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JENSEN-OSTROWSKI TYPE INEQUALITIES AND APPLICATIONS FOR f-DIVERGENCE MEASURES

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Abstract. In this paper, we provide inequalities of Jensen-Ostrowski type, by investigating the magnitude of the quantity

$$\int_\Omega (f\circ g)\,d\mu - f(\zeta) - \int_\Omega (g-\zeta) f'\circ g\,d\mu + \frac{1}{2}\lambda\int_\Omega (g-\zeta)^2\,d\mu,$$

for various assumptions on the absolutely continuous function $f:[a,b]\to\mathbb{C}$, $\zeta\in[a,b]$, $\lambda\in\mathbb{C}$ and a μ-measurable function g on Ω. Special cases are considered to provide some inequalities of Jensen type, as well as Ostrowski type, in measure-theoretic (probabilistic) form. Application for f-divergence measure in Information Theory are also considered.

1. Introduction

In 1905 (1906) Jensen defined convex functions [15] as follows: f is convex if

$$f\left(\frac{a+b}{2}\right)\le\frac{f(a)+f(b)}{2} \tag{1}$$

for all $a,b\in D(f)$ (here $D(f)$ is the domain of f). Inequality (1) is the simplest form of Jensen's inequality. In general, the inequality takes the following form: if f is a convex function, then we have Jensen's discrete inequality:

$$f\left(\sum_{i=1}^n p_i x_i\right)\le\sum_{i=1}^n p_i f(x_i), \tag{2}$$

where $x_i\in D(f)$ for all $i\in\{1,\dots,n\}$, and $p_1+\dots+p_n=1$, with $p_i\ge 0$, $i\in\{1,\dots,n\}$. Jensen's inequality has been widely applied in many areas of research, e.g. Probability Theory, Statistical Physics and Information Theory.

A more general version of Jensen's inequality is given in the measure-theoretic and probabilistic form. Let $(\Omega,\mathcal{A},\mu)$ be a measurable space such that $\int_\Omega d\mu=1$, consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in the set of extended real numbers. Consider the Lebesgue space

$$L(\Omega,\mu):=\{f:\Omega\to\mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\,d\mu(t)<\infty\}.$$

For simplicity of notation, we write in the text $\int_\Omega w\,d\mu$ instead of $\int_\Omega w(t)\,d\mu(t)$. The Jensen (integral) inequality now takes the following form: for a μ-integrable function $g:\Omega\to[m,M]\subset\mathbb{R}$, and the convex function $f:[m,M]\subset\mathbb{R}\to\mathbb{R}$, we have

$$f\left(\int_\Omega g\,d\mu\right)\le\int_\Omega f\circ g\,d\mu. \tag{3}$$

In order to provide a reverse of the Jensen integral inequality for convex functions, Dragomir [5] obtained in 2002, the following result:

Theorem 1. *Let $f : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $g : \Omega \to [m, M]$ so that $f \circ g,\ g,\ f' \circ g,\ (f' \circ g) \cdot g \in L(\Omega, \mu)$. Then,*

$$\begin{aligned} 0 &\leq \int_\Omega f \circ g \, d\mu - f\left(\int_\Omega g \, d\mu\right) \\ &\leq \int_\Omega g \cdot (f' \circ g) \, d\mu - \int_\Omega f' \circ g d\mu \int_\Omega g \, d\mu \\ &\leq \frac{1}{2}\left[f'(M) - f'(m)\right] \int_\Omega \left| g - \int_\Omega g \, d\mu \right| d\mu. \end{aligned} \tag{4}$$

The following reverse of the Jensen's inequality also holds [6]:

Theorem 2. *Let $f : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, where $\mathring{I}$ is the interior of I. If $g : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \leq g(t) \leq M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega$$

and such that $g,\ f \circ g \in L(\Omega, \mu)$, then

$$\begin{aligned} 0 &\leq \int_\Omega f \circ g \, d\mu - f\left(\int_\Omega g \, d\mu\right) \\ &\leq \left(M - \int_\Omega g \, d\mu\right)\left(\int_\Omega g \, d\mu - m\right) \frac{f'_-(M) - f'_+(m)}{M - m} \\ &\leq \frac{1}{4}(M - m)\left[f'_-(M) - f'_+(m)\right], \end{aligned} \tag{5}$$

where f'_- is the left and f'_+ is the right derivative of the convex function f.

For other reverses of Jensen inequality and applications to divergence measures see [6].

In 1938, A. Ostrowski [14], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x \in [a, b]$.

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) such that $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right] \|f'\|_\infty (b - a), \tag{6}$$

for all $x \in [a, b]$ and the constant $\frac{1}{4}$ is the best possible.

Milovanović and Pečarić proved a generalisation of Ostrowski's inequality for n-time differentiable mappings [13]. The case of twice differentiable mappings is mentioned in Theorem 1.3 of [2], as follows:

Theorem 4. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping such that $f'' : (a, b) \to \mathbb{R}$ is bounded on (a, b), that is, $\|f''\|_\infty = \sup_{t \in [a,b]} |f''(t)| < \infty$. Then, we have the inequality:*

$$\begin{aligned} &\left| \frac{1}{2}\left[f(x) + \frac{(x-a)f(a) + (b-x)f(b)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\leq \frac{\|f''\|_\infty}{4}(b-a)^2\left[\frac{1}{12} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right] \end{aligned} \tag{7}$$

for all $x \in (a, b)$.

Dragomir [8] introduced some inequalities which combines the two aforementioned inequalities, i.e. the Jensen-Ostrowski type inequalities. These inequalities are established to give bounds for the magnitude of the quantity

$$\int_\Omega f \circ g \, d\mu - f(\zeta) - \lambda \left(\int_\Omega g d\mu - \zeta \right), \quad \zeta \in [a,b],$$

for various assumptions on the absolutely continuous function $f : [a,b] \to \mathbb{C}$ and a μ-measurable function g and $\lambda \in \mathbb{C}$. A particular case of $\zeta = \int_\Omega g \, d\mu$ provides some results connected with Jensen's inequality. Furthermore, in the case $\lambda = 0$, some generalizations of Ostrowski's inequality (6) are also given.

In the same spirit, we investigate in this paper, the magnitude of the quantity

$$\int_\Omega (f \circ g) \, d\mu - f(\zeta) - \int_\Omega (g-\zeta) f' \circ g \, d\mu + \frac{1}{2}\lambda \int_\Omega (g-\zeta)^2 \, d\mu, \quad \zeta \in [a,b],$$

for various assumptions on the absolutely continuous function $f : [a,b] \to \mathbb{C}$ and a μ-measurable function g and $\lambda \in \mathbb{C}$; to provide further inequalities of Jensen-Ostrowski type. We consider a particular case of $\zeta = \int_\Omega g \, d\mu$ to get some Jensen type inequalities; and a case of $\lambda = 0$ to get some Ostrowski type inequalities. In particular, we provide a generalised version Ostrowski's inequality (7), in the measure-theoretic (and probabilistic) form.

The paper is organised as follows. We provide some identities in Section 2 to assist us in the proofs of the main results. Inequalities with bounds involving the p-norms ($1 \le p \le \infty$) are given in Section 3. Inequalities for bounded second derivatives and convex second derivatives are given in Sections 4 and 5, respectively. Finally, an application for f-divergence measure in Information Theory are provided in Section 6.

2. Some identities

In this section, we give some identities which we use to assist us in proving the main results in Sections 3, 4 and 5.

Lemma 5. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, $f' : [a,b] \subset \mathring{I} \to \mathbb{C}$ is absolutely continuous on $[a,b]$ and $\zeta \in [a,b]$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω such that $f \circ g$, $(g-\zeta)f' \circ g$, $(g-\zeta)^2 \in L(\Omega,\mu)$, with $\int_\Omega d\mu = 1$, then*

$$\begin{aligned} & f(\zeta) + \int_\Omega (g-\zeta) f' \circ g \, d\mu - \frac{1}{2}\lambda \int_\Omega (g-\zeta)^2 \, d\mu - \int_\Omega (f \circ g) \, d\mu \\ & = \int_\Omega \left[(g-\zeta)^2 \int_0^1 s \big[f''\big((1-s)\zeta + sg\big) - \lambda \big] \, ds \right] d\mu \qquad (8) \\ & = \int_0^1 s \left[\int_\Omega (g-\zeta)^2 \big[f''\big((1-s)\zeta + sg\big) - \lambda \big] \, d\mu \right] ds, \qquad (9) \end{aligned}$$

for $\lambda \in \mathbb{C}$. In particular, we have

$$\begin{aligned} & f(\zeta) + \int_\Omega (g-\zeta) f' \circ g \, d\mu - \int_\Omega (f \circ g) \, d\mu \\ & = \int_\Omega \left[(g-\zeta)^2 \int_0^1 s \big[f''\big((1-s)\zeta + sg\big) \big] \, ds \right] d\mu \qquad (10) \\ & = \int_0^1 s \left[\int_\Omega (g-\zeta)^2 f''\big((1-s)\zeta + sg\big) \, d\mu \right] ds. \qquad (11) \end{aligned}$$

Proof. Since f is differentiable on $[a,b]$, thus for any u,v, we have

$$f(u)-f(v)=(u-v)\int_0^1 f'\big((1-s)v+su\big)\,ds. \tag{12}$$

Since f' is absolutely continuous on $[a,b]$, we have the following, using integration by parts

$$\begin{aligned}
&\int_0^1 f'\big((1-s)v+su\big)\,ds\\
&=sf'\big((1-s)v+su\big)\Big|_0^1-\int_0^1\frac{d}{ds}\big[f'\big((1-s)v+su\big)\big]s\,ds\\
&=f'(u)-(u-v)\int_0^1 sf''\big((1-s)v+su\big)\,ds.
\end{aligned} \tag{13}$$

Replacing (13) in (12), we get

$$f(u)-f(v)=(u-v)f'(u)-(u-v)^2\int_0^1 sf''\big((1-s)v+su\big)\,ds \tag{14}$$

for any $u,v\in[a,b]$. Letting $u=g(t)$ and $v=\zeta$ in (14), we get

$$f(g(t))-f(\zeta)=(g(t)-\zeta)f'(g(t))-(g(t)-\zeta)^2\int_0^1 sf''\big((1-s)\zeta+sg(t)\big)\,ds.$$

Integrating with respect to μ on Ω, we get

$$\begin{aligned}
&\int_\Omega f\circ g\,d\mu-f(\zeta)\\
&=\int_\Omega(g-\zeta)f'\circ g\,d\mu-\int_\Omega\left[(g-\zeta)^2\int_0^1 sf''\big((1-s)\zeta+sg\big)\,ds\right]d\mu.
\end{aligned} \tag{15}$$

Now, observe that for $\zeta\in[a,b]$ and $\lambda\in\mathbb{C}$, we have

$$\begin{aligned}
&\int_\Omega\left[(g-\zeta)^2\int_0^1 s\big[f''\big((1-s)\zeta+sg\big)-\lambda\big]\,ds\right]d\mu\\
&=\int_\Omega\left[(g-\zeta)^2\int_0^1 sf''\big((1-s)\zeta+sg\big)\,ds\right]d\mu-\lambda\int_\Omega(g-\zeta)^2\,d\mu\int_0^1 s\,ds\\
&=\int_\Omega\left[(g-\zeta)^2\int_0^1 sf''\big((1-s)\zeta+sg\big)\,ds\right]d\mu-\frac{1}{2}\lambda\int_\Omega(g-\zeta)^2\,d\mu.
\end{aligned} \tag{16}$$

By (15) and (16), we have

$$\begin{aligned}
&\int_\Omega f\circ g\,d\mu-f(\zeta)=\int_\Omega(g-\zeta)f'\circ g\,d\mu\\
&-\frac{1}{2}\lambda\int_\Omega(g-\zeta)^2\,d\mu-\int_\Omega\left[(g-\zeta)^2\int_0^1 s\big[f''\big((1-s)\zeta+sg\big)-\lambda\big]\,ds\right]d\mu,
\end{aligned}$$

and this is equivalent to (8). We obtain (9) from (8) by Fubini's theorem. We obtain (10) and (11) by letting $\lambda=0$ in (8) and (9), respectively. □

By $\sigma^2(g)$ we mean the dispersion of function g defined on Ω, that is

$$\sigma^2(g):=\int_\Omega g^2\,d\mu-\left(\int_\Omega g\,d\mu\right)^2=\int_\Omega\left(g-\int_\Omega g\,d\mu\right)^2 d\mu.$$

Corollary 6. *Under the assumptions of Lemma 5, we have*

$$\begin{aligned} &f\left(\int_\Omega g\,d\mu\right)+\int_\Omega\left(g-\int_\Omega g\,d\mu\right)f'\circ g\,d\mu-\frac{1}{2}\lambda\sigma^2(g)-\int_\Omega(f\circ g)\,d\mu \\ &=\int_0^1 s\left[\int_\Omega\left(g-\int_\Omega g\,d\mu\right)^2\left[f''\left((1-s)\int_\Omega g\,d\mu+sg\right)-\lambda\right]d\mu\right]ds \end{aligned} \tag{17}$$

for $\lambda\in\mathbb{C}$. In particular, for $\lambda=0$, we have

$$\begin{aligned} &f\left(\int_\Omega g\,d\mu\right)+\int_\Omega\left(g-\int_\Omega g\,d\mu\right)f'\circ g\,d\mu-\int_\Omega(f\circ g)\,d\mu \\ &=\int_0^1 s\left[\int_\Omega\left(g-\int_\Omega g\,d\mu\right)^2\left[f''\left((1-s)\int_\Omega g\,d\mu+sg\right)\right]d\mu\right]ds \end{aligned} \tag{18}$$

Proof. We observe that since $g:\Omega\to[a,b]$ and $\int_\Omega d\mu=1$ then $\int_\Omega g\,d\mu\in[a,b]$ and by taking $\zeta=\int_\Omega g\,d\mu$ in (8), we get (17). □

3. Bounds in terms of p-norms

We use the notation

$$\|k\|_{\Omega,p}:=\begin{cases}\left(\int_\Omega |k(t)|^p\,d\mu(t)\right)^{1/p}, & p\ge 1,\ k\in L_p(\Omega,\mu);\\ \operatorname*{ess\,sup}_{t\in\Omega}|k(t)|, & p=\infty,\ k\in L_\infty(\Omega,\mu);\end{cases}$$

and

$$\|f\|_{[0,1],p}:=\begin{cases}\left(\int_0^1 |f(s)|^p\,ds\right)^{1/p}, & p\ge 1,\ f\in L_p([0,1]);\\ \operatorname*{ess\,sup}_{s\in[0,1]}|f(s)|, & p=\infty,\ f\in L_\infty([0,1]).\end{cases}$$

We denote by ℓ, the identity function on $[0,1]$, namely, $\ell(t)=t$ $(t\in[0,1])$; and we have

$$\operatorname*{ess\,sup}_{s\in[0,1]}|f''\big((1-s)\zeta+sg(t)\big)-\lambda|=\|f''\big((1-\ell)\zeta+\ell g(t)\big)-\lambda\|_{[0,1],\infty},$$

for $t\in\Omega$, $\zeta\in[a,b]$ and $\lambda\in\mathbb{C}$.

Theorem 7. *Let $f:I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$, $f':[a,b]\subset\mathring{I}\to\mathbb{C}$ be absolutely continuous on $[a,b]$ and $\zeta\in[a,b]$. If $g:\Omega\to[a,b]$ is Lebesgue μ-measurable on Ω such that $f\circ g$, $(g-\zeta)f'\circ g$, $(g-\zeta)^2\in L(\Omega,\mu)$, with $\int_\Omega d\mu=1$, then*

$$\begin{aligned} &\left|\int_\Omega(f\circ g)\,d\mu-f(\zeta)-\int_\Omega(g-\zeta)f'\circ g\,d\mu+\frac{1}{2}\lambda\int_\Omega(g-\zeta)^2\,d\mu\right| \\ &\le\frac{1}{2}\int_\Omega(g-\zeta)^2\|f''\big((1-\ell)\zeta+\ell g\big)-\lambda\|_{[0,1],\infty}\,d\mu \\ &\le\begin{cases}\frac{1}{2}\|g-\zeta\|_{\Omega,\infty}^2\Big\|\|f''\big((1-\ell)\zeta+\ell g\big)-\lambda\|_{[0,1],\infty}\Big\|_{\Omega,1};\\ \frac{1}{2}\|(g-\zeta)^2\|_{\Omega,p}\Big\|\|f''\big((1-\ell)\zeta+\ell g\big)-\lambda\|_{[0,1],\infty}\Big\|_{\Omega,q};\\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \frac{1}{2}\|(g-\zeta)^2\|_{\Omega,1}\Big\|\|f''\big((1-\ell)\zeta+\ell g\big)-\lambda\|_{[0,1],\infty}\Big\|_{\Omega,\infty};\end{cases} \end{aligned} \tag{19}$$

for any $\lambda \in \mathbb{C}$. In particular, we have the following Ostrowki type inequality:

$$\begin{aligned}
(20)\qquad & \left| \int_\Omega (f\circ g)\,d\mu - f(\zeta) - \int_\Omega (g-\zeta) f'\circ g\,d\mu \right| \\
&\le \frac{1}{2}\int_\Omega (g-\zeta)^2 \|f''\big((1-\ell)\zeta + \ell g\big)\|_{[0,1],\infty}\,d\mu \\
&\le \begin{cases} \frac{1}{2}\|g-\zeta\|^2_{\Omega,\infty}\Big\| \|f''\big((1-\ell)\zeta+\ell g\big)\|_{[0,1],\infty}\Big\|_{\Omega,1}; \\[2mm] \frac{1}{2}\|(g-\zeta)^2\|_{\Omega,p}\Big\| \|f''\big((1-\ell)\zeta+\ell g\big)\|_{[0,1],\infty}\Big\|_{\Omega,q}; \\ \hfill p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}\|(g-\zeta)^2\|_{\Omega,1}\Big\| \|f''\big((1-\ell)\zeta+\ell g\big)\|_{[0,1],\infty}\Big\|_{\Omega,\infty}. \end{cases}
\end{aligned}$$

Proof. Taking the modulus in the equality (9), we have

$$\begin{aligned}
(21)\qquad & \left| \int_\Omega (f\circ g)\,d\mu - f(\zeta) - \int_\Omega (g-\zeta) f'\circ g\,d\mu + \frac{1}{2}\lambda\int_\Omega (g-\zeta)^2\,d\mu \right| \\
&= \left| \int_0^1 s\left[\int_\Omega (g-\zeta)^2\big[f''\big((1-s)\zeta+sg\big)-\lambda\big]\,d\mu\right] ds \right| \\
&\le \int_0^1 \left| s\left[\int_\Omega (g-\zeta)^2\big[f''\big((1-s)\zeta+sg\big)-\lambda\big]\,d\mu\right] \right| ds \\
&\le \int_0^1 s\left[\int_\Omega (g-\zeta)^2\big|f''\big((1-s)\zeta+sg\big)-\lambda\big|\,d\mu\right] ds \\
&\le \int_0^1 s\left[\int_\Omega (g-\zeta)^2\big\|f''\big((1-\ell)\zeta+\ell g\big)-\lambda\big\|_{[0,1],\infty}\,d\mu\right] ds \\
&= \int_0^1 s\,ds \int_\Omega (g-\zeta)^2\big\|f''\big((1-\ell)\zeta+\ell g\big)-\lambda\big\|_{[0,1],\infty}\,d\mu \\
&= \frac{1}{2}\int_\Omega (g-\zeta)^2\big\|f''\big((1-\ell)\zeta+\ell g\big)-\lambda\big\|_{[0,1],\infty}\,d\mu
\end{aligned}$$

for any $\lambda \in \mathbb{C}$. Utilising Hölder's inequality for the μ-measurable functions $F, G : \Omega \to \mathbb{C}$,

$$\left|\int_\Omega FG\,d\mu\right| \le \left(\int_\Omega |F|^p\,d\mu\right)^{\frac{1}{p}} \left(\int_\Omega |G|^q\,d\mu\right)^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1,$$

and

$$\left|\int_\Omega FG\,d\mu\right| \le \operatorname*{ess\,sup}_{t\in\Omega}|F(t)|\int_\Omega |G|\,d\mu,$$

we get (19) from (21). By letting $\lambda = 0$ in (19), we obtain (20). $\square$

Remark 8. If we take $\zeta = \int_\Omega g\,d\mu$ in (19), then we get the following Jensen type inequalities:

$$\begin{aligned}
&\left| \int_\Omega (f\circ g)\,d\mu - f\left(\int_\Omega g\,d\mu\right) - \int_\Omega \left(g - \int_\Omega g\,d\mu\right) f'\circ g\,d\mu + \frac{1}{2}\lambda\sigma^2(g) \right| \\
&\le \frac{1}{2}\int_\Omega \left(g-\int_\Omega g\,d\mu\right)^2 \left\| f''\left((1-\ell)\int_\Omega g\,d\mu + \ell g\right) - \lambda\right\|_{[0,1],\infty} d\mu
\end{aligned}$$

$$\leq \begin{cases} \frac{1}{2}\left\|g-\int_\Omega g\,d\mu\right\|_{\Omega,\infty}^2\left\|\left\|f''\left((1-\ell)\int_\Omega g\,d\mu+\ell g\right)-\lambda\right\|_{[0,1],\infty}\right\|_{\Omega,1}; \\ \frac{1}{2}\|\left(g-\int_\Omega g\,d\mu\right)^2\|_{\Omega,p}\left\|\left\|f''\left((1-\ell)\int_\Omega g\,d\mu+\ell g\right)-\lambda\right\|_{[0,1],\infty}\right\|_{\Omega,q}; \\ \qquad p>1,\frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}\sigma^2(g)\left\|\left\|f''\left((1-\ell)\int_\Omega g\,d\mu+\ell g\right)-\lambda\right\|_{[0,1],\infty}\right\|_{\Omega,\infty}; \end{cases}$$

and in particular when $\lambda=0$, we have

$$\begin{aligned}&\left|\int_\Omega (f\circ g)\,d\mu - f\left(\int_\Omega g\,d\mu\right)-\int_\Omega\left(g-\int_\Omega g\,d\mu\right)f'\circ g\,d\mu\right|\\ &\leq \frac{1}{2}\int_\Omega\left(g-\int_\Omega g\,d\mu\right)^2\left\|f''\left((1-\ell)\int_\Omega g\,d\mu+\ell g\right)\right\|_{[0,1],\infty}d\mu\\ &\leq \begin{cases} \frac{1}{2}\left\|g-\int_\Omega g\,d\mu\right\|_{\Omega,\infty}^2\left\|\left\|f''\left((1-\ell)\int_\Omega g\,d\mu+\ell g\right)\right\|_{[0,1],\infty}\right\|_{\Omega,1}; \\ \frac{1}{2}\|\left(g-\int_\Omega g\,d\mu\right)^2\|_{\Omega,p}\left\|\left\|f''\left((1-\ell)\int_\Omega g\,d\mu+\ell g\right)\right\|_{[0,1],\infty}\right\|_{\Omega,q}; \\ \qquad p>1,\frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}\sigma^2(g)\left\|\left\|f''\left((1-\ell)\int_\Omega g\,d\mu+\ell g\right)\right\|_{[0,1],\infty}\right\|_{\Omega,\infty}. \end{cases}\end{aligned}$$

Corollary 9. *Let $f:I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$, $f':[a,b]\subset\mathring{I}\to\mathbb{C}$ be absolutely continuous on $[a,b]$ and $\zeta\in[a,b]$. If $g:\Omega\to[a,b]$ is Lebesgue μ-measurable on Ω such that $f\circ g$, $(g-\zeta)f'\circ g$, $(g-\zeta)^2\in L(\Omega,\mu)$, with $\int_\Omega d\mu=1$, then we have the following Ostrowski type inequality:*

$$\begin{aligned}&\left|\int_\Omega (f\circ g)\,d\mu - f(\zeta)-\int_\Omega (g-\zeta)f'\circ g\,d\mu\right|\\ &\leq \frac{1}{2}\|f''\|_{[a,b],\infty}\left[\sigma^2(g)+\left(\int_\Omega g\,d\mu-\zeta\right)^2\right].\end{aligned}\tag{22}$$

We also have the following Jensen type inequality:

$$\begin{aligned}&\left|\int_\Omega (f\circ g)\,d\mu - f\left(\int_\Omega g\,d\mu\right)-\int_\Omega\left(g-\int_\Omega g\,d\mu\right)f'\circ g\,d\mu\right|\\ &\leq \frac{1}{2}\|f''\|_{[a,b],\infty}\;\sigma^2(g)\end{aligned}\tag{23}$$

which is the best inequality one can get from (22).

Proof. We have from (20) with $\lambda=0$

$$\begin{aligned}&\left|\int_\Omega (f\circ g)\,d\mu - f(\zeta)-\int_\Omega (g-\zeta)f'\circ g\,d\mu\right|\\ &\leq \frac{1}{2}\int_\Omega (g-\zeta)^2\|f''\big((1-\ell)\zeta+\ell g\big)\|_{[0,1],\infty}\,d\mu.\end{aligned}$$

However for any $t\in\Omega$ and almost every $s\in[0,1]$, we have

$$\left|f''\big((1-s)\zeta+sg(t)\big)\right|\leq \operatorname*{ess\,sup}_{u\in[a,b]}|f''(u)|=\|f''\|_{[a,b],\infty}.$$

It implies that

$$\left\|f''\big((1-\ell)\zeta+\ell g\big)\right\|_{[0,1],\infty} = \operatorname{ess}\sup_{s\in[0,1]}\left|f''\big((1-s)\zeta+sg(t)\big)\right| \le \|f''\|_{[a,b],\infty}.$$

Therefore, we have

$$\left|\int_\Omega (f\circ g)\,d\mu - f(\zeta) - \int_\Omega (g-\zeta)f'\circ g\,d\mu\right| \le \frac{1}{2}\|f''\|_{[a,b],\infty}\int_\Omega (g-\zeta)^2\,d\mu.$$

We also note that

$$\begin{aligned}
\int_\Omega (g-\zeta)^2\,d\mu &= \int_\Omega\left(g-\int_\Omega g\,d\mu+\int_\Omega g\,d\mu-\zeta\right)^2 d\mu \\
&= \int_\Omega\left(g-\int_\Omega g\,d\mu\right)^2 d\mu + \left(\int_\Omega g\,d\mu-\zeta\right)^2 \\
&= \sigma^2(g) + \left(\int_\Omega g\,d\mu-\zeta\right)^2 .
\end{aligned} \tag{24}$$

and this proves (22). By choosing $\zeta = \int_\Omega g\,d\mu$ in (22), we obtain (23). □

Remark 10. We recall the quantity:

$$\int_\Omega (g-\zeta)^2\,d\mu = \int_\Omega\left(g-\int_\Omega g\,d\mu\right)^2 d\mu + \left(\int_\Omega g\,d\mu-\zeta\right)^2.$$

In the case that $\Omega=[a,b]$, $g:[a,b]\to[a,b]$ defined by $g(t)=t$, and $\mu(t)=\frac{t}{b-a}$, we have

$$\int_\Omega g\,d\mu = \frac{1}{b-a}\int_a^b t\,dt = \frac{a+b}{2},$$

and

$$\begin{aligned}
&\int_\Omega\left(g-\int_\Omega g\,d\mu\right)^2 d\mu + \left(\int_\Omega g\,d\mu-\zeta\right)^2 \\
&= \frac{1}{b-a}\int_a^b\left(g-\frac{a+b}{2}\right)^2 dt + \left(\zeta-\frac{a+b}{2}\right)^2 \\
&= \frac{1}{12}(b-a)^2 + \left(\zeta-\frac{a+b}{2}\right)^2.
\end{aligned}$$

Under this assumption, the left-hand side of (22) becomes

$$\begin{aligned}
&\left|f(\zeta)+\frac{1}{b-a}\int_a^b (t-\zeta)f'(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&= \left|f(\zeta)+\frac{1}{b-a}\left[(t-\zeta)f(t)\Big|_a^b - \int_a^b f(t)\,dt\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&= \left|f(\zeta)+\frac{(\zeta-a)f(a)+(b-\zeta)f(b)}{b-a} - \frac{2}{b-a}\int_a^b f(t)\,dt\right|,
\end{aligned}$$

and the right-hand side of (22) becomes

$$\begin{aligned}
&\frac{1}{2}\|f''\|_{[a,b],\infty}\left[\int_\Omega\left(g-\int_\Omega g\,d\mu\right)^2 d\mu+\left(\int_\Omega g\,d\mu-\zeta\right)^2\right]\\
&=\frac{1}{2}\|f''\|_{[a,b],\infty}\left[\frac{1}{12}(b-a)^2+\left(\zeta-\frac{a+b}{2}\right)^2\right]\\
&=\frac{\|f''\|_{[a,b],\infty}}{2}(b-a)^2\left[\frac{1}{12}+\frac{(\zeta-\frac{a+b}{2})^2}{(b-a)^2}\right].
\end{aligned}$$

Thus, (22) becomes

$$\begin{aligned}
&\left|f(\zeta)+\frac{(\zeta-a)f(a)+(b-\zeta)f(b)}{b-a}-\frac{2}{b-a}\int_a^b f(t)\,dt\right|\\
&\leq\frac{\|f''\|_{[a,b],\infty}}{2}(b-a)^2\left[\frac{1}{12}+\frac{(\zeta-\frac{a+b}{2})^2}{(b-a)^2}\right]
\end{aligned}$$

for $\zeta\in[a,b]$, which recovers the result by Milovanović and Pečarić (by dividing the above inequality by 2) as stated in Theorem 4.

4. Inequalities for Bounded Second Derivatives

Now, for $\gamma,\Gamma\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions [8]

$$\begin{aligned}
&\bar{U}_{[a,b]}(\gamma,\Gamma)\\
:=\ &\left\{h:[a,b]\to\mathbb{C}\middle|\ \operatorname{Re}\left[(\Gamma-h(t))\left(\overline{h(t)}-\overline{\gamma}\right)\right]\geq 0 \text{ for almost every } t\in[a,b]\right\}
\end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{h:[a,b]\to\mathbb{C}\middle|\ \left|h(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|\ \text{ for a.e. } t\in[a,b]\right\}.$$

The following representation result may be stated [8].

Proposition 11. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma)=\bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{25}$$

On making use of the complex numbers field properties we can also state that:

Proposition 12. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that*

$$\begin{aligned}
\bar{U}_{[a,b]}(\gamma,\Gamma)\ =\ &\{h:[a,b]\to\mathbb{C}\mid\big(\operatorname{Re}(\Gamma)-\operatorname{Re}(h(t))\big)\big(\operatorname{Re}(h(t))-\operatorname{Re}(\gamma)\big)\\
&+\big(\operatorname{Im}(\Gamma)-\operatorname{Im}(h(t))\big)\big(\operatorname{Im}(h(t))-\operatorname{Im}(\gamma)\big)\geq 0 \textit{ for a.e. } t\in[a,b]\}.
\end{aligned}\tag{26}$$

If we assume that $\operatorname{Re}(\Gamma)\geq\operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma)\geq\operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned}
\bar{S}_{[a,b]}(\gamma,\Gamma)\ :=\ &\{h:[a,b]\to\mathbb{C}\mid\operatorname{Re}(\Gamma)\geq\operatorname{Re}(h(t))\geq\operatorname{Re}(\gamma)\\
&\text{and }\operatorname{Im}(\Gamma)\geq\operatorname{Im}(h(t))\geq\operatorname{Im}(\gamma)\text{ for a.e. } t\in[a,b]\}.
\end{aligned}\tag{27}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset\neq\bar{S}_{[a,b]}(\gamma,\Gamma)\subseteq\bar{U}_{[a,b]}(\gamma,\Gamma). \tag{28}$$

Theorem 13. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, $f' : [a,b] \subset \mathring{I} \to \mathbb{C}$ be absolutely continuous on $[a,b]$ and $\zeta \in [a.b]$. For some $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, assume that $f'' \in \bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω such that $f \circ g$, $(g-\zeta)f' \circ g$, $(g-\zeta)^2 \in L(\Omega,\mu)$, with $\int_\Omega d\mu = 1$, then*

$$\left|\int_\Omega (f\circ g)\,d\mu - f(\zeta) - \int_\Omega (g-\zeta)f'\circ g\,d\mu + \frac{\gamma+\Gamma}{4}\int_\Omega (g-\zeta)^2\,d\mu\right| \le \frac{1}{4}|\Gamma-\gamma|\left[\int_\Omega\left(g-\int_\Omega g\,d\mu\right)^2 d\mu + \left(\int_\Omega g\,d\mu - \zeta\right)^2\right]. \tag{29}$$

In particular, we have the following Ostrowski type inequality:

$$\begin{aligned}&\left|\int_\Omega (f\circ g)\,d\mu - f\left(\frac{a+b}{2}\right)\right.\\ &\left.-\int_\Omega\left(g-\frac{a+b}{2}\right)f'\circ g\,d\mu + \frac{\gamma+\Gamma}{4}\int_\Omega\left(g-\frac{a+b}{2}\right)^2 d\mu\right|\\ &\le \frac{1}{4}|\Gamma-\gamma|\left[\int_\Omega\left(g-\int_\Omega g\,d\mu\right)^2 d\mu + \left(\int_\Omega g\,d\mu - \frac{a+b}{2}\right)^2\right],\end{aligned}$$

and we have the following Jensen type inequality:

$$\begin{aligned}&\left|\int_\Omega (f\circ g)\,d\mu - f\left(\int_\Omega g\,d\mu\right) - \int_\Omega\left(g-\int_\Omega g\,d\mu\right)f'\circ g\,d\mu + \frac{\gamma+\Gamma}{4}\sigma^2(g)\right|\\ &\le \frac{1}{4}|\Gamma-\gamma|\sigma^2(g).\end{aligned}$$

Proof. By equality (8), for $\lambda = \frac{\gamma+\Gamma}{2}$ we have

$$\begin{aligned}&f(\zeta) + \int_\Omega (g-\zeta)f'\circ g\,d\mu - \frac{\gamma+\Gamma}{4}\int_\Omega (g-\zeta)^2\,d\mu - \int_\Omega (f\circ g)\,d\mu\\ &= \int_\Omega\left[(g-\zeta)^2\int_0^1 s\left[f''\big((1-s)\zeta+sg\big) - \frac{\gamma+\Gamma}{2}\right]ds\right]d\mu.\end{aligned} \tag{30}$$

Since $f'' \in \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, we have

$$\left|f''\big((1-s)\zeta+sg\big) - \frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}|\Gamma-\gamma|, \tag{31}$$

for almost every $s \in [0,1]$ and any $t \in \Omega$. Multiply (31) with $s > 0$ and integrate over $[0,1]$, we obtain

$$\int_0^1 s\left|f''\big((1-s)\zeta+sg\big) - \frac{\gamma+\Gamma}{2}\right| ds \le \frac{1}{2}|\Gamma-\gamma|\int_0^1 s\,ds = \frac{1}{4}|\Gamma-\gamma|, \tag{32}$$

for any $t \in \Omega$. Taking the modulus of (30), we get the following, by (32) that

$$\begin{aligned}&\left|\int_\Omega (f\circ g)\,d\mu - f(\zeta) - \int_\Omega (g-\zeta)f'\circ g\,d\mu + \frac{\gamma+\Gamma}{4}\int_\Omega (g-\zeta)^2\,d\mu\right|\\ &\le \int_\Omega\left[(g-\zeta)^2\int_0^1 s\left|f''\big((1-s)\zeta+sg\big) - \frac{\gamma+\Gamma}{2}\right| ds\right]d\mu\\ &\le \frac{1}{4}|\Gamma-\gamma|\int_\Omega (g-\zeta)^2\,d\mu,\end{aligned}$$

and the proof is completed (note the use of (24)). □

Corollary 14. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, $f' : [a,b] \subset \mathring{I} \to \mathbb{C}$ be a convex function on $[a,b]$ and $\zeta \in [a,b]$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω such that $f \circ g$, $(g-\zeta)f' \circ g$, $(g-\zeta)^2 \in L(\Omega,\mu)$, with $\int_\Omega d\mu = 1$, then*

$$\left|\int_\Omega (f\circ g)\,d\mu - f(\zeta) - \int_\Omega (g-\zeta)f'\circ g\,d\mu + \frac{f''_+(a)+f''_-(b)}{4}\int_\Omega (g-\zeta)^2\,d\mu\right| \\ \leq \frac{1}{4}|f''_-(b)-f''_+(a)|\left[\sigma^2(g) + \left(\int_\Omega g\,d\mu - \zeta\right)^2\right]. \tag{33}$$

In particular, we have

$$\left|\int_\Omega (f\circ g)\,d\mu - f\left(\frac{a+b}{2}\right) \right. \\ \left. - \int_\Omega \left(g-\frac{a+b}{2}\right) f'\circ g\,d\mu + \frac{f''_+(a)+f''_-(b)}{4}\int_\Omega \left(g-\frac{a+b}{2}\right)^2 d\mu\right| \\ \leq \frac{1}{4}|f''_-(b)-f''_+(a)|\left[\sigma^2(g) + \left(\int_\Omega g\,d\mu - \frac{a+b}{2}\right)^2\right], \tag{34}$$

and

$$\left|\int_\Omega (f\circ g)\,d\mu - f\left(\int_\Omega g\,d\mu\right) \right. \\ \left. - \int_\Omega \left(g - \int_\Omega g\,d\mu\right) f'\circ g\,d\mu + \frac{f''_+(a)+f''_-(b)}{4}\sigma^2(g)\right| \\ \leq \frac{1}{4}|f''_-(b)-f''_+(a)|\sigma^2(g).$$

5. Inequalities for Convex Second Derivatives

In this section, we assume that $|f''|$ satisfies some convexity properties. Recall the following definitions of convexity:

Definition 15. Let $h : I \subset \mathbb{R} \to \mathbb{R}$ be a real-valued function.

(1) h is convex, if for any $x, y \in I$ and $s \in [0,1]$, we have
$$h\big((1-s)x+sy\big) \leq (1-s)h(x) + sh(y).$$
(2) h is quasi-convex, if for any $x, y \in I$ and $s \in [0,1]$, we have
$$h\big((1-s)x+sy\big) \leq \max\{h(x), h(y)\}.$$
(3) h is log-convex, if for any $x, y \in I$ and $s \in [0,1]$, we have
$$h\big((1-s)x+sy\big) \leq h(x)^{1-s}h(y)^s.$$
(4) for a fixed $q \in (0,1]$, h is q-convex, if for any $x, y \in I$ and $s \in [0,1]$, we have
$$h\big((1-s)x+sy\big) \leq (1-s)^q h(x) + s^q h(y).$$

For further background on these notions of convexity and some integral inequalities for functions with some convexity properties, we refer the reader to the paper by Dragomir [7].

Theorem 16. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, $f' : [a,b] \subset \mathring{I} \to \mathbb{C}$ be absolutely continuous on $[a,b]$ and $\zeta \in [a,b]$. Let $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω such that $f \circ g$, $(g-\zeta)f' \circ g$, $(g-\zeta)^2 \in L(\Omega,\mu)$, with $\int_\Omega d\mu = 1$.*

(i) If $|f''|$ is convex, then we have

$$\left|\int_\Omega (f\circ g)\,d\mu - f(\zeta) - \int_\Omega (g-\zeta)(f'\circ g)\,d\mu\right| \leq \frac{1}{3}\left[\frac{1}{2}|f''(\zeta)|\int_\Omega (g-\zeta)^2\,d\mu + \int_\Omega (g-\zeta)^2|f''\circ g|\,d\mu\right]. \tag{35}$$

(ii) If $|f''|$ is quasi-convex, then we have

$$\begin{aligned}&\left|\int_\Omega (f\circ g)\,d\mu - f(\zeta) - \int_\Omega (g-\zeta)(f'\circ g)\,d\mu\right| \\ &\leq \frac{1}{2}\int_\Omega (g-\zeta)^2 \max\{|f''(\zeta)|,|f''\circ g|\}\,d\mu \\ &\leq \begin{cases} \dfrac{|f''(\zeta)|}{2}\displaystyle\int_\Omega (g-\zeta)^2\,d\mu, & |f''\circ g|\leq |f''(\zeta)|; \\ \dfrac{1}{2}\displaystyle\int_\Omega (g-\zeta)^2|f''\circ g|\,d\mu, & |f''\circ g|\geq |f''(\zeta)|.\end{cases}\end{aligned} \tag{36}$$

(iii) If $|f''|$ is log-convex, then we have

$$\left|\int_\Omega (f\circ g)\,d\mu - f(\zeta) - \int_\Omega (g-\zeta)(f'\circ g)\,d\mu\right| \leq \int_\Omega (g-\zeta)^2\left|\frac{|f''(\zeta)|-|f''\circ g|-|f''\circ g|\left[\log(|f''(\zeta)|)-\log(|f''\circ g|)\right]}{\left[\log(|f''(\zeta)|)-\log(|f''\circ g|)\right]^2}\right|\,d\mu. \tag{37}$$

(iv) If $|f''|$ is q-convex (for a fixed $q\in(0,1]$), then we have

$$\left|\int_\Omega (f\circ g)\,d\mu - f(\zeta) - \int_\Omega (g-\zeta)(f'\circ g)\,d\mu\right| \leq \frac{1}{(q+2)}\left[\frac{1}{q+1}|f''(\zeta)|\int_\Omega (g-\zeta)^2\,d\mu + \int_\Omega (g-\zeta)^2|f''\circ g|\,d\mu\right]. \tag{38}$$

Proof. (i) If $|f''|$ is convex, then

$$\left|f''\big((1-s)\zeta+sg\big)\right| \leq (1-s)|f''(\zeta)| + s|f''\circ g|.$$

This implies that

$$\begin{aligned}&\int_0^1 s\left|f''\big((1-s)\zeta+sg(t)\big)\right|\,ds \\ &\leq \left[\int_0^1 s(1-s)\,ds\right]|f''(\zeta)| + \left[\int_0^1 s^2\,ds\right]|f''\circ g(t)| \\ &= \frac{1}{6}|f''(\zeta)| + \frac{1}{3}|f''\circ g(t)|.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_\Omega \left|(g-\zeta)^2\int_0^1 s\left[f''\big((1-s)\zeta+sg(t)\big)\right]\,ds\right|\,d\mu \\ &\leq \frac{1}{6}|f''(\zeta)|\int_\Omega (g-\zeta)^2\,d\mu + \frac{1}{3}\int_\Omega (g-\zeta)^2|f''\circ g|\,d\mu.\end{aligned}$$

The proof is completed by (10).

(ii) If $|f''|$ is quasi-convex, then

$$\left|f''\big((1-s)\zeta+sg\big)\right| \leq \max\{|f''(\zeta)|,|f''\circ g|\}.$$

This implies that

$$\begin{aligned}\int_0^1 s\left|f''\big((1-s)\zeta+sg(t)\big)\right|\,ds &\leq \left[\int_0^1 s\,ds\right]\max\{|f''(\zeta)|,|f''\circ g|\}\\ &= \frac{1}{2}\max\{|f''(\zeta)|,|f''\circ g|\}.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_\Omega\left|(g-\zeta)^2\int_0^1 s\left[f''\big((1-s)\zeta+sg(t)\big)\right]\,ds\right|\,d\mu\\ &\leq\frac{1}{2}\int_\Omega(g-\zeta)^2\max\{|f''(\zeta)|,|f''\circ g|\}\,d\mu.\end{aligned}$$

The proof is completed by (10).

(iii) If $|f''|$ is log-convex, then

$$\left|f''\big((1-s)\zeta+sg\big)\right|\leq|f''(\zeta)|^{1-s}|f''\circ g|^s.$$

This implies that

$$\begin{aligned}&\int_0^1 s\left|f''\big((1-s)\zeta+sg(t)\big)\right|\,ds\\ &\leq\left[\int_0^1 s|f''(\zeta)|^{1-s}|f''\circ g|^s ds\right]\\ &=\frac{|f''(\zeta)|-|f''\circ g|-|f''\circ g|\left[\log(|f''(\zeta)|)-\log(|f''\circ g|)\right]}{\left[\log(|f''(\zeta)|)-\log(|f''\circ g|)\right]^2}.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_\Omega\left|(g-\zeta)^2\int_0^1 s\left[f''\big((1-s)\zeta+sg(t)\big)\right]\,ds\right|\,d\mu\\ &\leq\int_\Omega(g-\zeta)^2\left|\frac{|f''(\zeta)|-|f''\circ g|-|f''\circ g|\left[\log(|f''(\zeta)|)-\log(|f''\circ g|)\right]}{\left[\log(|f''(\zeta)|)-\log(|f''\circ g|)\right]^2}\right|\,d\mu.\end{aligned}$$

The proof is completed by (10).

(iv) If $|f''|$ is q-convex (for a fixed $q\in(0,1]$), then

$$\left|f''\big((1-s)\zeta+sg\big)\right|\leq(1-s)^q|f''(\zeta)|+s^q|f''\circ g|.$$

This implies that

$$\begin{aligned}&\int_0^1 s\left|f''\big((1-s)\zeta+sg(t)\big)\right|\,ds\\ &\leq\left[\int_0^1 s(1-s)^q\,ds\right]|f''(\zeta)|+\left[\int_0^1 s^{q+1}\,ds\right]|f''\circ g(t)|\\ &=\frac{1}{(q+1)(q+2)}|f''(\zeta)|+\frac{1}{q+2}|f''\circ g(t)|.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_\Omega\left|(g-\zeta)^2\int_0^1 s\left[f''\big((1-s)\zeta+sg(t)\big)\right]\,ds\right|\,d\mu\\ &\leq\frac{1}{(q+1)(q+2)}|f''(\zeta)|\int_\Omega(g-\zeta)^2\,d\mu+\frac{1}{q+2}\int_\Omega(g-\zeta)^2|f''\circ g|\,d\mu.\end{aligned}$$

The proof is completed by (10). □

Corollary 17 (Jensen type inequalities)**.** *Under the assumptions of Theorem 16:*

(i) *If $|f''|$ is convex, then we have*

$$\left|\int_\Omega (f\circ g)\,d\mu - f\left(\int_\Omega g\,d\mu\right) - \int_\Omega \left(g - \int_\Omega g\,d\mu\right)(f'\circ g)\,d\mu\right| \tag{39}$$
$$\leq \frac{1}{3}\left[\frac{1}{2}\left|f''\left(\int_\Omega g\,d\mu\right)\right|\sigma^2(g) + \int_\Omega \left(g - \int_\Omega g\,d\mu\right)^2 |f''\circ g|\,d\mu\right].$$

(ii) *If $|f''|$ is quasi-convex, then we have*

$$\left|\int_\Omega (f\circ g)\,d\mu - f\left(\int_\Omega g\,d\mu\right) - \int_\Omega \left(g - \int_\Omega g\,d\mu\right)(f'\circ g)\,d\mu\right|$$
$$\leq \frac{1}{2}\int_\Omega \left(g - \int_\Omega g\,d\mu\right)^2 \max\left\{\left|f''\left(\int_\Omega g\,d\mu\right)\right|, |f''\circ g|\right\}d\mu \tag{40}$$
$$\leq \begin{cases} \frac{1}{2}\left|f''\left(\int_\Omega g\,d\mu\right)\right|\sigma^2(g), & |f''\circ g| \leq |f''(\zeta)|; \\ \frac{1}{2}\int_\Omega \left(g - \int_\Omega g\,d\mu\right)^2 |f''\circ g|\,d\mu, & |f''\circ g| \geq |f''(\zeta)|. \end{cases}$$

(iii) *If $|f''|$ is log-convex, then we have*

$$\left|\int_\Omega (f\circ g)\,d\mu - f\left(\int_\Omega g\,d\mu\right) - \int_\Omega \left(g - \int_\Omega g\,d\mu\right)(f'\circ g)\,d\mu\right| \tag{41}$$
$$\leq \int_\Omega \left(g - \int_\Omega g\,d\mu\right)^2$$
$$\times \left|\frac{|f''(\int_\Omega g\,d\mu)| - |f''\circ g| - |f''\circ g|\left[\log(|f''(\int_\Omega g\,d\mu)|) - \log(|f''\circ g|)\right]}{\left[\log(|f''(\int_\Omega g\,d\mu)|) - \log(|f''\circ g|)\right]^2}\right| d\mu.$$

(iv) *If $|f''|$ is q-convex (for a fixed $q\in(0,1]$), then we have*

$$\left|\int_\Omega (f\circ g)\,d\mu - f\left(\int_\Omega g\,d\mu\right) - \int_\Omega \left(g - \int_\Omega g\,d\mu\right)(f'\circ g)\,d\mu\right| \tag{42}$$
$$\leq \frac{1}{(q+2)}\left[\frac{1}{q+1}\left|f''\left(\int_\Omega g\,d\mu\right)\right|\sigma^2(g) + \int_\Omega \left(g - \int_\Omega g\,d\mu\right)^2 |f''\circ g|\,d\mu\right].$$

6. Applications for f-Divergence

One of the important issues in many applications of Probability Theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures for this purpose have been proposed, extensively studied and have been applied in a variety of fields such as: anthropology, genetics, finance, economics, and political science, biology, the analysis of contingency tables, approximation of probability distributions, signal processing and pattern recognition. A number of these measures of distance are specific cases of Csiszár f-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set Ω and the σ-finite measure μ are given. Consider the set of all probability densities on μ to be $\mathcal{P} := \left\{p | p:\Omega\to\mathbb{R},\, p(t)\geq 0,\, \int_\Omega p(t)\,d\mu(t) = 1\right\}$. The Kullback-Leibler divergence [12] is well known among the information divergences. It is defined as:

$$D_{KL}(p,q) := \int_\Omega p(t)\log\left[\frac{p(t)}{q(t)}\right]d\mu(t), \quad p,q\in\mathcal{P}. \tag{43}$$

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These include the variation distance, Hellinger distance [9], χ^2-divergence, α-divergence, Bhattacharyya distance [1], Harmonic distance, Jeffrey's distance [10], triangular discrimination [17]. We recall the definition of χ^2-divergence, due to its usage in this text:

$$D_{\chi^2}(p,q) := \int_\Omega p(t)\left[\left(\frac{q(t)}{p(t)}\right)^2 - 1\right] d\mu(t), \quad p,q \in \mathcal{P}. \tag{44}$$

For other divergence measures, see the paper [11] by Kapur or the book on line [16] by Taneja.

Csiszár f-divergence is defined as follows [3]

$$I_f(p,q) := \int_\Omega p(t) f\left[\frac{q(t)}{p(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}, \tag{45}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived. The χ^2-divergence and the above mention distances are particular instances of Csiszár f-divergence. There are also many others which are not in this class [16]. For the basic properties of Csiszár f-divergence, we refer the readers to [3], [4] and [18].

Proposition 18. *Let $f : (0,\infty) \to \mathbb{R}$ be a convex function with the property that $f(1) = 0$. Assume that $p, q \in \mathcal{P}$ and there exists constants $0 < r < 1 < R < \infty$ such that*

$$r \le \frac{q(t)}{p(t)} \le R, \quad \text{for } \mu\text{-a.e. } t \in \Omega. \tag{46}$$

If $\zeta \in [r,R]$, then we have the inequalities

$$\left|I_f(p,q) - I_{\Phi_\zeta}(p,q) - f(\zeta)\right| \le \frac{1}{2}\|f''\|_{[r,R],\infty}\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right], \tag{47}$$

where $\Phi_\zeta(x) = (x-\zeta)f'(x)$, $x > 0$. In particular, we have

$$\begin{aligned} &\left|I_f(p,q) - I_{\Phi_{(r+R)/2}}(p,q) - f\left(\frac{r+R}{2}\right)\right| \\ &\le \frac{1}{2}\|f''\|_{[r,R],\infty}\left[D_{\chi^2}(p,q) + \left(\frac{r+R}{2} - 1\right)^2\right], \end{aligned} \tag{48}$$

and when $\zeta = 1$,

$$|I_f(p,q) - I_{\Phi_1}(p,q)| \le \frac{1}{2}\|f''\|_{[r,R],\infty} D_{\chi^2}(p,q). \tag{49}$$

Proof. We choose $g(t) = q(t)/p(t)$ and noting that $\int_\Omega p(t)\,d\mu = 1$, in inequality (22), we have

$$\begin{aligned}
&\left|\int_\Omega f\left(\frac{q(t)}{p(t)}\right) p(t)\,d\mu - f(\zeta) - \int_\Omega \big(q(t)-\zeta p(t)\big) f'\left(\frac{q(t)}{p(t)}\right)\,d\mu\right| \\
&= \left|I_f(p,q) - f(\zeta) - \int_\Omega \big(q(t)-\zeta p(t)\big) f'\left(\frac{q(t)}{p(t)}\right)\,d\mu\right| \\
&\le \frac{1}{2}\|f''\|_{[r,R],\infty}\left[\int_\Omega \left(\frac{q(t)}{p(t)} - \int_\Omega q(t)\,d\mu\right)^2 p(t)\,d\mu + \left(\int_\Omega q(t)\,d\mu - \zeta\right)^2\right] \\
&= \frac{1}{2}\|f''\|_{[r,R],\infty}\left[\int_\Omega \left(\frac{q(t)}{p(t)} - 1\right)^2 p(t)\,d\mu + (\zeta-1)^2\right] \\
&= \frac{1}{2}\|f''\|_{[r,R],\infty}\left[\int_\Omega \left(\frac{q^2(t)}{p(t)} - 2q(t) + p(t)\right) d\mu + (\zeta-1)^2\right] \\
&= \frac{1}{2}\|f''\|_{[r,R],\infty}\left[\int_\Omega \frac{q^2(t)}{p(t)}\,d\mu - 1 + (\zeta-1)^2\right] \\
&= \frac{1}{2}\|f''\|_{[r,R],\infty}\left[\int_\Omega \left(\frac{q^2(t)}{p(t)} - p(t)\right) d\mu + (\zeta-1)^2\right] \\
&= \frac{1}{2}\|f''\|_{[r,R],\infty}\left[\int_\Omega \left(\frac{q^2(t)}{p^2(t)} - 1\right) p(t)\,d\mu + (\zeta-1)^2\right] \\
&= \frac{1}{2}\|f''\|_{[r,R],\infty}\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right];
\end{aligned}$$

and this completes the proof. □

Proposition 19. *With the assumptions of Proposition 18, we have*

$$\begin{aligned}
(50)\qquad &\left|I_f(p,q) - I_{\Phi_\zeta}(p,q) - f(\zeta) + \frac{f''_+(r)+f''_-(R)}{4}\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right]\right| \\
&\le \frac{1}{4}|f''_-(R) - f''_+(r)|\left[D_{\chi^2}(p,q) + (\zeta-1)^2\right],
\end{aligned}$$

for $\zeta \in [r,R]$. In particular, we have

$$\begin{aligned}
&\Bigg|I_f(p,q) - I_{\Phi_{r+R/2}}(p,q) - f\left(\frac{r+R}{2}\right) \\
(51)\qquad &+\frac{f''_+(r)+f''_-(R)}{4}\left[D_{\chi^2}(p,q) + \left(\frac{r+R}{2} - 1\right)^2\right]\Bigg| \\
&\le \frac{1}{4}|f''_-(R) - f''_+(r)|\left[D_{\chi^2}(p,q) + \left(\frac{r+R}{2} - 1\right)^2\right],
\end{aligned}$$

and when $\zeta = 1$,

$$\begin{aligned}
(52)\qquad &\left|I_f(p,q) - I_{\Phi_1}(p,q) + \frac{f''_+(r)+f''_-(R)}{4} D_{\chi^2}(p,q)\right| \\
&\le \frac{1}{4}|f''_-(R) - f''_+(r)| D_{\chi^2}(p,q).
\end{aligned}$$

Proof. Utilising inequality (33) for $g(t)=q(t)/p(t)$ and noting that $\int_\Omega p(t)\,d\mu=1$, we have

$$\begin{aligned}
&\left|\int_\Omega f\left(\frac{q(t)}{p(t)}\right)p(t)\,d\mu-f(\zeta)-\int_\Omega (q(t)-\zeta p(t))\,f'\left(\frac{q(t)}{p(t)}\right)\,d\mu\right.\\
&\left.+\frac{f''_+(r)+f''_-(R)}{4}\int_\Omega\left(\frac{q(t)}{p(t)}-\zeta\right)^2p(t)\,d\mu\right|\\
&=\left|I_f(p,q)-f(\zeta)-\int_\Omega (q(t)-\zeta p(t))\,f'\left(\frac{q(t)}{p(t)}\right)\,d\mu\right.\\
&\left.+\frac{f''_+(r)+f''_-(R)}{4}\left[D_{\chi^2}(p,q)+(\zeta-1)^2\right]\right|\\
&=\left|I_f(p,q)-I_{\Phi_\zeta}(p,q)-f(\zeta)+\frac{f''_+(r)+f''_-(R)}{4}\left[D_{\chi^2}(p,q)+(\zeta-1)^2\right]\right|\\
&\le\frac{1}{4}|f''_-(R)-f''_+(r)|\left[\int_\Omega\left(\frac{q(t)}{p(t)}-1\right)^2p(t)\,d\mu+(1-\zeta)^2\right]\\
&=\frac{1}{4}|f''_-(R)-f''_+(r)|\left[D_{\chi^2}(p,q)+(\zeta-1)^2\right].
\end{aligned}$$

Note that we make use of the following:

$$\begin{aligned}
\int_\Omega\left(\frac{q(t)}{p(t)}-\zeta\right)^2p(t)\,d\mu &= \int_\Omega\left(\frac{q(t)}{p(t)}\right)^2p(t)\,d\mu-2\zeta+\zeta^2\\
&= \int_\Omega\left(\frac{q(t)}{p(t)}\right)^2p(t)\,d\mu-1+(\zeta-1)^2\\
&= \int_\Omega\left(\left(\frac{q(t)}{p(t)}\right)^2p(t)-p(t)\right)\,d\mu+(\zeta-1)^2\\
&= \int_\Omega\left(\left(\frac{q(t)}{p(t)}\right)^2-1\right)p(t)\,d\mu+(\zeta-1)^2\\
&= D_{\chi^2}(p,q)+(\zeta-1)^2;
\end{aligned}$$

and this completes the proof. □

Example 20. If we consider the convex function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t\log(t)$, then

$$I_f(p,q)=\int_\Omega p(t)\frac{q(t)}{p(t)}\log\left(\frac{q(t)}{p(t)}\right)d\,\mu(t)=\int_\Omega q(t)\log\left(\frac{q(t)}{p(t)}\right)d\,\mu(t)=D_{KL}(q,p).$$

We also have $f'(t)=\log(t)+1$ and $f''(t)=1/t$; and therefore

$$\begin{aligned}
I_{\Phi_\zeta}(p,q) &= \int_\Omega (q(t)-\zeta p(t))\left[\log\left(\frac{q(t)}{p(t)}\right)+1\right]\,d\mu\\
&= \int_\Omega q(t)\log\left(\frac{q(t)}{p(t)}\right)\,d\mu+\int_\Omega q(t)\,d\mu\\
&\quad -\zeta\int_\Omega p(t)\log\left(\frac{q(t)}{p(t)}\right)\,d\mu-\zeta\int_\Omega p(t)\,d\mu\\
&= D_{KL}(q,p)+D_{KL}(p,q)+1-\zeta,
\end{aligned}$$

for all $\zeta \in [r, R]$. By Proposition 18, we have the following inequalities

$$\begin{aligned} |D_{KL}(p,q) + \zeta [\log(\zeta) - 1] + 1| &\leq \frac{1}{2} \left[\sup_{x \in [r,R]} \frac{1}{x} \right] \left[D_{\chi^2}(p,q) + (\zeta - 1)^2 \right] \\ &= \frac{1}{2r} \left[D_{\chi^2}(p,q) + (\zeta - 1)^2 \right], \end{aligned}$$

for all $\zeta \in [r, R]$; and

$$\begin{aligned} &\left| D_{KL}(p,q) + \left(\frac{r+R}{2} \right) \left[\log \left(\frac{r+R}{2} \right) - 1 \right] + 1 \right| \\ &\leq \frac{1}{2r} \left[D_{\chi^2}(p,q) + \left(\frac{r+R}{2} - 1 \right)^2 \right], \end{aligned}$$

and when $\zeta = 1$,

$$|D_{KL}(p,q) + 1| \leq \frac{1}{2r} D_{\chi^2}(p,q),$$

which is equivalent to

$$0 \leq D_{KL}(p,q) \leq \frac{1}{2r} D_{\chi^2}(p,q) - 1.$$

Furthermore, by Proposition 19, we have the inequalities:

$$\begin{aligned} &\left| D_{KL}(p,q) + \zeta [\log(\zeta) - 1] + 1 - \frac{r+R}{4rR} \left[D_{\chi^2}(p,q) + (\zeta - 1)^2 \right] \right| \\ &\leq \frac{R-r}{4rR} \left[D_{\chi^2}(p,q) + (\zeta - 1)^2 \right], \end{aligned}$$

for $\zeta \in [r, R]$; and

$$\begin{aligned} &\left| D_{KL}(p,q) + \left(\frac{r+R}{2} \right) \left[\log \left(\frac{r+R}{2} \right) - 1 \right] + 1 \right. \\ &\left. - \frac{r+R}{4rR} \left[D_{\chi^2}(p,q) + \left(\frac{r+R}{2} - 1 \right)^2 \right] \right| \\ &\leq \frac{R-r}{4rR} \left[D_{\chi^2}(p,q) + \left(\frac{r+R}{2} - 1 \right)^2 \right], \end{aligned}$$

and when $\zeta = 1$,

$$\left| D_{KL}(p,q) + 1 - \frac{r+R}{4rR} D_{\chi^2}(p,q) \right| \leq \frac{R-r}{4rR} D_{\chi^2}(p,q).$$

Example 21. If we consider the convex function $f : (0, \infty) \to \mathbb{R}$, $f(t) = -\log(t)$, then

$$I_f(p,q) = -\int_\Omega p(t) \log \left(\frac{q(t)}{p(t)} \right) d\,\mu(t) = \int_\Omega p(t) \log \left(\frac{p(t)}{q(t)} \right) d\,\mu(t) = D_{KL}(p,q).$$

We also have $f'(t) = -1/t$ and $f''(t) = 1/t^2$; therefore

$$\begin{aligned} I_{\Phi_\zeta}(p,q) &= \int_\Omega (-q(t) + \zeta p(t)) \left(\frac{p(t)}{q(t)} \right) d\mu \\ &= \zeta \int_\Omega \frac{p^2(t)}{q(t)} d\mu - \int_\Omega p(t)\, d\mu = \zeta \int_\Omega \frac{p^2(t)}{q(t)} d\mu - 1, \end{aligned}$$

for all $\zeta \in [r, R]$. We note that

$$\int_\Omega \frac{p^2(t)}{q(t)} d\mu = D_{\chi^2}(q,p) + 1.$$

By Proposition 18, we have the following inequalities

$$\begin{aligned}
&\left| D_{KL}(p,q) + \log(\zeta) - \zeta \int_\Omega \frac{p^2(t)}{q(t)}\, d\mu + 1 \right| \\
&= \left| D_{KL}(p,q) + \log(\zeta) - \zeta \left[D_{\chi^2}(q,p) + 1 \right] + 1 \right| \\
&\leq \frac{1}{2} \left[\sup_{x\in[r,R]} \frac{1}{t^2} \right] \left[D_{\chi^2}(p,q) + (\zeta - 1)^2 \right] \\
&= \frac{1}{2r^2} \left[D_{\chi^2}(p,q) + (\zeta - 1)^2 \right],
\end{aligned}$$

for all $\zeta \in [r, R]$; and

$$\begin{aligned}
&\left| D_{KL}(p,q) + \log\left(\frac{r+R}{2} \right) - \frac{r+R}{2} \left[D_{\chi^2}(q,p) + 1 \right] + 1 \right| \\
&\leq \frac{1}{2r^2} \left[D_{\chi^2}(p,q) + \left(\frac{r+R}{2} - 1 \right)^2 \right],
\end{aligned}$$

and when $\zeta = 1$,

$$\left| D_{KL}(p,q) - D_{\chi^2}(q,p) \right| \leq \frac{1}{2r^2} D_{\chi^2}(p,q).$$

Furthermore, by Proposition 19, we have the inequalities:

$$\begin{aligned}
&\Big| D_{KL}(p,q) + \log(\zeta) - \zeta \left[D_{\chi^2}(q,p) + 1 \right] + 1 \\
&+ \frac{r^2+R^2}{4r^2R^2} \left[D_{\chi^2}(p,q) + (\zeta - 1)^2 \right] \Big| \\
&\leq \frac{R^2 - r^2}{4r^2R^2} \left[D_{\chi^2}(p,q) + (\zeta - 1)^2 \right],
\end{aligned}$$

for $\zeta \in [r, R]$; and

$$\begin{aligned}
&\Bigg| D_{KL}(p,q) + \log\left(\frac{r+R}{2} \right) - \frac{r+R}{2} \left[D_{\chi^2}(q,p) + 1 \right] + 1 \\
&+ \frac{r^2+R^2}{4r^2R^2} \left[D_{\chi^2}(p,q) + \left(\frac{r+R}{2} - 1 \right)^2 \right] \Bigg| \\
&\leq \frac{R^2 - r^2}{4r^2R^2} \left[D_{\chi^2}(p,q) + \left(\frac{r+R}{2} - 1 \right)^2 \right],
\end{aligned}$$

and when $\zeta = 1$,

$$\left| D_{KL}(p,q) - D_{\chi^2}(q,p) + \frac{r^2+R^2}{4r^2R^2} \left[D_{\chi^2}(p,q) \right] \right| \leq \frac{R^2 - r^2}{4r^2R^2} D_{\chi^2}(p,q).$$

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