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A NOTE ON YOUNG'S INEQUALITY

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Abstract. In this note we obtain two new reverses of Young's inequality.

1. Introduction

The famous *Young inequality* for scalars says that if a, $b>0$ and $\nu\in[0,1]$, then

$$a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b \tag{1.1}$$

with equality if and only if $a=b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality*.

Kittaneh and Manasrah [3], [4] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^{2}\leq(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq R\left(\sqrt{a}-\sqrt{b}\right)^{2} \tag{1.2}$$

where a, $b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$. The case $\nu=\frac{1}{2}$ reduces (1.2) to an identity and is of no interest.

We recall that *Specht's ratio* is defined by [6]

$$S\left(h\right):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)}\text{ if }h\in(0,1)\cup(1,\infty),\\[2ex] 1\text{ if }h=1.\end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.4}$$

where a, $b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$.

The second inequality in (1.4) is due to Tominaga [7] while the first one is due to Furuichi [1].

It is an open question for the author if in the right hand side of (1.4) we can replace $S\left(\frac{a}{b}\right)$ by $S\left(\left(\frac{a}{b}\right)^{R}\right)$ where $R=\max\{1-\nu,\nu\}$.

We consider the *Kantorovich's constant* defined by

$$K\left(h\right):=\frac{(h+1)^{2}}{4h},\ h>0. \tag{1.5}$$

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The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\geq 1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^{r}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{1.6}$$

where $a,\,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [8] while the second by Liao et al. [5].

In [8] the authors also showed that $K^{r}(h)\geq S(h^{r})$ for $h>0$ and $r\in\left[0,\frac{1}{2}\right]$ implying that the lower bound in (1.6) is better than the lower bound from (1.4).

In this note we obtain two new reverses of Young's inequality.

2. Main Results

We have the following result:

Lemma 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a convex function on the interval I, $a,\,b\in\mathring{I}$, the interior of I, with $a<b$ and $\nu\in[0,1]$. Then*

$$\begin{aligned}\nu(1-\nu)(b-a)&\left[f'_{+}((1-\nu)a+\nu b)-f'_{-}((1-\nu)a+\nu b)\right]\\&\leq(1-\nu)f(a)+\nu f(b)-f((1-\nu)a+\nu b)\\&\leq\nu(1-\nu)(b-a)\left[f'_{-}(b)-f'_{+}(a)\right].\end{aligned} \tag{2.1}$$

In particular, we have

$$\begin{aligned}\frac{1}{4}(b-a)\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right]&\leq\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\\&\leq\frac{1}{4}(b-a)\left[f'_{-}(b)-f'_{+}(a)\right].\end{aligned} \tag{2.2}$$

The constant $\frac{1}{4}$ is best possible in both inequalities from (2.2).

Proof. The case $\nu=0$ or $\nu=1$ reduces to equality in (2.1).

Since f is convex on I it follows that the function is differentiable on $\mathring{I}$ except a countably number of points, the lateral derivatives $f'_{\pm}$ exists in each point of $\mathring{I}$, they are increasing on $\mathring{I}$ and $f'_{-}\leq f'_{+}$ on $\mathring{I}$.

For any $x,\,y\in\mathring{I}$ we have for the Lebesgue integral

$$f(x)=f(y)+\int_{y}^{x}f'(s)\,ds=f(y)+(x-y)\int_{0}^{1}f'((1-t)y+tx)\,dt. \tag{2.3}$$

Assume that $a<b$ and $\nu\in(0,1)$. By (2.3) we have

$$\begin{aligned}&f((1-\nu)a+\nu b)\\&=f(a)+\nu(b-a)\int_{0}^{1}f'((1-t)a+t((1-\nu)a+\nu b))\,dt\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}&f((1-\nu)a+\nu b)\\&=f(b)-(1-\nu)(b-a)\int_{0}^{1}f'((1-t)b+t((1-\nu)a+\nu b))\,dt.\end{aligned} \tag{2.5}$$

If we multiply (2.4) by $1-\nu$, (2.4) by ν and add the obtained equalities, then we get

$$\begin{aligned} f\left((1-\nu)a+\nu b\right) &= (1-\nu)f(a)+\nu f(b) \\ &\quad + (1-\nu)\nu(b-a)\int_0^1 f'\left((1-t)a+t\left((1-\nu)a+\nu b\right)\right)dt \\ &\quad - (1-\nu)\nu(b-a)\int_0^1 f'\left((1-t)b+t\left((1-\nu)a+\nu b\right)\right)dt, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &(1-\nu)f(a)+\nu f(b)-f\left((1-\nu)a+\nu b\right) = (1-\nu)\nu(b-a) \\ &\times\int_0^1 \left[f'\left((1-t)b+t\left((1-\nu)a+\nu b\right)\right)-f'\left((1-t)a+t\left((1-\nu)a+\nu b\right)\right)\right]dt. \end{aligned} \tag{2.6}$$

That is an equality of interest in itself.

Since $a<b$ and $\nu\in(0,1)$, then $(1-\nu)a+\nu b\in(a,b)$ and

$$(1-t)a+t\left((1-\nu)a+\nu b\right)\in\left[a,(1-\nu)a+\nu b\right]$$

while

$$(1-t)b+t\left((1-\nu)a+\nu b\right)\in\left[(1-\nu)a+\nu b,b\right]$$

for any $t\in[0,1]$.

By the monotonicity of the derivative we have

$$f'_+\left((1-\nu)a+\nu b\right)\le f'\left((1-t)b+t\left((1-\nu)a+\nu b\right)\right)\le f'_-(b) \tag{2.7}$$

and

$$f'_+(a)\le f'\left((1-t)a+t\left((1-\nu)a+\nu b\right)\right)\le f'_-\left((1-\nu)a+\nu b\right) \tag{2.8}$$

for any $t\in[0,1]$.

By integrating the inequalities (2.7) and (2.8) we get

$$f'_+\left((1-\nu)a+\nu b\right)\le\int_0^1 f'\left((1-t)b+t\left((1-\nu)a+\nu b\right)\right)dt\le f'_-(b)$$

and

$$f'_+(a)\le\int_0^1 f'\left((1-t)a+t\left((1-\nu)a+\nu b\right)\right)dt\le f'_-\left((1-\nu)a+\nu b\right),$$

which implies that

$$\begin{aligned} &f'_+\left((1-\nu)a+\nu b\right)-f'_-\left((1-\nu)a+\nu b\right)\le\int_0^1 f'\left((1-t)b+t\left((1-\nu)a+\nu b\right)\right)dt \\ &-\int_0^1 f'\left((1-t)a+t\left((1-\nu)a+\nu b\right)\right)dt\le f'_-(b)-f'_+(a). \end{aligned}$$

Making use of the equality (2.6) we the obtain the desired result (2.1).

If we consider the convex function $f:[a,b]\to\mathbb{R}$, $f(x)=\left|x-\frac{a+b}{2}\right|$, then we have $f'_+\left(\frac{a+b}{2}\right)=1$, $f'_-\left(\frac{a+b}{2}\right)=-1$ and by replacing in (2.2) we get in all terms the same quantity $\frac{1}{2}(b-a)$ which show that the constant $\frac{1}{4}$ is best possible in both inequalities from (2.2). □

Corollary 1. *If the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on $\mathring{I}$, then for any $a, b \in \mathring{I}$ and $\nu \in [0,1]$ we have*

$$\begin{aligned} 0 &\leq (1-\nu) f(a) + \nu f(b) - f((1-\nu) a + \nu b) \\ &\leq \nu (1-\nu)(b-a)[f'(b) - f'(a)]. \end{aligned} \tag{2.9}$$

Proof. If $a < b$, then the inequality (2.9) follows by (2.1). If $b < a$, then by (2.1) we get

$$\begin{aligned} 0 &\leq (1-\nu) f(b) + \nu f(a) - f((1-\nu) b + \nu a) \\ &\leq \nu (1-\nu)(b-a)[f'(b) - f'(a)] \end{aligned} \tag{2.10}$$

for any $\nu \in [0,1]$. If we replace ν by $1-\nu$ in (2.10), then we get (2.9). □

Theorem 1. *For any $a, b > 0$ and $\nu \in [0,1]$ we have*

$$0 \leq (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \leq \nu (1-\nu)(a-b)(\ln a - \ln b) \tag{2.11}$$

and

$$1 \leq \frac{(1-\nu) a + \nu b}{a^{1-\nu} b^{\nu}} \leq \exp\left[4\nu (1-\nu)\left(K\left(\frac{a}{b}\right) - 1\right)\right], \tag{2.12}$$

where K is Kantorovich's constant.

Proof. If we write the inequality (2.9) for the convex function $f : \mathbb{R} \to (0, \infty)$, $f(x) = \exp(x)$, then we have

$$\begin{aligned} 0 &\leq (1-\nu) \exp(x) + \nu \exp(y) - \exp((1-\nu) x + \nu y) \\ &\leq \nu (1-\nu)(x-y)[\exp(x) - \exp(y)] \end{aligned} \tag{2.13}$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0,1]$.

Let $a, b > 0$. If we take $x = \ln a$, $y = \ln b$ in (2.13), then we get the desired inequality (2.11).

Now, if we write the inequality (2.9) for the convex function $f : (0, \infty) \to \mathbb{R}$, $f(x) = -\ln x$, then we get

$$0 \leq \ln((1-\nu) a + \nu b) - (1-\nu) \ln a - \nu \ln b \leq \nu (1-\nu) \frac{(b-a)^2}{ab},$$

namely

$$\ln\left[\frac{(1-\nu) a + \nu b}{a^{1-\nu} b^{\nu}}\right] \leq \nu (1-\nu) \frac{(b-a)^2}{ab}.$$

This is equivalent to the desired result (2.12). □

Remark 1. *In particular, we have from (2.11) and (2.12) that*

$$0 \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{1}{4}(a-b)(\ln a - \ln b) \tag{2.14}$$

and

$$1 \leq \frac{\frac{a+b}{2}}{\sqrt{ab}} \leq \exp\left[K\left(\frac{a}{b}\right) - 1\right]. \tag{2.15}$$

It is natural to ask which upper bound, provided by Kittaneh-Manasrah inequality (1.2) and our inequality (2.11) is better.

Consider the functions

$$g_{\nu}(x) := \max\{1-\nu,\nu\}\left(\sqrt{x}-1\right)^2, \ h_{\nu}(x) := \nu(1-\nu)(x-1)\ln x$$

defined for $x>0$. Let $\nu \in [0,1]$ and consider the difference

$$d_{\nu}(x) := h_{\nu}(x) - g_{\nu}(x), \ x>0.$$

The $3D$ plot of function $d_{\nu}(x)$ for $x \in [0,10]$ and $\nu \in [0,1]$ reveals that it takes both negative and positive values, showing that some time inequality (1.2) is better and other time worse than (2.11).

Consider also the functions

$$S(x) := \frac{x^{\frac{1}{x-1}}}{e\ln\left(x^{\frac{1}{x-1}}\right)}, \ K_{\nu}(x) := \left(\frac{(x+1)^2}{4x}\right)^{\max\{1-\nu,\nu\}}$$

and

$$D_{\nu}(x) := \exp\left[\nu(1-\nu)\frac{(x-1)^2}{x}\right]$$

defined for $x>0$ and $\nu \in [0,1]$.

The $3D$ plots of the differences $K_{\nu}(x)-S(x)$, $K_{\nu}(x)-D_{\nu}(x)$ and $S(x)-D_{\nu}(x)$ for $x \in [0,10]$ and $\nu \in [0,1]$ reveal that they take both negative and positive values, showing that, in general, there is no ordering for the upper bounds of the quantity $\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}$ as provided by the inequalities (1.4), (1.6) and (2.12).

References

[1] S. Furuichi, Refined Young inequalities with Specht's ratio, *Journal of the Egyptian Mathematical Society* **20**(2012) , 46–49.

[2] S. Furuichi, On refined Young inequalities and reverse inequalities, *J. Math. Inequal.*, **5** (2011), 21-31.

[3] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.,* **361** (2010), 262-269

[4] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra.,* **59** (2011), 1031-1037.

[5] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.

[6] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.*, **74** (1960), pp. 91-98.

[7] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.H.

[8] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.

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