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SOME RESULTS FOR ISOTONIC FUNCTIONALS VIA TWO NEW REVERSES OF YOUNG'S INEQUALITY

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Abstract. In this paper we obtain some inequalities for isotonic functionals via two new reverses of Young's inequality. Applications for integrals and n-tuples of real numbers are provided as well.

1. Introduction

Let L be a *linear class* of real-valued functions $g : E \to \mathbb{R}$ having the properties:

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1$, $t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalised if*
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2], [18] and [19]). For other inequalities for isotonic functionals see [1], [4]-[17] and [20]-[23].

We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second ($p_k \geq 0, k \in E$).

The famous *Young's inequality* for scalars says that if $a, b > 0$ and $\nu \in [0, 1]$, then

$$a^{1-\nu} b^{\nu} \leq (1 - \nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality.*

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We recall that *Specht's ratio* is defined by [22]

$$S(h):=\begin{cases}\frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty)\\ \\ 1 \text{ if } h=1.\end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to 1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^r\right)a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.3}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$.

The second inequality in (1.3) is due to Tominaga [24] while the first one is due to Furuichi [13].

It is an open question for the author if in the right hand side of (1.3) we can replace $S\left(\frac{a}{b}\right)$ by $S\left(\left(\frac{a}{b}\right)^R\right)$ where $R=\max\{1-\nu,\nu\}$.

Kittaneh and Manasrah [14], [15] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^2\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\leq R\left(\sqrt{a}-\sqrt{b}\right)^2 \tag{1.4}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

We also consider the *Kantorovich's constant* defined by

$$K(h):=\frac{(h+1)^2}{4h},\ h>0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\geq 1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^r\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq K^R\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{1.6}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [25] while the second by Liao et al. [16].

In the recent paper [9] we obtained the following reverses of Young's inequality as well:

$$0\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\leq\nu(1-\nu)(a-b)(\ln a-\ln b) \tag{1.7}$$

and

$$1\leq\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\leq\exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{1.8}$$

where $a,b>0$, $\nu\in[0,1]$.

It has been shown in [9] that there is no ordering for the upper bounds of the quantity $(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}$ as provided by the inequalities (1.4) and (1.7). The same conclusion is true for the upper bounds of the quantity $\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}$ incorporated in the inequalities (1.3), (1.6) and (1.8).

In this paper we obtain some inequalities for isotonic functionals via the reverse of Young's inequalities (1.7) and (1.8). Applications for integrals and n-tuples of real numbers are also provided.

2. Reverses of Callebaut's Inequality

The functional version of *Callebaut's inequality* states that

$$A^{2}(fg)\leq A\left(f^{2-\nu}g^{\nu}\right)A\left(f^{\gamma}g^{2-\nu}\right)\leq A\left(f^{2}\right)A\left(g^{2}\right) \tag{2.1}$$

provided that f^2, g^2, $f^{2-\nu}g^{\nu}$, $f^{\nu}g^{2-\nu}$, $fg\in L$ for some $\nu\in[0,2]$. For the discrete and integral of one real variable versions see [3].

If $a,b\in[m,M]\subset(0,\infty)$, then by (1.7) we have the inequality

$$0\leq(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq\nu\,(1-\nu)\,(M-m)\ln\left(\frac{M}{m}\right) \tag{2.2}$$

for any $\nu\in[0,1]$.

We start with the following result:

Theorem 1. *Let A, $B:L\to\mathbb{R}$ be two normalised isotonic functionals. If f, $g:E\to\mathbb{R}$ are such that $f\geq 0$, $g>0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$ and*

$$0<m\leq\frac{f}{g}\leq M<\infty \tag{2.3}$$

for some constants m,M, then

$$\begin{aligned}0&\leq(1-\nu)\,A\left(f^{2}\right)B\left(g^{2}\right)+\nu A\left(g^{2}\right)B\left(f^{2}\right)\\&\quad-A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right)\\&\leq2\nu\,(1-\nu)\left(M^{2}-m^{2}\right)\ln\left(\frac{M}{m}\right)A\left(g^{2}\right)B\left(g^{2}\right).\end{aligned} \tag{2.4}$$

In particular, we have

$$\begin{aligned}0&\leq A\left(f^{2}\right)A\left(g^{2}\right)-A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right)\\&\leq2\nu\,(1-\nu)\left(M^{2}-m^{2}\right)\ln\left(\frac{M}{m}\right)A^{2}\left(g^{2}\right).\end{aligned} \tag{2.5}$$

Proof. For any x, $y\in E$ we have

$$m^{2}\leq\frac{f^{2}(x)}{g^{2}(x)},\frac{f^{2}(y)}{g^{2}(y)}\leq M^{2}.$$

If we use the inequalities (1.1) and (2.2) for

$$a=\frac{f^{2}(x)}{g^{2}(x)},\ b=\frac{f^{2}(y)}{g^{2}(y)},$$

then we get

$$\begin{aligned}0&\leq(1-\nu)\frac{f^{2}(x)}{g^{2}(x)}+\nu\frac{f^{2}(y)}{g^{2}(y)}-\left(\frac{f^{2}(x)}{g^{2}(x)}\right)^{1-\nu}\left(\frac{f^{2}(y)}{g^{2}(y)}\right)^{\nu}\\&\leq2\nu\,(1-\nu)\left(M^{2}-m^{2}\right)\ln\left(\frac{M}{m}\right)\end{aligned} \tag{2.6}$$

for any $x,y\in E$.

Now, if we multiply (2.6) by $g^2(x)g^2(y)>0$ then we get

$$0\le(1-\nu)f^2(x)g^2(y)+\nu g^2(x)f^2(y) - f^{2(1-\nu)}(x)g^{2\nu}(x)f^{2\nu}(y)g^{2(1-\nu)}(y) \le 2\nu(1-\nu)\left(M^2-m^2\right)\ln\left(\frac{M}{m}\right)g^2(x)g^2(y) \tag{2.7}$$

for any $x,y\in E$.

Fix $y\in E$. Then by (2.7) we have in the order of L that

$$0\le(1-\nu)g^2(y)f^2+\nu f^2(y)g^2-f^{2\nu}(y)g^{2(1-\nu)}(y)f^{2(1-\nu)}g^{2\nu} \le 2\nu(1-\nu)\left(M^2-m^2\right)\ln\left(\frac{M}{m}\right)g^2(y)g^2. \tag{2.8}$$

If we take the functional A in (2.8) then we get

$$0\le(1-\nu)g^2(y)A\left(f^2\right)+\nu f^2(y)A\left(g^2\right) - f^{2\nu}(y)g^{2(1-\nu)}(y)A\left(f^{2(1-\nu)}g^{2\nu}\right) \le 2\nu(1-\nu)\left(M^2-m^2\right)\ln\left(\frac{M}{m}\right)g^2(y)A\left(g^2\right) \tag{2.9}$$

for any $y\in E$.

This inequality can be written in the order of L as

$$0\le(1-\nu)A\left(f^2\right)g^2+\nu A\left(g^2\right)f^2-A\left(f^{2(1-\nu)}g^{2\nu}\right)f^{2\nu}g^{2(1-\nu)} \le 2\nu(1-\nu)\left(M^2-m^2\right)\ln\left(\frac{M}{m}\right)A\left(g^2\right)g^2. \tag{2.10}$$

Now, if we take the functional B in (2.10), then we get the desired result (2.4). □

The following reverse of two functional Cauchy-Bunyakovsky-Schwarz inequality for isotonic functionals holds:

Corollary 1. *Let A, $B:L\to\mathbb{R}$ be two normalised isotonic functionals. If f, $g:E\to\mathbb{R}$ are such that $f\ge0$, $g>0$, f^2, g^2, $fg\in L$ and the condition (2.3) holds true, then*

$$0\le\frac{1}{2}\left[A\left(f^2\right)B\left(g^2\right)+A\left(g^2\right)B\left(f^2\right)\right]-A(fg)B(fg) \le\frac{1}{2}\left(M^2-m^2\right)\ln\left(\frac{M}{m}\right)A\left(g^2\right)B\left(g^2\right). \tag{2.11}$$

In particular, we have

$$0\le A\left(f^2\right)A\left(g^2\right)-A^2(fg)\le\frac{1}{2}\left(M^2-m^2\right)\ln\left(\frac{M}{m}\right)A^2\left(g^2\right). \tag{2.12}$$

Let $a,b\in[m,M]\subset(0,\infty)$, then $\frac{m}{M}\le\frac{a}{b}\le\frac{M}{m}$ with $\frac{m}{M}<1<\frac{M}{m}$. If $\frac{a}{b}\in\left[\frac{m}{M},1\right)$ then $K\left(\frac{a}{b}\right)\le K\left(\frac{m}{M}\right)=K\left(\frac{M}{m}\right)$. If $\frac{a}{b}\in\left(1,\frac{M}{m}\right]$ then also $K\left(\frac{a}{b}\right)\le K\left(\frac{M}{m}\right)$. Therefore for any $a,b\in[m,M]$ we have from (1.8) that

$$1\le\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\le\exp\left[4\nu(1-\nu)\left(K\left(\frac{M}{m}\right)-1\right)\right], \tag{2.13}$$

for any $\nu\in[0,1]$.

Theorem 2. *With the assumptions of Theorem 1 we have*

$$(1-\nu) A\left(f^2\right) B\left(g^2\right)+\nu A\left(g^2\right) B\left(f^2\right) \leq \exp\left[4\nu(1-\nu)\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right] A\left(f^{2(1-\nu)} g^{2\nu}\right) B\left(f^{2\nu} g^{2(1-\nu)}\right). \tag{2.14}$$

In particular, we have

$$A\left(f^2\right) A\left(g^2\right) \leq \exp\left[4\nu(1-\nu)\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right] \times A\left(f^{2(1-\nu)} g^{2\nu}\right) A\left(f^{2\nu} g^{2(1-\nu)}\right). \tag{2.15}$$

Proof. For any $x, y \in E$ we have

$$m^2 \leq \frac{f^2(x)}{g^2(x)}, \frac{f^2(y)}{g^2(y)} \leq M^2.$$

If we use the inequality (2.13) for

$$a=\frac{f^2(x)}{g^2(x)},\ b=\frac{f^2(y)}{g^2(y)},$$

then we get

$$(1-\nu) \frac{f^2(x)}{g^2(x)}+\nu \frac{f^2(y)}{g^2(y)} \leq \exp\left[4\nu(1-\nu)\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right]\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu} \tag{2.16}$$

for any $x, y \in E$.

Now, if we multiply (2.16) by $g^2(x) g^2(y)>0$ then we get

$$(1-\nu) f^2(x) g^2(y)+\nu g^2(x) f^2(y) \leq \exp\left[4\nu(1-\nu)\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right] \times f^{2(1-\nu)}(x) g^{2\nu}(x) f^{2\nu}(y) g^{2(1-\nu)}(y) \tag{2.17}$$

for any $x, y \in E$.

Fix $y \in E$. Then by (2.17) we have in the order of L that

$$(1-\nu) g^2(y) f^2+\nu f^2(y) g^2 \leq \exp\left[4\nu(1-\nu)\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right] \times f^{2\nu}(y) g^{2(1-\nu)}(y) f^{2(1-\nu)} g^{2\nu}. \tag{2.18}$$

If we take the functional A in (2.18) then we get

$$(1-\nu) g^2(y) A\left(f^2\right)+\nu f^2(y) A\left(g^2\right) \leq \exp\left[4\nu(1-\nu)\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right] \times f^{2\nu}(y) g^{2(1-\nu)}(y) A\left(f^{2(1-\nu)} g^{2\nu}\right),$$

for any $y \in E$.

This inequality can be written in the order of L as

$$(1-\nu) A\left(f^{2}\right) g^{2}+\nu A\left(g^{2}\right) f^{2} \leq \exp \left[4 \nu(1-\nu)\left(K\left(\left(\frac{M}{m}\right)^{2}\right)-1\right)\right] A\left(f^{2(1-\nu)} g^{2 \nu}\right) f^{2 \nu} g^{2(1-\nu)}. \tag{2.19}$$

Now, if we take the functional B in (2.19), then we get the desired result (2.14). □

The following reverse of two functional Cauchy-Bunyakovsky-Schwarz inequality for isotonic functionals holds:

Corollary 2. *Let $A, B: L \rightarrow \mathbb{R}$ be two normalised isotonic functionals. If $f, g: E \rightarrow \mathbb{R}$ are such that $f \geq 0$, $g>0$, f^{2}, g^{2}, $fg \in L$ and the condition (2.3) holds true, then*

$$\frac{1}{2}\left[A\left(f^{2}\right) B\left(g^{2}\right)+A\left(g^{2}\right) B\left(f^{2}\right)\right] \leq \exp \left[K\left(\left(\frac{M}{m}\right)^{2}\right)-1\right] A(f g) B(f g). \tag{2.20}$$

In particular,

$$A\left(f^{2}\right) A\left(g^{2}\right) \leq \exp \left[K\left(\left(\frac{M}{m}\right)^{2}\right)-1\right] A^{2}(f g). \tag{2.21}$$

3. Reverses of Hölder's Inequality

We have the following reverse of *Hölder's inequality* for isotonic functionals:

Theorem 3. *Let $A: L \rightarrow \mathbb{R}$ be a normalised isotonic functional and $p, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $f, g: E \rightarrow \mathbb{R}$ are such that fg, f^{p}, $g^{q} \in L$ and*

$$0<m_{1} \leq f \leq M_{1}<\infty,\ 0<m_{2} \leq g \leq M_{2}<\infty, \tag{3.1}$$

for some constants m_{1}, M_{1}, m_{2}, M_{2}, then by putting

$$M_{p, q}:=\max \left\{\left(\frac{M_{1}}{m_{1}}\right)^{p},\left(\frac{M_{2}}{m_{2}}\right)^{q}\right\}$$

we have

$$0 \leq 1-\frac{A(f g)}{\left[A\left(f^{p}\right)\right]^{1 / p}\left[A\left(g^{q}\right)\right]^{1 / q}} \leq \frac{2}{p q}\left(\frac{M_{p, q}^{2}-1}{M_{p, q}}\right) \ln \left(M_{p, q}\right). \tag{3.2}$$

Proof. Observe that, by (3.1) we have

$$m_{1}^{p} \leq A\left(f^{p}\right) \leq M_{1}^{p} \text{ and } m_{2}^{q} \leq A\left(g^{q}\right) \leq M_{2}^{q}.$$

Also

$$\left(\frac{m_{1}}{M_{1}}\right)^{p} \leq \frac{f^{p}}{A\left(f^{p}\right)} \leq\left(\frac{M_{1}}{m_{1}}\right)^{p}$$

and

$$\left(\frac{m_{2}}{M_{2}}\right)^{q} \leq \frac{g^{q}}{A\left(g^{q}\right)} \leq\left(\frac{M_{2}}{m_{2}}\right)^{q}$$

giving that

$$m_{p,q}\le\frac{f^p}{A(f^p)},\frac{g^q}{A(g^q)}\le M_{p,q},$$

where

$$\begin{aligned}m_{p,q} &: =\min\left\{\left(\frac{m_1}{M_1}\right)^p,\left(\frac{m_2}{M_2}\right)^q\right\}=\min\left\{\frac{1}{\left(\frac{M_1}{m_1}\right)^p},\frac{1}{\left(\frac{M_2}{m_2}\right)^q}\right\}\\ &= \frac{1}{\max\left\{\left(\frac{M_1}{m_1}\right)^p,\left(\frac{M_2}{m_2}\right)^q\right\}}=\frac{1}{M_{p,q}}.\end{aligned}$$

Using the inequality (2.2) for $\nu=\frac{1}{q}$, $a=\frac{f^p}{A(f^p)}$, $b=\frac{g^q}{A(g^q)}$, $m=m_{p,q}$ and $M=M_{p,q}$ we get

$$\begin{aligned}0&\le\frac{1}{p}\frac{f^p}{A(f^p)}+\frac{1}{q}\frac{g^q}{A(g^q)}-\frac{fg}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}\\&\le\frac{2}{pq}\left(M_{p,q}-\frac{1}{M_{p,q}}\right)\ln(M_{p,q}).\end{aligned}\tag{3.3}$$

If we take the functional A in (3.3), then we get

$$0\le\frac{1}{p}\frac{A(f^p)}{A(f^p)}+\frac{1}{q}\frac{A(g^q)}{A(g^q)}-\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}\le\frac{2}{pq}\left(\frac{M_{p,q}^2-1}{M_{p,q}}\right)\ln(M_{p,q}),$$

which is equivalent to the desired result (3.2). □

We also have:

Theorem 4. *With the assumptions of Theorem 3 we have*

$$1\le\frac{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}{A(fg)}\le\exp\left[\frac{4}{pq}\left(K\left(M_{p,q}^2\right)-1\right)\right].\tag{3.4}$$

Proof. From (2.13) we have for $\nu=\frac{1}{q}$, $a=\frac{f^p}{A(f^p)}$, $b=\frac{g^q}{A(g^q)}$, $m=\frac{1}{M_{p,q}}$ and $M=M_{p,q}$ that

$$\frac{1}{p}\frac{f^p}{A(f^p)}+\frac{1}{q}\frac{g^q}{A(g^q)}\le\exp\left[\frac{4}{pq}\left(K\left(M_{p,q}^2\right)-1\right)\right]\frac{fg}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}.\tag{3.5}$$

If we take the functional A in (3.5), then we get

$$1=\frac{1}{p}\frac{A(f^p)}{A(f^p)}+\frac{1}{q}\frac{A(g^q)}{A(g^q)}\le\exp\left[\frac{4}{pq}\left(K\left(M_{p,q}^2\right)-1\right)\right]\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}},\tag{3.6}$$

which is equivalent to (3.4). □

4. Some Related Inequalities

We have:

Theorem 5. *Let $A:L\to\mathbb{R}$ be a normalised isotonic functional and $\nu\in[0,1]$. If $f,g:E\to\mathbb{R}$ are such that $f^\nu, g^{-\nu}, f^{\nu-1}, g^{1-\nu}\in L$ and*

$$0<m_1\le f\le M_1<\infty,\ 0<m_2\le g\le M_2<\infty,\tag{4.1}$$

for some constants m_1, M_1, m_2, M_2, then by putting

$$M:=\max\{M_1,M_2\}\ \text{ and } m:=\min\{m_1,m_2\}$$

we have

$$0\le(1-\nu)A(f^{\nu})B\left(g^{-\nu}\right)+\nu A\left(f^{\nu-1}\right)B\left(g^{1-\nu}\right)-1 \le\nu(1-\nu)(M-m)\ln\left(\frac{M}{m}\right)A\left(f^{1-\nu}\right)B(g^{\nu}) \tag{4.2}$$

and

$$1\le(1-\nu)A(f^{\nu})B\left(g^{-\nu}\right)+\nu A\left(f^{\nu-1}\right)B\left(g^{1-\nu}\right) \le\exp\left[4\nu(1-\nu)\left(K\left(\frac{M}{m}\right)-1\right)\right]. \tag{4.3}$$

Proof. From (2.2) we have

$$0\le(1-\nu)f(x)+\nu g(y)-f^{1-\nu}(x)g^{\nu}(y)\le\nu(1-\nu)(M-m)\ln\left(\frac{M}{m}\right)$$

for any $\nu\in[0,1]$ and $x,y\in E$, which is equivalent to

$$0\le(1-\nu)f^{\nu}(x)g^{-\nu}(y)+\nu f^{\nu-1}(x)g^{1-\nu}(y)-1 \le\nu(1-\nu)(M-m)\ln\left(\frac{M}{m}\right)f^{1-\nu}(x)g^{\nu}(y) \tag{4.4}$$

for any $\nu\in[0,1]$ and $x,y\in E$.

Now, if we apply to (4.4) the functional A and then the functional B we get (4.2).

From the inequality (2.13) we also have

$$1\le(1-\nu)f^{\nu}(x)g^{-\nu}(y)+\nu f^{\nu-1}(x)g^{1-\nu}(y) \le\exp\left[4\nu(1-\nu)\left(K\left(\frac{M}{m}\right)-1\right)\right],$$

for any $\nu\in[0,1]$ and $x,y\in E$.

This yields in a similar way to (4.3) and the details are omitted. □

Remark 1. *If we take in (4.2) and (4.3) $\nu=\frac{1}{2}$, then we get*

$$0\le\frac{1}{2}\left[A\left(f^{\frac{1}{2}}\right)B\left(g^{-\frac{1}{2}}\right)+A\left(f^{-\frac{1}{2}}\right)B\left(g^{\frac{1}{2}}\right)\right]-1 \le\frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right)A\left(f^{\frac{1}{2}}\right)B\left(g^{\frac{1}{2}}\right) \tag{4.5}$$

and

$$1\le\frac{1}{2}\left[A\left(f^{\frac{1}{2}}\right)B\left(g^{-\frac{1}{2}}\right)+A\left(f^{-\frac{1}{2}}\right)B\left(g^{\frac{1}{2}}\right)\right]\le\exp\left[\left(K\left(\frac{M}{m}\right)-1\right)\right], \tag{4.6}$$

provided $f^{\frac{1}{2}}$, $g^{-\frac{1}{2}},f^{-\frac{1}{2}},g^{\frac{1}{2}}\in L$ and f,g satisfy (4.1).

In particular, we have

$$0\le\frac{1}{2}\left[A\left(f^{\frac{1}{2}}\right)A\left(g^{-\frac{1}{2}}\right)+A\left(f^{-\frac{1}{2}}\right)A\left(g^{\frac{1}{2}}\right)\right]-1 \le\frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right)A\left(f^{\frac{1}{2}}\right)A\left(g^{\frac{1}{2}}\right) \tag{4.7}$$

and

$$1\le\frac{1}{2}\left[A\left(f^{\frac{1}{2}}\right)A\left(g^{-\frac{1}{2}}\right)+A\left(f^{-\frac{1}{2}}\right)A\left(g^{\frac{1}{2}}\right)\right]\le\exp\left[\left(K\left(\frac{M}{m}\right)-1\right)\right]. \tag{4.8}$$

In a similar way we can prove the following results:

Theorem 6. *With the assumptions of Theorem 5 we have*

$$
\begin{aligned}
(4.9)\qquad & 0 \le (1-\nu) A\left(f^{\nu} g^{-\nu}\right) + \nu A\left(f^{\nu-1} g^{1-\nu}\right) - 1 \\
& \le \nu (1-\nu)(M-m) \ln\left(\frac{M}{m}\right) A\left(f^{1-\nu} g^{\nu}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(4.10)\qquad & 1 \le (1-\nu) A\left(f^{\nu} g^{-\nu}\right) + \nu A\left(f^{\nu-1} g^{1-\nu}\right) \\
& \le \exp\left[4\nu (1-\nu)\left(K\left(\frac{M}{m}\right) - 1\right)\right].
\end{aligned}
$$

5. Integral Inequalities

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \ge 0$ for μ -a.e. (almost every) $x \in \Omega$, consider the *Lebesgue space*

$$
L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)| w(x)\, d\mu(x) < \infty\}.
$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\, d\mu(x)$. The same for other integrals involved below. We assume that $\int_\Omega w d\mu = 1$.

Let f, g be μ-measurable functions with the property that there exists the constants M, $m > 0$ such that

$$
0 < m \le \frac{f}{g} \le M < \infty \ \ \mu\text{-almost everywhere (a.e.) on } \Omega.
$$

If f^2, $g^2 \in L_w(\Omega, \mu)$, then by (2.5) we have for any $s \in [0,1]$ that

$$
\begin{aligned}
(5.1)\qquad & 0 \le \int_\Omega w f^2 d\mu \int_\Omega w g^2 d\mu - \int_\Omega w f^{2(1-s)} g^{2s} d\mu \int_\Omega w f^{2s} g^{2(1-s)} d\mu \\
& \le 2\nu (1-\nu)\left(M^2 - m^2\right) \ln\left(\frac{M}{m}\right)\left(\int_\Omega w g^2 d\mu\right)^2.
\end{aligned}
$$

From (2.15) we also have

$$
\begin{aligned}
(5.2)\qquad & \int_\Omega w f^2 d\mu \int_\Omega w g^2 d\mu \le \exp\left[4\nu (1-\nu)\left(K\left(\left(\frac{M}{m}\right)^2\right) - 1\right)\right] \\
& \times \int_\Omega w f^{2(1-s)} g^{2s} d\mu \int_\Omega w f^{2s} g^{2(1-s)} d\mu.
\end{aligned}
$$

Let f, g be μ-measurable functions with the property that there exists the constants M_1, M_2, m_1, $m_2 > 0$ such that

$$
(5.3)\qquad 0 < m_1 \le f \le M_1 < \infty,\ 0 < m_2 \le g \le M_2 < \infty,\ \mu\text{-a.e. on } \Omega.
$$

Then by putting

$$
M_{p,q} := \max\left\{\left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q\right\}
$$

where $p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, we have from (3.2) that

$$0\le 1-\frac{\int_\Omega wfgd\mu}{\left(\int_\Omega wf^p d\mu\right)^{1/p}\left(\int_\Omega wg^q d\mu\right)^{1/q}}\le \frac{2}{pq}\left(\frac{M_{p,q}^2-1}{M_{p,q}}\right)\ln\left(M_{p,q}\right), \tag{5.4}$$

while from (3.4) that

$$1\le \frac{\left(\int_\Omega wf^p d\mu\right)^{1/p}\left(\int_\Omega wg^q d\mu\right)^{1/q}}{\int_\Omega wfgd\mu}\le \exp\left[\frac{4}{pq}\left(K\left(M_{p,q}^2\right)-1\right)\right]. \tag{5.5}$$

If f, g are μ-measurable functions with the property that there exists the constants M_1, M_2, m_1, $m_2>0$ such that (5.3) is true, then by (4.7) and (4.8) we have

$$\begin{aligned} 0 &\le \frac{1}{2}\left[\int_\Omega wf^{\frac{1}{2}}d\mu\int_\Omega wg^{-\frac{1}{2}}d\mu+\int_\Omega wf^{-\frac{1}{2}}d\mu\int_\Omega wg^{\frac{1}{2}}d\mu\right]-1 \\ &\le \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right)\int_\Omega wf^{\frac{1}{2}}d\mu\int_\Omega wg^{\frac{1}{2}}d\mu \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} 1 &\le \frac{1}{2}\left[\int_\Omega wf^{\frac{1}{2}}d\mu\int_\Omega wg^{-\frac{1}{2}}d\mu+\int_\Omega wf^{-\frac{1}{2}}d\mu\int_\Omega wg^{\frac{1}{2}}d\mu\right] \\ &\le \exp\left[\left(K\left(\frac{M}{m}\right)-1\right)\right], \end{aligned} \tag{5.7}$$

where $M:=\max\{M_1,M_2\}$ and $m:=\min\{m_1,m_2\}$.

6. Inequalities for Real Numbers

We consider the n-tuples of positive numbers $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ and the probability distribution $p=(p_1,...,p_n)$, i.e. $p_i\ge 0$ for any $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$.

If there exist the constants m, $M>0$ such that

$$0<m\le\frac{a_i}{b_i}\le M<\infty \text{ for any } i\in\{1,...,n\},$$

then by (5.1) for the *counting discrete measure*, we have for any $s\in[0,1]$ that

$$\begin{aligned} 0 &\le \sum_{i=1}^n p_i a_i^2\sum_{i=1}^n p_i b_i^2-\sum_{i=1}^n p_i a_i^{2(1-s)}b_i^{2s}\sum_{i=1}^n p_i a_i^{2s}b_i^{2(1-s)} \\ &\le 2\nu(1-\nu)\left(M^2-m^2\right)\ln\left(\frac{M}{m}\right)\left(\sum_{i=1}^n p_i b_i^2\right)^2 \end{aligned} \tag{6.1}$$

while from (5.1)

$$\begin{aligned} \sum_{i=1}^n p_i a_i^2\sum_{i=1}^n p_i b_i^2 &\le \exp\left[4\nu(1-\nu)\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right] \\ &\times\sum_{i=1}^n p_i a_i^{2(1-s)}b_i^{2s}\sum_{i=1}^n p_i a_i^{2s}b_i^{2(1-s)}. \end{aligned} \tag{6.2}$$

If there exist the constants m_1,M_1,m_2,M_2 such that

$$0<m_1\le a_i\le M_1<\infty,\ 0<m_2\le b_i\le M_2<\infty \text{ for any } i\in\{1,...,n\} \tag{6.3}$$

and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by putting

$$M_{p,q} := \max \left\{ \left(\frac{M_1}{m_1} \right)^p, \left(\frac{M_2}{m_2} \right)^q \right\}$$

where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have from (5.4) that

$$0 \le 1 - \frac{\sum_{i=1}^n p_i a_i b_i}{\left(\sum_{i=1}^n p_i a_i^p\right)^{1/p} \left(\sum_{i=1}^n p_i b_i^q\right)^{1/q}} \le \frac{2}{pq} \left(\frac{M_{p,q}^2 - 1}{M_{p,q}} \right) \ln \left(M_{p,q} \right), \tag{6.4}$$

while from (5.5) that

$$1 \le \frac{\left(\sum_{i=1}^n p_i a_i^p\right)^{1/p} \left(\sum_{i=1}^n p_i b_i^q\right)^{1/q}}{\sum_{i=1}^n p_i a_i b_i} \le \exp \left[\frac{4}{pq} \left(K \left(M_{p,q}^2 \right) - 1 \right) \right]. \tag{6.5}$$

If there exist the constants m_1, M_1, m_2, M_2 such that (6.3) is valid, then by (5.6) and (5.7) we have

$$\begin{aligned} 0 &\le \frac{1}{2} \left[\sum_{i=1}^n p_i a_i^{\frac{1}{2}} \sum_{i=1}^n p_i b_i^{-\frac{1}{2}} + \sum_{i=1}^n p_i a_i^{-\frac{1}{2}} \sum_{i=1}^n p_i b_i^{\frac{1}{2}} \right] - 1 \\ &\le \frac{1}{4} (M - m) \ln \left(\frac{M}{m} \right) \sum_{i=1}^n p_i a_i^{\frac{1}{2}} \sum_{i=1}^n p_i b_i^{\frac{1}{2}} \end{aligned} \tag{6.6}$$

and

$$1 \le \frac{1}{2} \left[\sum_{i=1}^n p_i a_i^{\frac{1}{2}} \sum_{i=1}^n p_i b_i^{-\frac{1}{2}} + \sum_{i=1}^n p_i a_i^{-\frac{1}{2}} \sum_{i=1}^n p_i b_i^{\frac{1}{2}} \right] \le \exp \left[\left(K \left(\frac{M}{m} \right) - 1 \right) \right], \tag{6.7}$$

where $M := \max \{M_1, M_2\}$ and $m := \min \{m_1, m_2\}$.

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