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SOME RESULTS FOR ISOTONIC FUNCTIONALS VIA TWO REVERSES OF YOUNG'S INEQUALITY

S. S. DRAGOMIR[1,2]

Abstract. In this paper we obtain some inequalities for isotonic functionals via two reverses of Young's inequality due to Furuichi and Liao, Wu and Zhao.

1. Introduction

Let L be a *linear class* of real-valued functions $g : E \to \mathbb{R}$ having the properties:

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1$, $t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalised if*
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2], [18] and [19]). For other inequalities for isotonic functionals see [1], [4]-[17] and [20]-[23].

We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second $(p_k \geq 0, k \in E)$.

As is known to all, the famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0, 1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called ν-weighted arithmetic-geometric mean inequality.

Kittaneh and Manasrah [14], [15] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a} - \sqrt{b}\right)^2 \leq (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \leq R\left(\sqrt{a} - \sqrt{b}\right)^2 \tag{1.2}$$

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where $a,b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$. The case $\nu=\frac{1}{2}$ reduces (1.2) to an identity and is of no interest.

We recall that *Specht's ratio* is defined by

$$S(h):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty)\\ \\ 1 \text{ if } h=1.\end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to1}S(h)=1,\ S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0,\ h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's

$$S\left(\left(\frac{a}{b}\right)^r\right)a^{1-\nu}b^\nu\le(1-\nu)a+\nu b\le S\left(\frac{a}{b}\right)a^{1-\nu}b^\nu, \tag{1.4}$$

where $a,b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$.

The second inequality in (1.4) is due to Tominaga [24] while the first one is due to Furuichi [12].

It is an open question for the author if in the right hand side of (1.4) we can replace $S\left(\frac{a}{b}\right)$ by $S\left(\left(\frac{a}{b}\right)^R\right)$ where $R=\max\{1-\nu,\nu\}$.

We consider the *Kantorovich's ratio* defined by

$$K(h):=\frac{(h+1)^2}{4h},\ h>0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\ge1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^r\left(\frac{a}{b}\right)a^{1-\nu}b^\nu\le(1-\nu)a+\nu b\le K^R\left(\frac{a}{b}\right)a^{1-\nu}b^\nu \tag{1.6}$$

where $a,b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [25] while the second by Liao et al. [16].

In [25] the authors also showed that

$$K^r(h)\ge S(h^r)\ \text{ for } h>0 \text{ and } r\in\left[0,\frac{1}{2}\right]$$

implying that the lower bound in (1.6) is better than the lower bound from (1.4).

Furuichi [13] also proved the related inequality

$$(1-\nu)a+\nu b\le r\left(\sqrt{a}-\sqrt{b}\right)^2+S\left(\sqrt{\frac{a}{b}}\right)a^{1-\nu}b^\nu \tag{1.7}$$

where $a,b>0,\ \nu\in[0,1]\setminus\left\{\frac{1}{2}\right\},\ r=\min\{1-\nu,\nu\}$ while Liao et al. [16] showed that

$$(1-\nu)a+\nu b\le r\left(\sqrt{a}-\sqrt{b}\right)^2+K^{R'}\left(\sqrt{\frac{a}{b}}\right)a^{1-\nu}b^\nu, \tag{1.8}$$

where r is as above and $R'=\max\{2r,1-2r\}$.

The inequalities (1.7) and (1.8) can be put together as

$$(1.9)\qquad (1-\nu)\,a+\nu b\le r\left(\sqrt{a}-\sqrt{b}\right)^2+D\left(\sqrt{\frac{a}{b}}\right)a^{1-\nu}b^{\nu}$$

where

$$(1.10)\qquad D_\nu\,(h)=\min\left\{S\,(h)\,,K^{R'}\,(h)\right\},\ h>0$$

and $\nu\in[0,1]\setminus\left\{\frac{1}{2}\right\}$, $r=\min\{1-\nu,\nu\}$ and $R'=\max\{2r,1-2r\}$.

We observe that the function D_ν is decreasing on $(0,1)$ and increasing on $(1,\infty)$, $D_\nu\,(h)>1$ for any $h>0$, $\lim_{h\to1}D_\nu\,(h)=1$ and $D_\nu\,(h)=D_\nu\left(\frac{1}{h}\right)$ for any $h>0$.

In this paper we obtain some inequalities for isotonic functionals via two reverses of Young's inequality due to Furuichi and Liao, Wu and Zhao. Applications for integrals and n-tuples of real numbers are also provided.

2. Reverses of Callebaut's Inequality

The functional version of *Callebaut inequality* states that

$$(2.1)\qquad A^2\,(fg)\le A\left(f^{2-\nu}g^{\nu}\right)A\left(f^{\gamma}g^{2-\nu}\right)\le A\left(f^2\right)A\left(g^2\right)$$

provided that f^2, g^2, $f^{2-\nu}g^{\nu}$, $f^{\nu}g^{2-\nu}$, $fg\in L$ for some $\nu\in[0,2]$. For the discrete and integral of one real variable versions see [3].

Let $a,b\in[m,M]\subset(0,\infty)$, then $\sqrt{\frac{m}{M}}\le\sqrt{\frac{a}{b}}\le\sqrt{\frac{M}{m}}$ with $\frac{m}{M}<1<\frac{M}{m}$. If $\sqrt{\frac{a}{b}}\in\left[\sqrt{\frac{m}{M}},1\right)$ then $D_\nu\left(\sqrt{\frac{a}{b}}\right)\le D_\nu\left(\sqrt{\frac{m}{M}}\right)=D_\nu\left(\sqrt{\frac{M}{m}}\right)$. If $\sqrt{\frac{a}{b}}\in\left(1,\sqrt{\frac{M}{m}}\right]$ then also $D_\nu\left(\sqrt{\frac{a}{b}}\right)\le D_\nu\left(\sqrt{\frac{M}{m}}\right)$. Therefore for any $a,b\in[m,M]$ we have from (1.9)

$$(2.2)\qquad (1-\nu)\,a+\nu b\le r\left(\sqrt{a}-\sqrt{b}\right)^2+D_\nu\left(\sqrt{\frac{M}{m}}\right)a^{1-\nu}b^{\nu},$$

where $\nu\in[0,1]\setminus\left\{\frac{1}{2}\right\}$, $r=\min\{1-\nu,\nu\}$ and D_ν is defined by (1.10).

We have:

Theorem 1. *Let $A,B:L\to\mathbb{R}$ be two normalised isotonic functionals. If $f,g:E\to\mathbb{R}$ are such that $f\ge0$, $g>0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$ and*

$$(2.3)\qquad 0<m\le\frac{f}{g}\le M<\infty$$

for some constants m,M, then

$$(2.4)\qquad \begin{aligned}&(1-\nu)\,A\left(f^2\right)B\left(g^2\right)+\nu A\left(g^2\right)B\left(f^2\right)\\ &\le r\left(A\left(f^2\right)B\left(g^2\right)-2A\,(fg)\,B\,(fg)+A\left(g^2\right)B\left(f^2\right)\right)\\ &+D_\nu\left(\frac{M}{m}\right)A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right).\end{aligned}$$

Proof. For any $x,y\in E$ we have

$$m^2\le\frac{f^2\,(x)}{g^2\,(x)},\frac{f^2\,(y)}{g^2\,(y)}\le M^2.$$

If we use the inequalities (2.2) for

$$a=\frac{f^{2}(x)}{g^{2}(x)},\ b=\frac{f^{2}(y)}{g^{2}(y)},$$

then we get

$$\begin{aligned}(2.5)\qquad &(1-\nu)\frac{f^{2}(x)}{g^{2}(x)}+\nu\frac{f^{2}(y)}{g^{2}(y)}\\ &\leq r\left(\frac{f(x)}{g(x)}-\frac{f(y)}{g(y)}\right)^{2}+D_{\nu}\left(\frac{M}{m}\right)\left(\frac{f^{2}(x)}{g^{2}(x)}\right)^{1-\nu}\left(\frac{f^{2}(y)}{g^{2}(y)}\right)^{\nu}\end{aligned}$$

for any $x,y\in E$.

Now, if we multiply (2.5) by $g^{2}(x)\,g^{2}(y)>0$ then we get

$$\begin{aligned}(2.6)\qquad &(1-\nu)f^{2}(x)\,g^{2}(y)+\nu g^{2}(x)\,f^{2}(y)\\ &\leq r\left(f^{2}(x)\,g^{2}(y)-2f(x)\,g(x)\,f(y)\,g(y)+f^{2}(y)\,g^{2}(x)\right)\\ &+D_{\nu}\left(\frac{M}{m}\right)f^{2(1-\nu)}(x)\,g^{2\nu}(x)\,f^{2\nu}(y)\,g^{2(1-\nu)}(y)\end{aligned}$$

for any $x,y\in E$.

Fix $y\in E$. Then by (2.6) we have in the order of L that

$$\begin{aligned}(2.7)\qquad &(1-\nu)g^{2}(y)\,f^{2}+\nu f^{2}(y)\,g^{2}\\ &\leq r\left(g^{2}(y)\,f^{2}-2f(y)\,g(y)\,fg+f^{2}(y)\,g^{2}\right)\\ &+D_{\nu}\left(\frac{M}{m}\right)f^{2\nu}(y)\,g^{2(1-\nu)}(y)\,f^{2(1-\nu)}g^{2\nu}.\end{aligned}$$

If we take the functional A in (2.7) then we get

$$\begin{aligned}(2.8)\qquad &(1-\nu)g^{2}(y)\,A\left(f^{2}\right)+\nu f^{2}(y)\,A\left(g^{2}\right)\\ &\leq r\left(g^{2}(y)\,A\left(f^{2}\right)-2f(y)\,g(y)\,A(fg)+f^{2}(y)\,A\left(g^{2}\right)\right)\\ &+D_{\nu}\left(\frac{M}{m}\right)f^{2\nu}(y)\,g^{2(1-\nu)}(y)\,A\left(f^{2(1-\nu)}g^{2\nu}\right)\end{aligned}$$

for any $y\in E$.

This inequality can be written in the order of L as

$$\begin{aligned}(2.9)\qquad &(1-\nu)A\left(f^{2}\right)g^{2}+\nu A\left(g^{2}\right)f^{2}\\ &\leq r\left(A\left(f^{2}\right)g^{2}-2A(fg)\,fg+A\left(g^{2}\right)f^{2}\right)\\ &+D_{\nu}\left(\frac{M}{m}\right)A\left(f^{2(1-\nu)}g^{2\nu}\right)f^{2\nu}g^{2(1-\nu)}.\end{aligned}$$

Now, if we take the functional B in (2.9), then we get the desired result (2.4). $\square$

The following reverse Callebaut type inequality holds:

Corollary 1. *Let $A:L\to\mathbb{R}$ be a normalised isotonic functional. If $f,g:E\to\mathbb{R}$ are such that $f\geq 0$, $g>0$, f^{2}, g^{2}, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$ and the condition (2.3) is valid, then*

(2.10)

$$(1-2r)\,A\left(f^{2}\right)A\left(g^{2}\right)+2rA^{2}(fg)\leq D_{\nu}\left(\frac{M}{m}\right)A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right).$$

where $\nu \in [0,1] \setminus \left\{\frac{1}{2}\right\}$, $r = \min\{1-\nu, \nu\}$ *and* D_ν *is defined by (1.10).*

3. A Reverse of Hölder's and Related Inequalities

First, observe that if $a, b > 0$ and

$$0 < L^{-1} \le \frac{a}{b} \le L < \infty, \tag{3.1}$$

for some $L > 1$, then by (2.2) we have

$$(1-\nu)a + \nu b \le r\left(\sqrt{a} - \sqrt{b}\right)^2 + D_\nu\left(\sqrt{L}\right) a^{1-\nu} b^\nu, \tag{3.2}$$

where $\nu \in [0,1] \setminus \left\{\frac{1}{2}\right\}$, $r = \min\{1-\nu, \nu\}$ and D_ν is defined by (1.10).

Theorem 2. *Let* $A : L \to \mathbb{R}$ *be a normalised isotonic functional and* $p, q > 1$, $p, q \neq 2$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *If* f, $g : E \to \mathbb{R}$ *are such that* fg, f^p, g^q, $f^{\frac{p}{2}} g^{\frac{q}{2}} \in L$ *and for some constants* m_1, M_1, m_2, M_2,

$$0 < m_1 \le f \le M_1 < \infty, \; 0 < m_2 \le g \le M_2 < \infty, \tag{3.3}$$

then

$$\begin{aligned} \frac{A\left(f^{\frac{p}{2}} g^{\frac{q}{2}}\right)}{\sqrt{A(f^p)\, A(g^q)}} &\le \frac{(1-2s)\sqrt{A(f^p)\, A(g^q)} + 2sA\left(f^{\frac{p}{2}} g^{\frac{q}{2}}\right)}{\sqrt{A(f^p)\, A(g^q)}} \\ &\le D_{p,q}\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}} \left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right) \frac{A(fg)}{[A(f^p)]^{1/p}\, [A(g^q)]^{1/q}}, \end{aligned} \tag{3.4}$$

where $s = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$, $U = \max\{2s, 1-2r\}$ *and*

$$D_{p,q}(h) = \min\left\{S(h), K^U(h)\right\}, \; h > 0. \tag{3.5}$$

Proof. Observe that, by (3.3) we have

$$m_1^p \le A(f^p) \le M_1^p \text{ and } m_2^q \le A(g^q) \le M_2^q.$$

Also

$$\left(\frac{m_1}{M_1}\right)^p \le \frac{f^p}{A(f^p)} \le \left(\frac{M_1}{m_1}\right)^p$$

and

$$\left(\frac{m_2}{M_2}\right)^q \le \frac{g^q}{A(g^q)} \le \left(\frac{M_2}{m_2}\right)^q$$

giving that

$$\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1} \le \frac{\frac{f^p}{A(f^p)}}{\frac{g^q}{A(g^q)}} \le \left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q.$$

Using the inequality (3.2) for $\nu=\frac{1}{q}$, $a=\frac{f^p}{A(f^p)}$, $b=\frac{g^q}{A(g^q)}$ and $L=\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q$, we get

$$\begin{aligned}
&\frac{1}{p}\frac{f^p}{A\left(f^p\right)}+\frac{1}{q}\frac{g^q}{A\left(g^q\right)} \\
&\leq s\left(\frac{f^p}{A\left(f^p\right)}-2\frac{f^{\frac{p}{2}}g^{\frac{q}{2}}}{\sqrt{A\left(f^p\right)A\left(g^q\right)}}+\frac{g^q}{A\left(g^q\right)}\right) \\
&+D_{p,q}\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right)\frac{fg}{\left[A\left(f^p\right)\right]^{1/p}\left[A\left(g^q\right)\right]^{1/q}}.
\end{aligned} \tag{3.6}$$

If we take the functional A in (3.6), then we get

$$\begin{aligned}
&\frac{1}{p}\frac{A\left(f^p\right)}{A\left(f^p\right)}+\frac{1}{q}\frac{A\left(g^q\right)}{A\left(g^q\right)} \\
&\leq r\left(\frac{A\left(f^p\right)}{A\left(f^p\right)}-2\frac{A\left(f^{\frac{p}{2}}g^{\frac{q}{2}}\right)}{\sqrt{A\left(f^p\right)A\left(g^q\right)}}+\frac{A\left(g^q\right)}{A\left(g^q\right)}\right) \\
&+D_{\nu}\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right)\frac{A\left(fg\right)}{\left[A\left(f^p\right)\right]^{1/p}\left[A\left(g^q\right)\right]^{1/q}},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
1 &\leq 2s\left(1-\frac{A\left(f^{\frac{p}{2}}g^{\frac{q}{2}}\right)}{\sqrt{A\left(f^p\right)A\left(g^q\right)}}\right) \\
&+D_{p,q}\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right)\frac{A\left(fg\right)}{\left[A\left(f^p\right)\right]^{1/p}\left[A\left(g^q\right)\right]^{1/q}}.
\end{aligned} \tag{3.7}$$

The inequality (3.7) is equivalent with the second part of (3.4) upon a simple calculation.

The first inequality in (3.4) follows by Schwarz's inequality for functionals

$$A\left(h^2\right)A\left(\ell^2\right)\geq A^2\left(h\ell\right)$$

provided $h^2,\ell^2,h\ell\in L$. □

Further, observe that if $a,b>0$ and

$$0<l^{-1}\leq\frac{a}{b}\leq L<\infty, \tag{3.8}$$

for some $L,l>0$ with $Ll>1$,then

$$D_\nu\left(\frac{a}{b}\right)\leq\max\left\{D_\nu\left(l^{-1}\right),D_\nu\left(L\right)\right\}=\max\left\{D_\nu\left(l\right),D_\nu\left(L\right)\right\}$$

where $\nu\in[0,1]\setminus\left\{\frac{1}{2}\right\}$, $r=\min\{1-\nu,\nu\}$ and D_ν is defined by (1.10).

By (1.9) we have

$$(1-\nu)\,a+\nu b\leq r\left(\sqrt{a}-\sqrt{b}\right)^2+\max\left\{D_\nu\left(\sqrt{l}\right),D_\nu\left(\sqrt{L}\right)\right\}a^{1-\nu}b^\nu \tag{3.9}$$

where $\nu\in[0,1]\setminus\left\{\frac{1}{2}\right\}$, $r=\min\{1-\nu,\nu\}$ and D_ν is defined by (1.10).

The following result also holds:

Theorem 3. *Let A, $B : L \to \mathbb{R}$ be two normalised isotonic functionals and p, $q > 1$, $p, q \neq 2$ with $\frac{1}{p} + \frac{1}{q} = 1$. If f, g, $u, v : E \to \mathbb{R}$ are such that u, $v \geq 0$, u, v, uf, vg, uf^p, vg^q, $f^{\frac{p}{2}}u$, $g^{\frac{q}{2}}v \in L$ and the conditions (3.3) hold, then*

$$\begin{aligned}(3.10)\qquad &\frac{1}{p}A\left(f^p u\right)B\left(v\right) + \frac{1}{q}A\left(u\right)B\left(g^q v\right) \\ &\leq s\left(A\left(f^p u\right)B\left(v\right) - 2A\left(f^{\frac{p}{2}}u\right)B\left(g^{\frac{q}{2}}v\right) + A\left(u\right)B\left(g^q v\right)\right) \\ &+ \max\left\{D_{p,q}\left(\sqrt{\frac{M_2^q}{m_1^p}}\right), D_{p,q}\left(\sqrt{\frac{M_1^p}{m_2^q}}\right)\right\}A\left(fu\right)B\left(gv\right)\end{aligned}$$

where $s = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$, $U = \max\left\{2s, 1-2r\right\}$ and $D_{p,q}\left(h\right)$ is defined by (3.5).

In particular, we have

$$\begin{aligned}(3.11)\qquad &\frac{1}{p}A\left(f^p u\right)A\left(v\right) + \frac{1}{q}A\left(u\right)A\left(g^q v\right) \\ &\leq s\left(A\left(f^p u\right)A\left(v\right) - 2A\left(f^{\frac{p}{2}}u\right)A\left(g^{\frac{q}{2}}v\right) + A\left(u\right)A\left(g^q v\right)\right) \\ &+ \max\left\{D_{p,q}\left(\sqrt{\frac{M_2^q}{m_1^p}}\right), D_{p,q}\left(\sqrt{\frac{M_1^p}{m_2^q}}\right)\right\}A\left(fu\right)A\left(gv\right).\end{aligned}$$

Proof. Observe that, by (3.3) we have

$$\frac{m_1^p}{M_2^q} \leq \frac{f^p\left(x\right)}{g^q\left(y\right)} \leq \frac{M_1^p}{m_2^q}$$

for any $x, y \in E$.

Now, if we write the inequality (3.9) for $l = \frac{M_2^q}{m_1^p}$, $L = \frac{M_1^p}{m_2^q}$, $a = f^p\left(x\right)$, $b = g^q\left(y\right)$ and $\nu = \frac{1}{q}$, then we get

$$\begin{aligned}(3.12)\qquad &\frac{1}{p}f^p\left(x\right) + \frac{1}{q}g^q\left(y\right) \\ &\leq r\left(f^p\left(x\right) - 2f^{\frac{p}{2}}\left(x\right)g^{\frac{q}{2}}\left(y\right) + g^q\left(y\right)\right) \\ &+ \max\left\{D_\nu\left(\sqrt{\frac{M_2^q}{m_1^p}}\right), D_\nu\left(\sqrt{\frac{M_1^p}{m_2^q}}\right)\right\}f\left(x\right)g\left(y\right)\end{aligned}$$

for any $x, y \in E$.

If we multiply (3.12) by $u\left(x\right)v\left(y\right) \geq 0$ we get

$$\begin{aligned}(3.13)\qquad &\frac{1}{p}v\left(y\right)f^p u + \frac{1}{q}g^q\left(y\right)v\left(y\right)u \\ &\leq r\left(v\left(y\right)f^p u - 2g^{\frac{q}{2}}\left(y\right)v\left(y\right)f^{\frac{p}{2}}u + g^q\left(y\right)v\left(y\right)u\right) \\ &+ \max\left\{D_\nu\left(\sqrt{\frac{M_2^q}{m_1^p}}\right), D_\nu\left(\sqrt{\frac{M_1^p}{m_2^q}}\right)\right\}g\left(y\right)v\left(y\right)fu\end{aligned}$$

in the order of L, where $y \in E$.

If we take the functional A in (3.13), then we get

$$\begin{aligned}(3.14)\qquad &\frac{1}{p}A\left(f^{p}u\right)v+\frac{1}{q}A\left(u\right)g^{q}v\\ &\leq r\left(A\left(f^{p}u\right)v-2A\left(f^{\frac{p}{2}}u\right)g^{\frac{q}{2}}v+A\left(u\right)g^{q}v\right)\\ &+\max\left\{D_{\nu}\left(\sqrt{\frac{M_{2}^{q}}{m_{1}^{p}}}\right),D_{\nu}\left(\sqrt{\frac{M_{1}^{p}}{m_{2}^{q}}}\right)\right\}A\left(fu\right)gv\end{aligned}$$

in the order of L.

Finally, if we take the functional B in (3.14) then we get the desired result (3.10). □

Remark 1. *We observe that (3.10) can be written in an equivalent form as*

$$\begin{aligned}(3.15)\qquad &\left(\frac{1}{p}-s\right)A\left(f^{p}u\right)B\left(v\right)+\left(\frac{1}{q}-s\right)A\left(u\right)B\left(g^{q}v\right)+2sA\left(f^{\frac{p}{2}}u\right)B\left(g^{\frac{q}{2}}v\right)\\ &\leq\max\left\{D_{p,q}\left(\sqrt{\frac{M_{2}^{q}}{m_{1}^{p}}}\right),D_{p,q}\left(\sqrt{\frac{M_{1}^{p}}{m_{2}^{q}}}\right)\right\}A\left(fu\right)B\left(gv\right)\end{aligned}$$

where $s=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$, $U=\max\left\{2s,1-2s\right\}$ *and* $D_{p,q}\left(h\right)$ *is defined by (3.5).*

In particular, we have

$$\begin{aligned}(3.16)\qquad &\left(\frac{1}{p}-s\right)A\left(f^{p}u\right)A\left(v\right)+\left(\frac{1}{q}-s\right)A\left(u\right)A\left(g^{q}v\right)+2sA\left(f^{\frac{p}{2}}u\right)A\left(g^{\frac{q}{2}}v\right)\\ &\leq\max\left\{D_{p,q}\left(\sqrt{\frac{M_{2}^{q}}{m_{1}^{p}}}\right),D_{p,q}\left(\sqrt{\frac{M_{1}^{p}}{m_{2}^{q}}}\right)\right\}A\left(fu\right)A\left(gv\right).\end{aligned}$$

By taking $u=g$ and $v=f$ in (3.16) we get

$$\begin{aligned}(3.17)\qquad &\left(\frac{1}{p}-s\right)A\left(f^{p}g\right)A\left(f\right)+\left(\frac{1}{q}-s\right)A\left(g\right)A\left(g^{q}f\right)+2sA\left(f^{\frac{p}{2}}g\right)A\left(g^{\frac{q}{2}}f\right)\\ &\leq\max\left\{D_{p,q}\left(\sqrt{\frac{M_{2}^{q}}{m_{1}^{p}}}\right),D_{p,q}\left(\sqrt{\frac{M_{1}^{p}}{m_{2}^{q}}}\right)\right\}A^{2}\left(fg\right),\end{aligned}$$

provided $f,\ g,\ f^{p}g,\ g^{q}f,\ f^{\frac{p}{2}}g,\ g^{\frac{q}{2}}f,\ fg\in L$.

By taking $u=f$ and $v=g$ in (3.16) we get

$$\begin{aligned}(3.18)\qquad &\left(\frac{1}{p}-s\right)A\left(f^{p+1}\right)A\left(g\right)+\left(\frac{1}{q}-s\right)A\left(f\right)A\left(g^{q+1}\right)\\ &+2sA\left(f^{\frac{p}{2}+1}\right)A\left(g^{\frac{q}{2}+1}\right)\\ &\leq\max\left\{D_{p,q}\left(\sqrt{\frac{M_{2}^{q}}{m_{1}^{p}}}\right),D_{p,q}\left(\sqrt{\frac{M_{1}^{p}}{m_{2}^{q}}}\right)\right\}A\left(f^{2}\right)A\left(g^{2}\right),\end{aligned}$$

provided that $f,\ g,\ f^{2},\ g^{2},\ f^{p+1},\ g^{q+1},\ f^{\frac{p}{2}+1},\ g^{\frac{q}{2}+1}\in L$.

If we take in (3.16) $f=g=\frac{\ell}{h}$, $M_1=M_2=M$, $m_1=m_2=m$, and $u=v=h^2$, then we get

$$\begin{aligned}(3.19)\qquad &\left[\left(\frac{1}{p}-s\right)A\left(\ell^p h^{2-p}\right)+\left(\frac{1}{q}-s\right)A\left(\ell^q h^{2-q}\right)\right]A\left(h^2\right)\\ &+2sA\left(\ell^{\frac{p}{2}}h^{2-\frac{p}{2}}\right)A\left(\ell^{\frac{q}{2}}h^{2-\frac{q}{2}}\right)\\ &\le \max\left\{D_{p,q}\left(\sqrt{\frac{M^q}{m^p}}\right),D_{p,q}\left(\sqrt{\frac{M^p}{m^q}}\right)\right\}A^2\left(\ell h\right),\end{aligned}$$

provided that h^2, ℓh, $\ell^p h^{2-p}$, $\ell^q h^{2-q}$, $\ell^{\frac{p}{2}}h^{2-\frac{p}{2}}$, $\ell^{\frac{q}{2}}h^{2-\frac{q}{2}}\in L$ and $0<m\le\frac{\ell}{h}\le M<\infty$.

4. Applications for Integrals

Let $(\Omega,\mathcal{A},\mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\ge 0$ for μ -a.e.(almost every) $x\in\Omega$, consider the Lebesgue space

$$L_w(\Omega,\mu):=\{f:\Omega\to\mathbb{R},\ f\text{ is }\mu\text{-measurable and }\int_\Omega |f(x)|\,w(x)\,d\mu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega wd\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$. The same for other integrals involved below. We assume that $\int_\Omega wd\mu=1$.

Assume that f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L_w(\Omega,\mu)$ for some $\nu\in[0,1]$, then by (2.10) we have

$$\begin{aligned}(4.1)\qquad &(1-2r)\int_\Omega f^2wd\mu\int_\Omega g^2wd\mu+2r\left(\int_\Omega fgwd\mu\right)^2\\ &\le D_\nu\left(\frac{M}{m}\right)\int_\Omega f^{2(1-\nu)}g^{2\nu}wd\mu\int_\Omega f^{2\nu}g^{2(1-\nu)}wd\mu.\end{aligned}$$

provided

$$(4.2)\qquad 0<m\le\frac{f}{g}\le M<\infty\ \mu\text{-a.e. on }\Omega,$$

for some constants $m,M,$ where $\nu\in[0,1]\setminus\left\{\frac{1}{2}\right\}$, $r=\min\{1-\nu,\nu\}$ and D_ν is defined by (1.10).

If f,g satisfy (4.2) and g^2, fg, f^pg^{2-p}, f^qg^{2-q}, $f^{\frac{p}{2}}g^{2-\frac{p}{2}}$, $f^{\frac{q}{2}}g^{2-\frac{q}{2}}\in L_w(\Omega,\mu)$, where $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$, then by (3.19) we have

$$\begin{aligned}(4.3)\qquad &\left[\left(\frac{1}{p}-s\right)\int_\Omega f^pg^{2-p}wd\mu+\left(\frac{1}{q}-s\right)\int_\Omega f^qg^{2-q}wd\mu\right]\int_\Omega g^2wd\mu\\ &+2s\int_\Omega f^{\frac{p}{2}}g^{2-\frac{p}{2}}wd\mu\int_\Omega f^{\frac{q}{2}}g^{2-\frac{q}{2}}wd\mu\\ &\le\max\left\{D_{p,q}\left(\sqrt{\frac{M^q}{m^p}}\right),D_{p,q}\left(\sqrt{\frac{M^p}{m^q}}\right)\right\}\left(\int_\Omega fgwd\mu\right)^2.\end{aligned}$$

For some constants m_1, M_1, m_2, M_2, assume that

$$(4.4)\qquad 0<m_1\le f\le M_1<\infty,\ 0<m_2\le g\le M_2<\infty\ \mu\text{-a.e. on }\Omega,$$

then from (3.4) we have

$$\frac{\int_{\Omega}f^{\frac{p}{2}}g^{\frac{q}{2}}wd\mu}{\sqrt{\int_{\Omega}f^{p}wd\mu\int_{\Omega}g^{q}wd\mu}} \leq \frac{(1-2s)\sqrt{\int_{\Omega}f^{p}wd\mu\int_{\Omega}g^{q}wd\mu}+2s\int_{\Omega}f^{\frac{p}{2}}g^{\frac{q}{2}}wd\mu}{\sqrt{\int_{\Omega}f^{p}wd\mu\int_{\Omega}g^{q}wd\mu}} \leq D_{p,q}\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right)\frac{\int_{\Omega}fgwd\mu}{\left[\int_{\Omega}f^{p}wd\mu\right]^{1/p}\left[\int_{\Omega}g^{q}wd\mu\right]^{1/q}}, \tag{4.5}$$

where $s=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$, $U=\max\{2s,1-2r\}$ and

$$D_{p,q}(h)=\min\left\{S(h),K^{U}(h)\right\},\ h>0.$$

If f, g satisfy (4.4), then by (3.17) and (3.18) we have

$$\left(\frac{1}{p}-s\right)\int_{\Omega}f^{p}gwd\mu\int_{\Omega}fwd\mu+\left(\frac{1}{q}-s\right)\int_{\Omega}gwd\mu\int_{\Omega}g^{q}fwd\mu + 2s\int_{\Omega}f^{\frac{p}{2}}gwd\mu\int_{\Omega}g^{\frac{q}{2}}fwd\mu \leq \max\left\{D_{p,q}\left(\sqrt{\frac{M_2^{q}}{m_1^{p}}}\right),D_{p,q}\left(\sqrt{\frac{M_1^{p}}{m_2^{q}}}\right)\right\}\left(\int_{\Omega}fgwd\mu\right)^{2}, \tag{4.6}$$

and

$$\left(\frac{1}{p}-s\right)\int_{\Omega}f^{p+1}wd\mu\int_{\Omega}gwd\mu+\left(\frac{1}{q}-s\right)\int_{\Omega}fwd\mu\int_{\Omega}g^{q+1}wd\mu + 2s\int_{\Omega}f^{\frac{p}{2}+1}wd\mu\int_{\Omega}g^{\frac{q}{2}+1}wd\mu \leq \max\left\{D_{p,q}\left(\sqrt{\frac{M_2^{q}}{m_1^{p}}}\right),D_{p,q}\left(\sqrt{\frac{M_1^{p}}{m_2^{q}}}\right)\right\}\int_{\Omega}f^{2}wd\mu\int_{\Omega}g^{2}wd\mu \tag{4.7}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$, $s=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$, $U=\max\{2s,1-2r\}$ and

$$D_{p,q}(h)=\min\left\{S(h),K^{U}(h)\right\},\ h>0.$$

5. Applications for Real Numbers

We consider the n-tuples of positive numbers $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ and the probability distribution $p=(p_1,...,p_n)$, i.e. $p_i\geq 0$ for any $i\in\{1,...,n\}$ with $\sum_{i=1}^{n}p_i=1$.

If we use the inequality (4.1) for the counting discrete measure and assume that

$$0<m\leq\frac{a_i}{b_i}\leq M<\infty \text{ for any } i\in\{1,...,n\}, \tag{5.1}$$

some constants m, M, then we have

$$(1-2r)\sum_{i=1}^{n}p_i a_i^2\sum_{i=1}^{n}p_i^2 b_i+2r\left(\sum_{i=1}^{n}p_i a_i b_i\right)^2 \leq D_\nu\left(\frac{M}{m}\right)\sum_{i=1}^{n}p_i a_i^{2(1-\nu)}b_i^{2\nu}\sum_{i=1}^{n}p_i a_i^{2\nu}b_i^{2(1-\nu)}, \tag{5.2}$$

where $\nu\in[0,1]\setminus\left\{\frac{1}{2}\right\}$, $r=\min\{1-\nu,\nu\}$ and D_ν is defined by (1.10).

From the inequality (4.3), if $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ satisfy (5.1), then we also have

$$\left[\left(\frac{1}{p}-s\right)\sum_{i=1}^{n}p_i a_i^p b_i^{2-p}+\left(\frac{1}{q}-s\right)\sum_{i=1}^{n}p_i a_i^q b_i^{2-q}\right]\sum_{i=1}^{n}p_i b_i^2 + 2s\sum_{i=1}^{n}p_i a_i^{\frac{p}{2}}b_i^{2-\frac{p}{2}}\sum_{i=1}^{n}p_i a_i^{\frac{q}{2}}b_i^{2-\frac{q}{2}} \leq \max\left\{D_{p,q}\left(\sqrt{\frac{M^q}{m^p}}\right),D_{p,q}\left(\sqrt{\frac{M^p}{m^q}}\right)\right\}\left(\sum_{i=1}^{n}p_i a_i b_i\right)^2. \tag{5.3}$$

Assume that

$$0<m_1\leq a_i\leq M_1<\infty,\ 0<m_2\leq b_i\leq M_2<\infty \text{ for any } i\in\{1,...,n\}, \tag{5.4}$$

then by (4.5), (4.6) and (4.7) we have

$$\frac{\sum_{i=1}^{n}p_i a_i^{\frac{p}{2}}b_i^{\frac{q}{2}}}{\sqrt{\sum_{i=1}^{n}p_i a_i^p\sum_{i=1}^{n}p_i b_i^q}} \leq \frac{(1-2s)\sqrt{\sum_{i=1}^{n}p_i a_i^p\sum_{i=1}^{n}p_i b_i^q}+2s\sum_{i=1}^{n}p_i a_i^{\frac{p}{2}}b_i^{\frac{q}{2}}}{\sqrt{\sum_{i=1}^{n}p_i a_i^p\sum_{i=1}^{n}p_i b_i^q}} \leq D_{p,q}\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right)\frac{\sum_{i=1}^{n}p_i a_i b_i}{\left[\sum_{i=1}^{n}p_i a_i^p\right]^{1/p}\left[\sum_{i=1}^{n}p_i b_i^q\right]^{1/q}}, \tag{5.5}$$

$$\left(\frac{1}{p}-s\right)\sum_{i=1}^{n}p_i a_i^p b_i\sum_{i=1}^{n}p_i a_i+\left(\frac{1}{q}-s\right)\sum_{i=1}^{n}p_i b_i\sum_{i=1}^{n}p_i a_i b_i^q + 2s\sum_{i=1}^{n}p_i a_i^{\frac{p}{2}}b_i\sum_{i=1}^{n}p_i a_i b_i^{\frac{q}{2}} \leq \max\left\{D_{p,q}\left(\sqrt{\frac{M_2^q}{m_1^p}}\right),D_{p,q}\left(\sqrt{\frac{M_1^p}{m_2^q}}\right)\right\}\left(\sum_{i=1}^{n}p_i a_i b_i\right)^2, \tag{5.6}$$

and

$$\left(\frac{1}{p}-s\right)\sum_{i=1}^{n}p_ia_i^{p+1}\sum_{i=1}^{n}p_ib_i+\left(\frac{1}{q}-s\right)\sum_{i=1}^{n}p_ia_i\sum_{i=1}^{n}p_ib_i^{q+1}$$
$$+2s\sum_{i=1}^{n}p_ia_i^{\frac{p}{2}+1}\sum_{i=1}^{n}p_ib_i^{\frac{q}{2}+1}$$
$$\leq\max\left\{D_{p,q}\left(\sqrt{\frac{M_2^q}{m_1^p}}\right),D_{p,q}\left(\sqrt{\frac{M_1^p}{m_2^q}}\right)\right\}\sum_{i=1}^{n}p_ia_i^2\sum_{i=1}^{n}p_ib_i^2, \tag{5.7}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$, $s=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$, $U=\max\{2s,1-2r\}$ and

$$D_{p,q}(h)=\min\left\{S(h),K^U(h)\right\},\ h>0.$$

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[1]Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa