{D}(0,\infty)$ with all zero derivatives of odd orders at $x=0$. Denote this set as $\mathbb{D}_c(0,\infty)$ and equipped it with a norm

$$\|f\|_{\widetilde{W}_{2,k}^s} = \|f\|_{L_{2,k}} + \|B_k^{\frac{s}{2}}\|_{L_{2,k}} \tag{71}$$

here s is an even natural number, $B_k^{s/2}$ is an iteration of the Bessel operator. Define Kipriyanov spaces for even s as a closure of $D_c(0,\infty)$ in the norm (71). It is a known fact that equivalent to (71) norm may be defined by [51]

$$\|f\|_{\widetilde{W}_{2,k}^s} = \|f\|_{L_{2,k}} + \|x^s(-\frac{1}{x}\frac{d}{dx})^s f\|_{L_{2,k}} \tag{72}$$

So the norm $\widetilde{W}_{2,k}^s$ may be defined for all s. Essentially this approach is the same as in [51], another approach is based on usage of Hankel transform. Below we adopt the norm (72) for the space $\widetilde{W}_{2,k}^s$.

Define weighted Sobolev norm by

$$\|f\|_{W_{2,k}^s} = \|f\|_{L_{2,k}} + \|D_-^s f\|_{L_{2,k}} \tag{73}$$

and a space $W_{2,k}^s$ as a closure of $\mathbb{D}_c(0,\infty)$ in this norm.

Theorem 21. a) Let $k \neq -n,\ n \in \mathbb{N}$. Then the space $\widetilde{W}_{2,k}^s$ is continuously embedded into $W_{2,k}^s$, and there exist a constant $A_3 > 0$ such that

$$\|f\|_{W_{2,k}^s} \leqslant A_3 \|f\|_{\widetilde{W}_{2,k}^s}, \tag{74}$$

б) Let $k+s \neq -2m_1-1,\ k-s \neq -2m_2-2,\ m_1 \in \mathbb{N}_0,\ m_2 \in \mathbb{N}_0$. Then the inverse embedding holds true of $W_{2,k}^s$ into $\widetilde{W}_{2,k}^s$, and there exist a constant $A_4 > 0$, such that

$$\|f\|_{\widetilde{W}_{2,k}^s} \leqslant A_4 \|f\|_{W_{2,k}^s}. \tag{75}$$

в) If the above mentioned conditions are not valid then embedding theorems under considerations fail.

Corollary 1. Let the next conditions hold true: $k \neq -n,\ n \in \mathbb{N};\ k+s \neq -2m_1 - -1,\ m_1 \in \mathbb{N}_0;\ k-s \neq -2m_2-2,\ m_2 \in \mathbb{N}_0$. Then Kipriyanov spaces may be defined as closure of $D_c(0,\infty)$ in the weighted Sobolev norm (73).

Corollary 2. Sharp constants in embedding theorems (74)–(75) are

$$A_3 = \max(1, \|{}_1S_-{}^{s-1}\|_{L_{2,k}}),\ A_4 = \max(1, \|{}_1P_-{}^{s-1}\|_{L_{2,k}}).$$

It is obvious that the theorem above and its corollaries are direct consequences of estimates for Buschman–Erdelyi transmutations. Sharp constants in embedding theorems (74)–(75) are also direct consequences of estimates for Buschman–Erdelyi transmutations of zero order smoothness. Estimates in $L_{p,\alpha}$ not included in this article allow to consider embedding theorems for general Sobolev and Kipriyanov spaces.

So by applying Buschman–Erdelyi transmutations of zero order smoothness we received an answer to a problem which for a long time was discussed in "folklore": — are Kipriyanov spaces isomorphic to power weighted Sobolev spaces or not? Of course we investigated just the simplest case, results may be generalize to other seminorms, higher dimensions, bounded domains but the principal idea is clear. All that do not in any sense disparage neither essential role nor necessity for applications of Kipriyanov spaces in the theory of partial differential equations.

The importance of Kipriyanov spaces is a special case of the next general principle of L. Kudryavtsev:

"EVERY EQUATION MUST BE INVESTIGATED IN ITS OWN SPACE!"

The proved in this section embedding theorems may be applied to direct transfer of known solution estimates for B–elliptic equations in Kipriyanov spaces (cf. [15],[51]) to new estimates in weighted Sobolev spaces, it is a direct consequence of boundedness and transmutation properties of Buschman–Erdelyi transmutations.

5.2 Solution representations to partial differential equations with Bessel operators.

The above classes of transmutations may be used for deriving explicit formulas for solutions of partial differential equations with Bessel operators via unperturbed equation solutions. An example is the B–elliptic equation of the form

$$\sum_{k=1}^{n} B_{\nu,x_k} u(x_1,\ldots,x_n) = f, \tag{76}$$

and similar B–hyperbolic and B–parabolic equations. This idea early works by Sonine–Poisson–Delsarte transmutations, cf. [1]–[3], [9], [15]. New results follow automatically for new classes of transmutations.

5.3 Cauchy problem for Euler–Poisson–Delsarte equation (EPD).

Consider EPD equation in a half space

$$B_{\alpha,t}u(t,x) = \frac{\partial^2 u}{\partial t^2} + \frac{2\alpha+1}{t}\frac{\partial u}{\partial t} = \Delta_x u + F(t,x),$$

with $t > 0,\ x \in \mathbb{R}^n$. Let us consider a general plan for finding different initial and boundary conditions at $t = 0$ with guaranteed existence of solutions. Define any transmutations $X_{\alpha,t}$ and $Y_{\alpha,t}$ satisfying (19). Suppose that functions $X_{\alpha,t}u = = v(t,x)$, $X_{\alpha,t}F = G(t,x)$ exist. Suppose that unperturbed Cauchy problem

$$\frac{\partial^2 v}{\partial t^2} = \Delta_x v + G,\ v|_{t=0} = \varphi(x),\ v'_t|_{t=0} = \psi(x) \tag{77}$$

is correctly solvable in a half space. Then if $Y_{\alpha,t} = X^{-1}_{\alpha,t}$ then we receive the next initial conditions

$$X_\alpha u|_{t=0} = a(x),\ (X_\alpha u)'|_{t=0} = b(x). \tag{78}$$

By this method the choice of different classes of transmutations (Sonine–Poisson–Delsarte, Buschman–Erdelyi of the first, second and third kinds, Buschman–Erdelyi of the zero order smoothness, unitary transmutations of Sonine–Katrakhov and Poisson–Katrakhov, transmutations with general kernels) will correspond different kinds of initial conditions [29].

In the monograph of Pskhu [58] this method is applied for solving an equation with fractional derivatives with the usage of Stankovic transform. Glushak applied Buschman–Erdelyi operators in [59].

The Buschman–Erdelyi operators were first introduced exactly for EPD equation by Copson. We formulate his result now.

Copson lemma.

Consider partial differential equation with two variables on the plane

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{2\alpha}{x}\frac{\partial u(x,y)}{\partial x} = \frac{\partial^2 u(x,y)}{\partial y^2} + \frac{2\beta}{y}\frac{\partial u(x,y)}{\partial y}$$

(this is EPD equation or B–hyperbolic one in Kipriyanov's terminology) for $x > > 0,\ y > 0$ and $\beta > \alpha > 0$ with boundary conditions on characteristics

$$u(x,0) = f(x), u(0,y) = g(y), f(0) = g(0).$$

It is supposed that the solution $u(x,y)$ is continuously differentiable in the closed first quadrant and has second derivatives in this open quadrant, boundary functions $f(x), g(y)$ are differentiable.

Then if the solution exist the next formulas hold true

$$\frac{\partial u}{\partial y} = 0, y = 0, \frac{\partial u}{\partial x} = 0, x = 0, \tag{79}$$

$$2^\beta \Gamma(\beta + \frac{1}{2}) \int\limits_0^1 f(xt) t^{\alpha+\beta+1} \left(1 - t^2\right)^{\frac{\beta-1}{2}} P^{1-\beta}_{-\alpha} t\, dt = \tag{80}$$

$$= 2^\alpha \Gamma(\alpha + \frac{1}{2}) \int\limits_0^1 g(xt) t^{\alpha+\beta+1} \left(1 - t^2\right)^{\frac{\alpha-1}{2}} P^{1-\alpha}_{-\beta} t\, dt, \tag{81}$$

$$\Downarrow$$

$$g(y) = \frac{2\Gamma(\beta+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta-\alpha)} y^{1-2\beta} \int_0^y x^{2\alpha-1} f(x) \left(y^2-x^2\right)^{\beta-\alpha-1} x\, dx, \qquad (82)$$

here $P_\nu^\mu(z)$ is the Legendre function of the first kind [7].

So the main conclusion from Copson lemma is that data on characteristics can not be taken arbitrary, these functions must be connected by Buschman–Erdelyi operators of the first kind, for more detailed consideration cf. [7].

5.4 Applications to generalized translations.

This class of operators was thoroughly studied by Levitan [16]–[17]. It has many applications to partial differential equations, including Bessel operators [8]. Generalized translations are used for moving singular point from the origin to any location. They are explicitly expressed via transmutations [8]. Due to this fact new classes of transmutations lead to new classed of generalized translations.

5.5 Applications to Dunkl operators.

In recent years Dunkl operators were thoroughly studied. These are difference–differentiation operators consisting of combinations of classical derivatives and finite differences. In higher dimensions Dunkl operators are defined by symmetry and reflection groups. For this class there are many results on transmutations which are of Sonine–Poisson–Delsarte and Buschman–Erdelyi types, cf. [63] and references therein.

5.6 Applications of Buschman–Erdelyi operators to the Radon transform.

It was proved by Ludwig in [55] that the Radon transform in terms of spherical harmonics acts in every harmonics at radial components as Buschman–Erdelyi operators. Let us formulate this result.

Theorem 22. Ludwig theorem ([55],[56]). Let the function $f(x)$ being expanded in $\mathbb{R}^n$ by spherical harmonics

$$f(x) = \sum_{k,l} f_{k,l}(r) Y_{k,l}(\theta). \qquad (83)$$

Then the Radon transform of this function may be calculated as another series in spherical harmonics

$$Rf(x) = g(r,\theta) = \sum_{k,l} g_{k,l}(r) Y_{k,l}(\theta), \qquad (84)$$

$$g_{k,l}(r) = \mathrm{c}(n) \int_r^\infty \left(1-\frac{s^2}{r^2}\right)^{\frac{n-3}{2}} C_l^{\frac{n-2}{2}}\left(\frac{s}{r}\right) f_{k,l}(r) r^{n-2}\, ds, \qquad (85)$$

here $\mathrm{c}(n)$ is some known constant, $C_l^{\frac{n-2}{2}}\left(\frac{s}{r}\right)$ is the Gegenbauer function [49]. The inverse formula is also valid of representing values $f_{k,l}(r)$ via $g_{k,l}(r)$.

The Gegenbauer function may be easily reduced to the Legendre function [49]. So the Ludwig's formula (85) reduce the Radon transform in terms of spherical harmonics series and up to unimportant power and constant terms to Buschman–Erdelyi operators of the first kind.

Exactly this formula in dimension two was developed by Cormack as the first step to the Nobel prize. Special cases of Ludwig's formula proved in 1966 are for any special spherical harmonics and in the simplest case on pure radial function, in this case it is reduced to Sonine–Poisson–Delsarte transmutations of Erdelyi–Kober type. Besides the fact that such formulas are known for about half a century they are rediscovered still... As consequences of the above connections the results may be proved for integral representations, norm estimates, inversion formulas for the Radon transform via Buschman–Erdelyi operators. In particular it makes clear that different kinds of inversion formulas for the Radon transform are at the same time inversion formulas for the Buschman–Erdelyi transmutations of the first kind and vice versa. A useful reference for this approach is [57].

5.7 Application of Buschman–Erdelyi operators to generalized polynomials and spherical harmonics.

It was known for many years that a problem of describing polynomial solutions for B–elliptic equation do not need the new theory. The answer is in the transmutation theory. A simple fact that Sonine–Poisson–Delsarte transmutations transform power function into another power function means that they also transform explicitly so called B–harmonic polynomials into classical harmonic polynomials and vice versa. The same is true for generalized B–harmonics because they are restrictions of B–harmonic polynomials onto the unit sphere. This approach is thoroughly applied by Rubin [64]–[65]. Usage of Buschman–Erdelyi operators refresh this theory with new possibilities.

5.8 Application of Buschman–Erdelyi transmutations for estimation of generalized Hardy operators.

We proved unitarity of shifted Hardy operators (41) and mentioned that it is a known fact from [50]. It is interesting that Hardy operators naturally arise in transmutation theory. Use the theorem 7 with integer parameter which guarantees unitarity for finding more unitary in $L_2(0,\infty)$ integral operators of very simple form.

Theorem 23. The next are pair of unitary mutually inverse integral operators in $L_2(0,\infty)$:

$$U_3 f = f + \int_0^x f(y)\,\frac{dy}{y},\ U_4 f = f + \frac{1}{x}\int_x^\infty f(y)\,dy,$$

$$U_5 f = f + 3x\int_0^x f(y)\,\frac{dy}{y^2},\ U_6 f = f - \frac{3}{x^2}\int_0^x y f(y)\,dy,$$

$$U_7 f = f + \frac{3}{x^2}\int_x^\infty y f(y)\,dy,\ U_8 f = f - 3x\int_x^\infty f(y)\frac{dy}{y^2},$$

$$U_9 f = f + \frac{1}{2}\int_0^x \left(\frac{15x^2}{y^3} - \frac{3}{y}\right) f(y)\,dy,$$

$$U_{10} f = f + \frac{1}{2}\int_x^\infty \left(\frac{15y^2}{x^3} - \frac{3}{x}\right) f(y)\,dy.$$

5.9 Integral operators with more general functions as kernels.

Consider an operator ${}_1S^{\nu}_{0+}$. It has the form

$$ {}_1S^{\nu}_{0+} = \frac{d}{dx}\int_0^x K\left(\frac{x}{y}\right) f(y)\,dy, \tag{86}$$

with kernel K expressed by $K(z) = P_\nu(z)$. Simple properties of special functions lead to the fact that ${}_1S^{\nu}_{0+}$ is a special case of (86) with Gegenbauer function kernel

$$K(z) = \frac{\Gamma(\alpha+1)\ \Gamma(2\beta)}{2^{p-\frac{1}{2}}\Gamma(\alpha+2\beta)\ \Gamma(\beta+\frac{1}{2})}(z^\alpha - 1)^{\beta-\frac{1}{2}} C^{\beta}_{\alpha}(z) \tag{87}$$

with $\alpha = \nu,\ \beta = \frac{1}{2}$ or with Jacobi function kernel

$$K(z) = \frac{\Gamma(\alpha+1)}{2^\rho \Gamma(\alpha+\rho+1)}(z-1)^\rho (z+1)^\sigma P^{(\rho,\sigma)}_{\alpha}(z) \tag{88}$$

with $\alpha = \nu,\ \rho = \sigma = 0$. More general are operators with Gauss hypergeometric function kernel ${}_2F_1$, Mayer G or Fox H function kernels, cf. [26], [52]. For studying such operators inequalities for kernel functions are very useful, e.g. [53]–[54].

Define the first class of generalized operators.

Definition 6. Define Gauss–Buschman–Erdelyi operators by formulas

$$ {}_1F_{0+}(a,b,c)[f] = \frac{1}{2^{c-1}\Gamma(c)}. \tag{89}$$

$$\int\limits_0^x \left(\frac{x}{y}-1\right)^{c-1}\left(\frac{x}{y}+1\right)^{a+b-c} {}_2F_1\left({}^{a,b}_{c}|\frac{1}{2}-\frac{1}{2}\frac{x}{y}\right) f(y)\,dy,$$

$${}_2F_{0+}(a,b,c)[f]=\frac{1}{2^{c-1}\Gamma(c)}. \quad (90)$$

$$\int\limits_0^x \left(\frac{y}{x}-1\right)^{c-1}\left(\frac{y}{x}+1\right)^{a+b-c} {}_2F_1\left({}^{a,b}_{c}|\frac{1}{2}-\frac{1}{2}\frac{y}{x}\right) f(y)\,dy,$$

$${}_1F_{-}(a,b,c)[f]=\frac{1}{2^{c-1}\Gamma(c)}. \quad (91)$$

$$\int\limits_0^x \left(\frac{y}{x}-1\right)^{c-1}\left(\frac{y}{x}+1\right)^{a+b-c} {}_2F_1\left({}^{a,b}_{c}|\frac{1}{2}-\frac{1}{2}\frac{y}{x}\right) f(y)\,dy,$$

$${}_2F_{-}(a,b,c)[f]=\frac{1}{2^{c-1}\Gamma(c)}. \quad (92)$$

$$\int\limits_0^x \left(\frac{x}{y}-1\right)^{c-1}\left(\frac{x}{y}+1\right)^{a+b-c} {}_2F_1\left({}^{a,b}_{c}|\frac{1}{2}-\frac{1}{2}\frac{x}{y}\right) f(y)\,dy,$$

$${}_3F_{0+}[f]=\tfrac{d}{dx}{}_1F_{0+}[f],\ \ {}_4F_{0+}[f]={}_2F_{0+}\frac{d}{dx}[f], \quad (93)$$

$${}_3F_{-}[f]={}_1F_{-}(-\tfrac{d}{dx})[f], {}_4F_{-}[f]=(-\frac{d}{dx}){}_2F_{-}[f]. \quad (94)$$

Symbol ${}_2F_1$ in definitions (90) and (92) means Gauss hypergeometric function on natural domain and in (89) and (91) the main branch of its analytical continuation.

Operators (89)–(92) generalize Buschman–Erdelyi ones (8)–(11) respectively. They reduce to Buschman–Erdelyi for the choice of parameters $a=-(\nu+\mu),\ b==1+\nu-\mu,\ c=1-\mu$. For operators (89)–(92) the above results are generalized with necessary changes. For example they are factorized via more simple operators (93)–(94) with special choice of parameters.

Operators (93)–(94) are generalizations of (13)–(16). For them the next result is true.

Theorem 24. Operators (93)–(94) may be extended to isometric in $L_2(0,\infty)$ if and only if they coincide with Buschman–Erdelyi operators of zero order smoothness (13)–(16) for integer values of $\nu=\frac{1}{2}(b-a-1)$.

This theorem single out Buschman–Erdelyi operators of zero order smoothness at least in the class (89)–(94). Operators (89)–(92) are generalizations of fractional integrals. Analogically may be studied generalizations to (42)–(43), (52), (56)–(57).

More general are operators with G function kernel.

$$ {}_1G_{0+}(\alpha,\beta,\delta,\gamma)[f] = \frac{2^\delta}{\Gamma(1-\alpha)\Gamma(1-\beta)}\cdot \qquad (95)$$
$$\int_0^x (\frac{x}{y}-1)^{-\delta}(\frac{x}{y}+1)^{1+\delta-\alpha-\beta} G_{2\ 2}^{1\ 2}\left(\frac{x}{2y}-\frac{1}{2}|^{\alpha,\beta}_{\gamma,\delta}\right) f(y)\,dy.$$

Another operators are with different interval of integration and parameters of G function. For $\alpha=1-a,\ \beta=1-b, \delta=1-c, \gamma=0$ (95) reduce to (89), for $\alpha=1+ +\nu,\ \beta=-\nu, \delta=\gamma=0$ (95) reduce to Buschman–Erdelyi operators of zero order smoothness ${}_1S^{\nu}_{0+}$.

Further generalizations are in terms of Wright or Fox functions. They lead to Wright–Buschman–Erdelyi and Fox–Buschman–Erdelyi operators. These classes are connected with Sonine–Dimovski and Poisson–Dimovski transmutations [10], [11], and also with generalized fractional integrals introduced by Kiryakova [11].

5.10 Application of Buschman–Erdelyi transmutations in works of V. Katrakhov.

V. Katrakhov found a new approach for boundary value problems for elliptic equations with strong singularities of infinite order. For example for Poisson equation he studied problems with solutions of arbitrary growth. At singular point he proposed the new kind of boundary condition: K—trace. His results are based on constant usage of Buschman–Erdelyi transmutations of the first kind for definition of norms, solution estimates and correctness proofs [60]–[61].

Moreover in joint papers with I. Kipriyanov he introduced and studied new classes of pseudodifferential operators based on transmutational technics [62]. These results were paraphrased in reorganized manner also in [2].

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