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SOME NEW REVERSES OF YOUNG'S OPERATOR INEQUALITY

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ABSTRACT. In this paper we obtain some new reverses of Young's operator inequality. Some extensions for convex functions of operators are also provided.

1. INTRODUCTION

Throughout this paper A, B are positive operators on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. We use the following notations for operators

$$A\nabla_\nu B := (1-\nu)A + \nu B,$$

the *weighted operator arithmetic mean* and

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},$$

the *weighted operator geometric mean.* When $\nu = \frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^\nu \le (1-\nu)a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality*.

We recall that *Specht's ratio* is defined by [7]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \ne 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^r\right)a^{1-\nu}b^\nu \le (1-\nu)a + \nu b \le S\left(\frac{a}{b}\right)a^{1-\nu}b^\nu, \tag{1.3}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.3) is due to Tominaga [8] while the first one is due to Furuichi [1].

The operator version is as follows [1], [8] :

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Theorem 1. *For two positive operators A, B and positive real numbers m, m', M, M' satisfying the following conditions (i) or (ii):*
(i) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$;
(ii) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$;
we have

$$S\left((h')^r\right) A\sharp_\nu B \leq A\nabla_\nu B \leq S(h)\, A\sharp_\nu B \tag{1.4}$$

where $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ *and* $\nu \in [0,1]$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^r\left(\frac{a}{b}\right) a^{1-\nu} b^\nu \leq (1-\nu)a + \nu b \leq K^R\left(\frac{a}{b}\right) a^{1-\nu} b^\nu \tag{1.6}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [9] while the second by Liao et al. [6].

The operator version is as follows [9], [6]:

Theorem 2. *For two positive operators A, B and positive real numbers m, m', M, M' satisfying the following conditions (i) or (ii):*
(i) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$;
(ii) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$;
we have

$$K^r(h')\, A\sharp_\nu B \leq A\nabla_\nu B \leq K^R(h)\, A\sharp_\nu B \tag{1.7}$$

where $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$, $\nu \in [0,1]$ $r = \min\{1-\nu, \nu\}$ *and* $R = \max\{1-\nu, \nu\}$.

Kittaneh and Manasrah [3], [4] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a} - \sqrt{b}\right)^2 \leq (1-\nu)a + \nu b - a^{1-\nu} b^\nu \leq R\left(\sqrt{a} - \sqrt{b}\right)^2 \tag{1.8}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. The case $\nu = \frac{1}{2}$ reduces (1.8) to an identity.

For some operator versions of (1.8) see [3] and [4].

Motivated by the above results, in this paper we obtain some new reverses of Young's operator inequality. Extensions for convex functions of operators are also provided.

2. Preliminary Results

We have the following result for general convex functions:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I, $a, b \in \mathring{I}$, the interior of I, with $a < b$ and $\nu \in [0, 1]$. Then*

$$\begin{aligned} &\nu (1-\nu)(b-a)\left[f'_{+}((1-\nu)a+\nu b)-f'_{-}((1-\nu)a+\nu b)\right] \\ &\le (1-\nu) f(a)+\nu f(b)-f((1-\nu)a+\nu b) \\ &\le \nu(1-\nu)(b-a)\left[f'_{-}(b)-f'_{+}(a)\right]. \end{aligned} \tag{2.1}$$

In particular, we have

$$\begin{aligned} \frac{1}{4}(b-a)\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right] &\le \frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right) \\ &\le \frac{1}{4}(b-a)\left[f'_{-}(b)-f'_{+}(a)\right]. \end{aligned} \tag{2.2}$$

The constant $\frac{1}{4}$ is best possible in both inequalities from (2.2).

Proof. The case $\nu = 0$ or $\nu = 1$ reduces to equality in (2.1).

Since f is convex on I it follows that the function is differentiable on $\mathring{I}$ except a countably number of points, the lateral derivatives $f'_{\pm}$ exists in each point of $\mathring{I}$, they are increasing on $\mathring{I}$ and $f'_{-} \le f'_{+}$ on $\mathring{I}$.

For any $x, y \in \mathring{I}$ we have for the Lebesgue integral

$$f(x) = f(y) + \int_y^x f'(s)\,ds = f(y) + (x-y)\int_0^1 f'((1-t)y+tx)\,dt. \tag{2.3}$$

Assume that $a < b$ and $\nu \in (0, 1)$. By (2.3) we have

$$\begin{aligned} &f((1-\nu)a+\nu b) \\ &= f(a)+\nu(b-a)\int_0^1 f'((1-t)a+t((1-\nu)a+\nu b))\,dt \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &f((1-\nu)a+\nu b) \\ &= f(b)-(1-\nu)(b-a)\int_0^1 f'((1-t)b+t((1-\nu)a+\nu b))\,dt. \end{aligned} \tag{2.5}$$

If we multiply (2.4) by $1-\nu$, (2.4) by ν and add the obtained equalities, then we get

$$\begin{aligned} f((1-\nu)a+\nu b) = (1-\nu) f(a)+\nu f(b) & \\ +(1-\nu)\nu(b-a)\int_0^1 & f'((1-t)a+t((1-\nu)a+\nu b))\,dt \\ -(1-\nu)\nu(b-a)\int_0^1 & f'((1-t)b+t((1-\nu)a+\nu b))\,dt, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &(1-\nu) f(a)+\nu f(b)-f((1-\nu)a+\nu b) = (1-\nu)\nu(b-a) \\ &\times \int_0^1 \left[f'((1-t)b+t((1-\nu)a+\nu b))-f'((1-t)a+t((1-\nu)a+\nu b))\right]dt. \end{aligned} \tag{2.6}$$

That is an equality of interest in itself.

Since $a<b$ and $\nu\in(0,1)$, then $(1-\nu)a+\nu b\in(a,b)$ and

$$(1-t)a+t((1-\nu)a+\nu b)\in[a,(1-\nu)a+\nu b]$$

while

$$(1-t)b+t((1-\nu)a+\nu b)\in[(1-\nu)a+\nu b,b]$$

for any $t\in[0,1]$.

By the monotonicity of the derivative we have

$$f'_+((1-\nu)a+\nu b)\le f'((1-t)b+t((1-\nu)a+\nu b))\le f'_-(b) \tag{2.7}$$

and

$$f'_+(a)\le f'((1-t)a+t((1-\nu)a+\nu b))\le f'_-((1-\nu)a+\nu b) \tag{2.8}$$

for any $t\in[0,1]$.

By integrating the inequalities (2.7) and (2.8) we get

$$f'_+((1-\nu)a+\nu b)\le\int_0^1 f'((1-t)b+t((1-\nu)a+\nu b))\,dt\le f'_-(b)$$

and

$$f'_+(a)\le\int_0^1 f'((1-t)a+t((1-\nu)a+\nu b))\,dt\le f'_-((1-\nu)a+\nu b),$$

which implies that

$$f'_+((1-\nu)a+\nu b)-f'_-((1-\nu)a+\nu b)\le\int_0^1 f'((1-t)b+t((1-\nu)a+\nu b))\,dt$$
$$-\int_0^1 f'((1-t)a+t((1-\nu)a+\nu b))\,dt\le f'_-(b)-f'_+(a).$$

Making use of the equality (2.6) we the obtain the desired result (2.1).

If we consider the convex function $f:[a,b]\to\mathbb{R}$, $f(x)=\left|x-\frac{a+b}{2}\right|$, then we have $f'_+\left(\frac{a+b}{2}\right)=1$, $f'_-\left(\frac{a+b}{2}\right)=-1$ and by replacing in (2.2) we get in all terms the same quantity $\frac{1}{2}(b-a)$ which show that the constant $\frac{1}{4}$ is best possible in both inequalities from (2.2). □

Corollary 1. *If the function $f:I\subset\mathbb{R}\to\mathbb{R}$ is a differentiable convex function on $\mathring{I}$, then for any $a,b\in\mathring{I}$ and $\nu\in[0,1]$ we have*

$$\begin{aligned}0&\le(1-\nu)f(a)+\nu f(b)-f((1-\nu)a+\nu b)\\&\le\nu(1-\nu)(b-a)[f'(b)-f'(a)].\end{aligned} \tag{2.9}$$

Proof. If $a<b$, then the inequality (2.9) follows by (2.1). If $b<a$, then by (2.1) we get

$$\begin{aligned}0&\le(1-\nu)f(b)+\nu f(a)-f((1-\nu)b+\nu a)\\&\le\nu(1-\nu)(b-a)[f'(b)-f'(a)]\end{aligned} \tag{2.10}$$

for any $\nu\in[0,1]$. If we replace ν by $1-\nu$ in (2.10), then we get (2.9). □

Theorem 3. *For any $a,b>0$ and $\nu\in[0,1]$ we have*

$$0\le(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\le\nu(1-\nu)(a-b)(\ln a-\ln b) \tag{2.11}$$

and

$$1\le\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\le\exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{2.12}$$

where K is Kantorovich's constant.

Proof. If we write the inequality (2.9) for the convex function $f:\mathbb{R}\to(0,\infty)$, $f(x)=\exp(x)$, then we have

$$0\le(1-\nu)\exp(x)+\nu\exp(y)-\exp((1-\nu)x+\nu y)\le\nu(1-\nu)(x-y)[\exp(x)-\exp(y)]. \tag{2.13}$$

for any $x,y\in\mathbb{R}$ and $\nu\in[0,1]$.

Let $a,b>0$. If we take $x=\ln a$, $y=\ln b$ in (2.13), then we get the desired inequality (2.11).

Now, if we write the inequality (2.9) for the convex function $f:(0,\infty)\to\mathbb{R}$, $f(x)=-\ln x$, then we get

$$0\le\ln((1-\nu)a+\nu b)-(1-\nu)\ln a-\nu\ln b\le\nu(1-\nu)\frac{(b-a)^2}{ab},$$

namely

$$\ln\left[\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\right]\le\nu(1-\nu)\frac{(b-a)^2}{ab}.$$

This is equivalent to the desired result (2.12). □

Remark 1. *In particular, we have from (2.11) and (2.12) that*

$$0\le\frac{a+b}{2}-\sqrt{ab}\le\frac{1}{4}(a-b)(\ln a-\ln b) \tag{2.14}$$

and

$$1\le\frac{\frac{a+b}{2}}{\sqrt{ab}}\le\exp\left(K\left(\frac{a}{b}\right)-1\right). \tag{2.15}$$

It is natural to ask which upper bound provided by Kittaneh-Manasrah inequality (1.8) and our inequality (2.11) is better.

Consider the functions

$$g_\nu(x):=\max\{1-\nu,\nu\}\left(\sqrt{x}-1\right)^2,\ h_\nu(x):=\nu(1-\nu)(x-1)\ln x$$

defined for $x>0$. Let $\nu=\frac{1}{4}$ and consider the difference

$$d_{\frac{1}{4}}(x):=h_{\frac{1}{4}}(x)-g_{\frac{1}{4}}(x)=\frac{3}{16}(x-1)\ln x-\frac{3}{4}\left(\sqrt{x}-1\right)^2,\ x>0.$$

If one plots the function $d_{\frac{1}{4}}$ on the interval $(0,5)$, that one realizes that the difference is positive on $(0,1)$ and negative on $(1,5)$ showing that some time inequality (1.8) is better and other time worse than (2.11).

Consider also the functions

$$S(x):=\frac{x^{\frac{1}{x-1}}}{e\ln\left(x^{\frac{1}{x-1}}\right)},\ K_\nu(x):=\left(\frac{(x+1)^2}{4x}\right)^{\max\{1-\nu,\nu\}}$$

and

$$D_\nu(x):=\exp\left[\nu(1-\nu)\frac{(x-1)^2}{x}\right]$$

defined for $x>0$ and $\nu\in[0,1]$.

The $3D$ plots of the differences $K_\nu(x)-S(x)$, $K_\nu(x)-D_\nu(x)$ and $S(x)-D_\nu(x)$ for $x>0$ and $\nu\in[0,1]$ reveal that they take both negative and positive values, showing that, in general, there is no ordering for the upper bounds of the quantity $\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}$ as provided by the inequalities (1.3), (1.6) and (2.12).

3. Operator Inequalities

We have the following result:

Theorem 4. *Let A, B be two positive operators. Then we have*

$$0 \leq A\nabla_{\nu}B - A\sharp_{\nu}B \leq \nu(1-\nu)\left(BA^{-1}-I\right)A\sharp_{\ln}B \tag{3.1}$$

where

$$A\sharp_{\ln}B := A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}.$$

In particular, we have

$$0 \leq A\nabla B - A\sharp B \leq \frac{1}{4}\left(BA^{-1}-I\right)A\sharp_{\ln}B. \tag{3.2}$$

Proof. From the inequality (2.11) we have

$$0 \leq (1-\nu) + \nu x - x^{\nu} \leq \nu(1-\nu)(x-1)\ln x \tag{3.3}$$

for any $x > 0$ and $\nu \in [0,1]$.

Using the functional calculus for continuous functions we have for any positive X that

$$0 \leq (1-\nu)I + \nu X - X^{\nu} \leq \nu(1-\nu)(X-I)\ln X$$

where I is the identity operator.

Substituting $A^{-1/2}BA^{-1/2}$ for X we have

$$\begin{aligned} 0 &\leq (1-\nu)I + \nu A^{-1/2}BA^{-1/2} - \left(A^{-1/2}BA^{-1/2}\right)^{\nu} \\ &\leq \nu(1-\nu)\left(A^{-1/2}BA^{-1/2}-I\right)\ln\left(A^{-1/2}BA^{-1/2}\right) \\ &= \nu(1-\nu)A^{-1/2}(B-A)A^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right) \end{aligned} \tag{3.4}$$

for any $\nu \in [0,1]$.

Multiplying both sides of (3.4) by $A^{1/2}$ we get

$$\begin{aligned} 0 &\leq (1-\nu)A + \nu B - A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2} \\ &\leq \nu(1-\nu)(B-A)A^{-1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} \\ &= \nu(1-\nu)(B-A)A^{-1}A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} \\ &= \nu(1-\nu)\left(BA^{-1}-I\right)A\sharp_{\ln}B \end{aligned}$$

and the inequality (3.1) is proved. □

We have:

Theorem 5. *Let A, B be two positive operators. For positive real numbers m, m', M, M', put $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ and let $\nu \in [0,1]$.*

(i) If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, then

$$0 \leq A\nabla_{\nu}B - A\sharp_{\nu}B \leq \nu(1-\nu)(h-1)\ln hA \tag{3.5}$$

and, in particular

$$0 \le A\nabla B - A\sharp B \le \frac{1}{4}(h-1)\ln hA \tag{3.6}$$

(ii) If $0 < mI \le B \le m'I < M'I \le A \le MI$, *then*

$$0 \le A\nabla_{\nu} B - A\sharp_{\nu} B \le \nu(1-\nu)\frac{h-1}{h}\ln hA \tag{3.7}$$

and, in particular

$$0 \le A\nabla B - A\sharp B \le \frac{1}{4}\frac{h-1}{h}\ln hA. \tag{3.8}$$

Proof. We consider the function $D : (0,\infty) \to [0,\infty)$ defined by $D(x) = (x-1)\ln x$. We have that $D'(x) = \ln x + 1 - \frac{1}{x}$ and $D''(x) = \frac{x+1}{x^2}$ for $x \in (0,\infty)$. This shows that the function is convex on $(0,\infty)$, monotonic decreasing on $(0,1)$ and monotonic increasing on $[1,\infty)$ with the minimum 0 realized in $x = 1$.

From the inequality (3.3) we have

$$0 \le (1-\nu) + \nu x - x^{\nu} \le \nu(1-\nu) D(x)$$

for any $x > 0$, and hence

$$0 \le (1-\nu) I + \nu X - X^{\nu} \le \nu(1-\nu) \max_{k' \le x \le k} D(x) \tag{3.9}$$

for the positive operator X that satisfies the condition $0 < k'I \le X \le kI$ for $0 < k' < k$.

If the condition (i) is valid, then for $X = A^{-1/2}BA^{-1/2}$ we have

$$I < \frac{M'}{m'}I = h'I \le X \le hI = \frac{M}{m}I$$

and by (3.9) we have

$$\begin{aligned} 0 &\le (1-\nu) I + \nu A^{-1/2}BA^{-1/2} - \left(A^{-1/2}BA^{-1/2}\right)^{\nu} \\ &\le \nu(1-\nu) \max_{h' \le x \le h} D(x). \end{aligned} \tag{3.10}$$

Since the function D is increasing on $(1,\infty)$ then $\max_{h' \le x \le h} D(x) = D(h) = (h-1)\ln h$ and by multiplying both sides of (3.10) by $A^{1/2}$ we get (3.5).

If the condition (ii) is valid, then for $X = A^{-1/2}BA^{-1/2}$ we have

$$0 < \frac{1}{h}I \le X \le \frac{1}{h'}I < I$$

and by (3.9) we have

$$\begin{aligned} 0 &\le (1-\nu) I + \nu A^{-1/2}BA^{-1/2} - \left(A^{-1/2}BA^{-1/2}\right)^{\nu} \\ &\le \nu(1-\nu) \max_{\frac{1}{h} \le x \le \frac{1}{h'}} D(x). \end{aligned} \tag{3.11}$$

Since the function D is decreasing on $(0,1)$ then

$$\max_{\frac{1}{h} \le x \le \frac{1}{h'}} D(x) = D\left(\frac{1}{h}\right) = \frac{h-1}{h} D(h)$$

and by multiplying both sides of (3.11) by $A^{1/2}$ we get (3.7). □

We also have:

Theorem 6. *For two positive operators A, B and positive real numbers m, m', M, M' satisfying the following conditions (i) or (ii):*
(i) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$;
(ii) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$;
we have

$$A\nabla_{\nu}B \leq \exp\left[4\nu\left(1-\nu\right)\left(K\left(h\right)-1\right)\right]A\sharp_{\nu}B \tag{3.12}$$

and in particular

$$A\nabla B \leq \exp\left[K\left(h\right)-1\right]A\sharp B. \tag{3.13}$$

Proof. From the inequality (2.12) we have for $a = 1$ and $b = x$ that

$$\begin{aligned}(1-\nu)+\nu x &\leq x^{\nu}\exp\left[4\nu\left(1-\nu\right)\left(K\left(\frac{1}{x}\right)-1\right)\right] \\ &= x^{\nu}\exp\left[4\nu\left(1-\nu\right)\left(K\left(x\right)-1\right)\right]\end{aligned} \tag{3.14}$$

for any $x > 0$ and hence

$$\begin{aligned}0 &\leq \left(1-\nu\right)I+\nu X \leq X^{\nu}\max_{k'\leq x\leq k}\exp\left[4\nu\left(1-\nu\right)\left(K\left(x\right)-1\right)\right] \\ &= X^{\nu}\exp\left[4\nu\left(1-\nu\right)\left(\max_{k'\leq x\leq k}K\left(x\right)-1\right)\right]\end{aligned} \tag{3.15}$$

for the positive operator X that satisfies the condition $0 < k'I \leq X \leq kI$ for $0 < k' < k$.

If the condition (i) is valid, then for $X = A^{-1/2}BA^{-1/2}$ we have

$$I < \frac{M'}{m'}I = h'I \leq X \leq hI = \frac{M}{m}I$$

and by (3.15) we have

$$\begin{aligned}0 &\leq \left(1-\nu\right)I+\nu A^{-1/2}BA^{-1/2} \\ &\leq \left(A^{-1/2}BA^{-1/2}\right)^{\nu}\exp\left[4\nu\left(1-\nu\right)\left(\max_{h'\leq x\leq h}K\left(x\right)-1\right)\right].\end{aligned} \tag{3.16}$$

Since K is monotonic increasing on $(1,\infty)$, then $\max_{h'\leq x\leq h}K\left(x\right) = K\left(h\right)$ and by multiplying both sides of (3.16) by $A^{1/2}$ we get (3.12).

If the condition (ii) is valid, then for $X = A^{-1/2}BA^{-1/2}$ we have

$$0 < \frac{1}{h}I \leq X \leq \frac{1}{h'}I < I$$

and by (3.15) we have

$$\begin{aligned}0 &\leq \left(1-\nu\right)I+\nu A^{-1/2}BA^{-1/2} \\ &\leq \left(A^{-1/2}BA^{-1/2}\right)^{\nu}\exp\left[4\nu\left(1-\nu\right)\left(\max_{\frac{1}{h}\leq x\leq\frac{1}{h'}}K\left(x\right)-1\right)\right].\end{aligned} \tag{3.17}$$

Since K is monotonic decreasing on $(0,1)$, then $\max_{\frac{1}{h}\leq x\leq\frac{1}{h'}}K\left(x\right) = K\left(\frac{1}{h}\right) = K\left(h\right)$ and by multiplying both sides of (3.17) by $A^{1/2}$ we get (3.12). □

4. Some Functional Extensions

We can extend some of the above results for functions of operators as follows.

Theorem 7. *Let $f : [\gamma, \Gamma] \to \mathbb{R}$ be a convex function on the real interval $[\gamma, \Gamma]$ and A a positive operator and B a selfadjoint operator such that*

$$\gamma I \leq A^{-1/2} B A^{-1/2} \leq \Gamma I, \tag{4.1}$$

then we have

$$\begin{aligned} 0 &\leq \frac{\Gamma A - B}{\Gamma - \gamma} f(\gamma) + \frac{B - A\gamma}{\Gamma - \gamma} f(\Gamma) - A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} \\ &\leq \left(\Gamma A^{1/2} - B A^{-1/2}\right)\left(A^{-1/2} B - A^{1/2}\gamma\right) \frac{f'_-(\Gamma) - f'_+(\gamma)}{\Gamma - \gamma} \\ &\leq \frac{1}{4}(\Gamma - \gamma)\left[f'_-(\Gamma) - f'_+(\gamma)\right] A. \end{aligned} \tag{4.2}$$

Proof. From the inequality (2.1) we have

$$\begin{aligned} 0 &\leq (1-\nu) f(\gamma) + \nu f(\Gamma) - f((1-\nu)\gamma + \nu\Gamma) \\ &\leq \nu(1-\nu)(\Gamma - \gamma)\left[f'_-(\Gamma) - f'_+(\gamma)\right], \end{aligned} \tag{4.3}$$

for any $\nu \in [0, 1]$.

If we take in (4.3) $\nu = \frac{x-\gamma}{\Gamma-\gamma} \in [0, 1]$ for $x \in [\gamma, \Gamma]$, then we get

$$\begin{aligned} 0 &\leq \frac{\Gamma - x}{\Gamma - \gamma} f(\gamma) + \frac{x - \gamma}{\Gamma - \gamma} f(\Gamma) - f(x) \leq (\Gamma - x)(x - \gamma) \frac{f'_-(\Gamma) - f'_+(\gamma)}{\Gamma - \gamma} \\ &\leq \frac{1}{4}(\Gamma - \gamma)\left[f'_-(\Gamma) - f'_+(\gamma)\right] \end{aligned} \tag{4.4}$$

for any $x \in [\gamma, \Gamma]$.

Using the functional calculus for continuous functions we have

$$\begin{aligned} 0 &\leq \frac{\Gamma I - X}{\Gamma - \gamma} f(\gamma) + \frac{X - I\gamma}{\Gamma - \gamma} f(\Gamma) - f(X) \\ &\leq (\Gamma I - X)(X - I\gamma) \frac{f'_-(\Gamma) - f'_+(\gamma)}{\Gamma - \gamma} \leq \frac{1}{4}(\Gamma - \gamma)\left[f'_-(\Gamma) - f'_+(\gamma)\right] I \end{aligned} \tag{4.5}$$

for any selfadjoint operator X with $\mathrm{Sp}(X) \subset [\gamma, \Gamma]$.

Now, if we write the inequality (4.5) for $X = A^{-1/2} B A^{-1/2}$ then we get

$$\begin{aligned} 0 &\leq \frac{\Gamma I - A^{-1/2} B A^{-1/2}}{\Gamma - \gamma} f(\gamma) + \frac{A^{-1/2} B A^{-1/2} - I\gamma}{\Gamma - \gamma} f(\Gamma) \\ &- f\left(A^{-1/2} B A^{-1/2}\right) \\ &\leq \left(\Gamma I - A^{-1/2} B A^{-1/2}\right)\left(A^{-1/2} B A^{-1/2} - I\gamma\right) \frac{f'_-(\Gamma) - f'_+(\gamma)}{\Gamma - \gamma} I \\ &\leq \frac{1}{4}(\Gamma - \gamma)\left[f'_-(\Gamma) - f'_+(\gamma)\right] I. \end{aligned} \tag{4.6}$$

By multiplying both sides of (4.6) by $A^{1/2}$ we get (4.2). □

If $\gamma > 0$ in (4.1) and take $f(t) = t^p$, $t > 0$ where $p \in (0,\infty) \cup (1,\infty)$ then we have from (4.2) that

$$
\begin{aligned}
(4.7) \qquad 0 &\le \frac{\Gamma A - B}{\Gamma - \gamma}\gamma^p + \frac{B - A\gamma}{\Gamma - \gamma}\Gamma^p - A\sharp_p B \\
&\le p\left(\Gamma A^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - A^{1/2}\gamma\right)\frac{\Gamma^{p-1} - \gamma^{p-1}}{\Gamma - \gamma} \\
&\le \frac{1}{4}p(\Gamma - \gamma)\left(\Gamma^{p-1} - \gamma^{p-1}\right)A,
\end{aligned}
$$

where $A\sharp_p B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^p A^{1/2}$, $p \in (0,\infty) \cup (1,\infty)$ and $A, B > 0$ satisfy the condition (4.1).

With the same assumptions for A and B and if we take $f(t) = -t^q$, $t > 0$ where $q \in (0,1)$ in (4.2), then we get

$$
\begin{aligned}
(4.8) \qquad 0 &\le A\sharp_q B - \frac{\Gamma A - B}{\Gamma - \gamma}\gamma^q - \frac{B - A\gamma}{\Gamma - \gamma}\Gamma^q \\
&\le \left(\Gamma A^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - A^{1/2}\gamma\right)\frac{\Gamma^{1-q} - \gamma^{1-q}}{(\gamma\Gamma)^{1-q}(\Gamma - \gamma)} \\
&\le \frac{1}{4}(\Gamma - \gamma)\frac{\Gamma^{1-q} - \gamma^{1-q}}{(\gamma\Gamma)^{1-q}}A.
\end{aligned}
$$

We also have:

Theorem 8. *Assume that $\gamma < 1 < \Gamma$ and let $f : [\gamma,\Gamma] \to \mathbb{R}$ be a continuously differentiable convex function on the real interval $[\gamma,\Gamma]$ and A a positive operator and B a selfadjoint operator such that (4.1) is valid. Then we have*

$$
\begin{aligned}
(4.9) \quad 0 &\le (1-\nu) f(1) A + \nu A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \\
&\quad - A^{1/2} f\left((1-\nu) I + \nu A^{-1/2}BA^{-1/2}\right)A^{1/2} \\
&\le \nu(1-\nu) A^{1/2}\left(A^{-1/2}BA^{-1/2} - I\right)\left[f'\left(A^{-1/2}BA^{-1/2}\right) - f'(1) I\right]A^{1/2} \\
&\le \nu(1-\nu) \sup_{x \in (\gamma,\Gamma)}\left\{(x-1)\left[f'(x) - f'(1)\right]\right\}A \\
&\le \nu(1-\nu)(\Gamma - \gamma)\left[f'(\Gamma) - f'(\gamma)\right]A
\end{aligned}
$$

and, in particular

$$
\begin{aligned}
(4.10) \quad 0 &\le \frac{f(1) A + A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}}{2} \\
&\quad - A^{1/2} f\left(\frac{I + A^{-1/2}BA^{-1/2}}{2}\right)A^{1/2} \\
&\le \frac{1}{4}\sup_{x \in (\gamma,\Gamma)}\left\{(x-1)\left[f'(x) - f'(1)\right]\right\}A \le \frac{1}{4}(\Gamma - \gamma)\left[f'(\Gamma) - f'(\gamma)\right]A.
\end{aligned}
$$

Proof. For any $a, b \in (\gamma,\Gamma)$ we have

$$
\begin{aligned}
(4.11) \qquad 0 &\le (1-\nu) f(a) + \nu f(b) - f((1-\nu) a + \nu b) \\
&\le \nu(1-\nu)(b-a)\left[f'(b) - f'(a)\right],
\end{aligned}
$$

for any $\nu \in [0,1]$.

If we take $a=1$ and $b=x$ in (4.11) then we get

$$\begin{aligned}0 &\leq (1-\nu) f(1) + \nu f(x) - f((1-\nu)+\nu x) \\ &\leq \nu(1-\nu)(x-1)\left[f'(x)-f'(1)\right],\end{aligned} \tag{4.12}$$

for any $x \in (\gamma, \Gamma)$.

Since f is continuously differentiable convex on (γ, Γ), hence f' is monotonic nondecreasing and then

$$\begin{aligned}0 &\leq (x-1)\left[f'(x)-f'(1)\right] = |x-1|\left|f'(x)-f'(1)\right| \\ &\leq (\Gamma-\gamma)\left[f'(\Gamma)-f'(\gamma)\right]\end{aligned}$$

for any $x \in (\gamma, \Gamma)$. This implies that

$$\sup_{x\in(\gamma,\Gamma)}\left\{(x-1)\left[f'(x)-f'(1)\right]\right\} \leq (\Gamma-\gamma)\left[f'(\Gamma)-f'(\gamma)\right].$$

From (4.12) we then have

$$\begin{aligned}0 &\leq (1-\nu) f(1) + \nu f(x) - f((1-\nu)+\nu x) \\ &\leq \nu(1-\nu)(x-1)\left[f'(x)-f'(1)\right] \\ &\leq \nu(1-\nu)\sup_{x\in(\gamma,\Gamma)}\left\{(x-1)\left[f'(x)-f'(1)\right]\right\} \\ &\leq \nu(1-\nu)(\Gamma-\gamma)\left[f'(\Gamma)-f'(\gamma)\right]\end{aligned} \tag{4.13}$$

for any $x \in (\gamma, \Gamma)$.

This implies in the operator order that

$$\begin{aligned}0 &\leq (1-\nu) f(1) I + \nu f(X) - f((1-\nu) I+\nu X) \\ &\leq \nu(1-\nu)(X-I)\left[f'(X)-f'(1) I\right] \\ &\leq \nu(1-\nu)\sup_{x\in(\gamma,\Gamma)}\left\{(x-1)\left[f'(x)-f'(1)\right]\right\} I \\ &\leq \nu(1-\nu)(\Gamma-\gamma)\left[f'(\Gamma)-f'(\gamma)\right] I\end{aligned} \tag{4.14}$$

for any selfadjoint operator X with $\operatorname{Sp}(X) \subset [\gamma, \Gamma]$.

If we take in (4.14) $X = A^{-1/2}BA^{-1/2}$ and multiply in both sides with $A^{1/2}$ then we get the desired result (4.9). □

If we take in (4.9) $0<\gamma<1<\Gamma$ and $f(t) = -\ln t$, then we get

$$\begin{aligned}0 &\leq A^{1/2}\left[\ln\left((1-\nu) I+\nu A^{-1/2}BA^{-1/2}\right)\right]A^{1/2} \\ &- \nu A^{1/2}\left[\ln\left(A^{-1/2}BA^{-1/2}\right)\right]A^{1/2} \\ &\leq \nu(1-\nu) A^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)\left(I-A^{1/2}B^{-1}A^{1/2}\right)A^{1/2} \\ &\leq \nu(1-\nu)\sup_{x\in(\gamma,\Gamma)}\left\{\frac{(x-1)^2}{x}\right\}A \leq \nu(1-\nu)\frac{(\Gamma-\gamma)^2}{\gamma\Gamma}A,\end{aligned} \tag{4.15}$$

provided $A, B > 0$ and satisfy (4.1) while $\nu \in [0,1]$.

Since the function $\delta(x) = \frac{(x-1)^2}{x}$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$ and $0<\gamma<1<\Gamma$ then

$$\sup_{x\in(\gamma,\Gamma)}\delta(x) = \max\left\{\frac{(\Gamma-1)^2}{\Gamma}, \frac{(\gamma-1)^2}{\gamma}\right\}$$

and from (4.15) we get

$$0 \leq A^{1/2}\left[\ln\left((1-\nu)I+\nu A^{-1/2}BA^{-1/2}\right)\right]A^{1/2} - \nu A^{1/2}\left[\ln\left(A^{-1/2}BA^{-1/2}\right)\right]A^{1/2} \leq \nu(1-\nu)A^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)\left(I-A^{1/2}B^{-1}A^{1/2}\right)A^{1/2} \leq \nu(1-\nu)\max\left\{\frac{(\Gamma-1)^2}{\Gamma},\frac{(\gamma-1)^2}{\gamma}\right\}A \leq \nu(1-\nu)\frac{(\Gamma-\gamma)^2}{\gamma\Gamma}A. \tag{4.16}$$

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