Received 09/09/15

SOME INEQUALITIES FOR ABSOLUTE VALUE IN HILBERT C^*-MODULES

Hamid Reza Moradi[1] Mohsen Erfanian Omidvar[2]
Silvestru Sever Dragomir[3]

Abstract

Some results related to the reverse Triangle inequality in the frame work of Hilbert C^*-module are given.

Keywords: Triangle inequality, Hilber C^*-module, C^*-algebra.
Mathematics Subject Classification [2010]: Primary 46L08, secondary 15A39, 26D15.

1 Introduction and Preliminaries

In Mathematical Analysis one of the most important inequalities is the (generalized) triangle inequality which states that, in a normed linear space $(\mathcal{X}; \|\cdot\|)$ we have the inequality

$$\left\| \sum_{i=1}^{n} x_i \right\| \leq \sum_{i=1}^{n} \|x_i\|,$$

for any vectors $x_i \in \mathcal{X}, i \in \{1, ..., n\}$.

A number of mathematicians have investigated the above inequality in various settings. We refer to interesting papers by Shrawan et al. [10] and Dadipour et al. [2]. Moreover, Farenick [5] have investigated the Triangle inequality over matrix algebras in Hilbert C^*-modules. Some versions of the Triangle inequality with simple conditions for the case of equality are presented in [1, 9].

A reverse of the generalized Triangle inequality in Hilbert space was given in [3] as follows:

Theorem 1.1. *Let $(\mathcal{H}; \langle \cdot, \cdot \rangle)$ be an inner product space, $x_i \in \mathcal{H}$, for all $i \in \{1, ..., n\}$ and $p_i \geq 0$ with $\sum\limits_{i=1}^{n} p_i = 1$ (probability distributioin). If there exists constants $r_i > 0, i \in \{1, ..., n\}$, so that*

$$\left\| x_i - \sum_{j=1}^{n} p_j x_j \right\| \leq r_i$$

for all $i \in \{1, ..., n\}$, then

$$\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \leq \sum_{i=1}^{n} p_i r_i^2.$$

Our aim is to extend some of these generalizations of the Triangle inequality in the framework of Hilbert C^*-modules. The notion of Hilbert C^*-module is a generalization of the notion of Hilbert space. This object was first used by I. Kaplansky [6]. Hilbert C^*-modules are useful tools in Kasparov's formulation of K-theory, theory of operator algebras, group representation theory, noncommutative geometry and theory of operator spaces.

In this section we recall some fundamental definitions in the theory of Hilbert modules that will be used in the sequel. Suppoose that $\mathcal{A}$ is a C^*-algebra and $\mathcal{X}$ is a linear space which is an algebric right $\mathcal{A}$-module satisfying $\lambda (xa) = x (\lambda a) = (\lambda x) a$ for all $x \in \mathcal{X}, a \in \mathcal{A}, \lambda \in \mathbb{C}$. The space $\mathcal{X}$ is called a pre-Hilbert $\mathcal{A}$-module (or inner product $\mathcal{A}$-module) if there exists an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying the following properties:

(a) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(b) $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$;

(c) $\langle x, ya \rangle = \langle x, y \rangle a$;

(d) $\langle x, y \rangle^* = \langle y, x \rangle$;

for all $x, y, z \in \mathcal{X}, a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. A pre-Hilbert $\mathcal{A}$-module is called a (right) Hilbert C^*-module over $\mathcal{A}$ (or a (right) Hilbert $\mathcal{A}$-module) if it is complete with respect to its norm. Two typical examples of Hilbert C^*-modules are as follows:

(a) Every Hilbert space is a Hilbert $\mathbb{C}$-module.

(b) Every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module via $\langle a, b \rangle = a^* b \ (a, b \in \mathcal{A})$.

Notice that the inner structure of a C^*-algebra is essentially more complicated than complex numbers. For instance, the notations such as orthogonality and theorems such as Riesz representation in the complex Hilbert space theory cannot simply be generalized or transferred to the theory of Hilbert C^*-modules.

For every $x \in \mathcal{X}$ we define the absolute value of x as the unique positive square root of $\langle x, x \rangle$, that is, $|x| = \langle x, x \rangle^{\frac{1}{2}}$.

We refere the reader to [7] for undefined notions on Hilbert C^*-modules.

2 Main Results

We start our work by presenting a reverse of the Triangle inequality for Hilbert C^*-modules.

Theorem 2.1. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and $x_i \in \mathcal{X}$ for all $i \in \{1, ..., n\}$, and p_i are in real or complex number field such that $\sum_{i=1}^{n} p_i = 1$. If there exist positive elements r_i, $i \in \{1, ..., n\}$ in $\mathcal{A}$, so that*

$$\left| x_i - \sum_{j=1}^{n} p_j x_j \right|^2 \leq r_i^2 \tag{1}$$

for $i \in \{1,..,n\}$, then

$$\sum_{i=1}^{n} p_i|x_i|^2 - \left|\sum_{i=1}^{n} p_i x_i\right|^2 \leq \sum_{i=1}^{n} p_i r_i^2. \qquad (2)$$

Proof. It follows from (1) that

$$\left\langle x_i - \sum_{j=1}^{n} p_j x_j, x_i - \sum_{j=1}^{n} p_j x_j \right\rangle \leq r_i^2$$

so

$$\langle x_i, x_i\rangle - \left\langle x_i, \sum_{j=1}^{n} p_j x_j \right\rangle - \left\langle \sum_{j=1}^{n} p_j x_j, x_i \right\rangle + \left\langle \sum_{j=1}^{n} p_j x_j, \sum_{j=1}^{n} p_j x_j \right\rangle \leq r_i^2, \qquad (3)$$

and

$$\langle x_i, x_i\rangle - 2\mathrm{Re}\left\langle x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + \left\langle \sum_{j=1}^{n} p_j x_j, \sum_{j=1}^{n} p_j x_j \right\rangle \leq r_i^2. \qquad (4)$$

If we multiply (4) by $p_i \geq 0$, and sum over i from 1 to n. We obtain

$$\sum_{i=1}^{n} p_i \langle x_i, x_i\rangle - 2\mathrm{Re}\left\langle \sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + \left\langle \sum_{j=1}^{n} p_j x_j, \sum_{j=1}^{n} p_j x_j \right\rangle \leq \sum_{i=1}^{n} p_i r_i^2$$

then

$$\sum_{i=1}^{n} p_i|x_i|^2 - 2\mathrm{Re}\left\langle \sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + \left|\sum_{j=1}^{n} p_j x_j\right|^2 \leq \sum_{i=1}^{n} p_i r_i^2,$$

by inner pruduct property we have

$$\left\langle \sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j \right\rangle = \left|\sum_{i=1}^{n} p_i x_i\right|^2$$

therefore

$$\sum_{i=1}^{n} p_i|x_i|^2 - \left|\sum_{i=1}^{n} p_i x_i\right|^2 \leq \sum_{i=1}^{n} p_i r_i^2. \qquad (5)$$

□

Now we prove a useful lemma, which is applied in the next theorem.

Lemma 2.2. *Let x, y in $\mathcal{X}$ be artbitrary. Then*

$$|x|^2 + |y|^2 \geq 2\,\mathrm{Re}\,\langle x, y\rangle\,. \qquad (6)$$

Proof. We have $\langle x-y, x-y\rangle \geq 0$, therefore

$$\langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \geq 0.$$

We know that $\langle x,y\rangle^* = \langle y,x\rangle$, so

$$\langle x,x\rangle + \langle y,y\rangle \geq 2\operatorname{Re}\langle x,y\rangle$$

hence

$$|x|^2 + |y|^2 \geq 2\operatorname{Re}\langle x,y\rangle .$$

$\square$

Theorem 2.3. *Let $\mathcal{X}$ be a Hilbert -module, e_1 and e_2 be elements in $\mathcal{X}$ with $|e_1| = |e_2| = 1$ and $x_i \in \mathcal{X}$, $i \in \{1,...,n\}$ and $\mathcal{A}$ be a unital C^*-algebra. If $M_i \geq m_i > 0$ for all $i \in \{1,...,n\}$, are such that*

$$\left| x_i - \frac{M_i + m_i}{2} e_1 \right|^2 \leq (M_i - m_i)^2, \tag{7}$$

then

$$\operatorname{Re}\left\langle \sum_{i=1}^n x_i, e_2 - e_1 \right\rangle \leq \sum_{i=1}^n \frac{(M_i - m_i)^2}{(M_i + m_i)}.$$

Proof. It follows from (7) that

$$\begin{aligned}
&\left\langle x_i - \frac{M_i + m_i}{2} e_1, x_i - \frac{M_i + m_i}{2} e_1 \right\rangle \\
&\quad = \langle x_i, x_i\rangle - \left\langle x_i, \frac{M_i + m_i}{2} e_1 \right\rangle - \left\langle \frac{M_i + m_i}{2} e_1, x_i \right\rangle + \left\langle \frac{M_i + m_i}{2} e_1, \frac{M_i + m_i}{2} e_1 \right\rangle \\
&\quad = |x_i|^2 + \left| \frac{M_i + m_i}{2} e_1 \right|^2 - 2\operatorname{Re}\left\langle x_i, \frac{M_i + m_i}{2} e_1 \right\rangle \\
&\quad = |x_i|^2 + \left| \frac{M_i + m_i}{2} \right|^2 |e_1|^2 - 2\operatorname{Re}\left\langle x_i, \frac{M_i + m_i}{2} e_1 \right\rangle \\
&\quad = |x_i|^2 + \left| \frac{M_i + m_i}{2} \right|^2 |e_2|^2 - 2\operatorname{Re}\left\langle x_i, \frac{M_i + m_i}{2} e_1 \right\rangle .
\end{aligned}$$

By assumption we conclude that

$$\begin{aligned}
&|x_i|^2 - 2\operatorname{Re}\left\langle x_i, \frac{M_i + m_i}{2} e_1 \right\rangle + \left| \frac{M_i + m_i}{2} e_2 \right|^2 \\
&\quad = |x_i|^2 - (M_i + m_i)\operatorname{Re}\langle x_i, e_1\rangle + \left| \frac{M_i + m_i}{2} e_2 \right|^2 \\
&\quad \leq (M_i - m_i)^2.
\end{aligned}$$

From the inequality (6) we deduce

$$2\operatorname{Re}\left\langle x_i, \frac{M_i + m_i}{2} e_2 \right\rangle - (M_i + m_i)\operatorname{Re}\langle x_i, e_1\rangle \leq (M_i - m_i)^2$$

therefore

$$(M_i + m_i)\operatorname{Re}\langle x_i, e_2\rangle - (M_i + m_i)\operatorname{Re}\langle x_i, e_1\rangle \leq (M_i - m_i)^2$$

or equivalently

$$\operatorname{Re}\langle x_i, e_2\rangle - \operatorname{Re}\langle x_i, e_1\rangle \leq \frac{(M_i - m_i)^2}{(M_i + m_i)}$$

and we obtain

$$\operatorname{Re}\langle x_i, e_2 - e_1\rangle \leq \frac{(M_i - m_i)^2}{(M_i + m_i)}.$$

Finally, if we sum in over i from 1 to n, then we get

$$\operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e_2 - e_1 \right\rangle \leq \sum_{i=1}^{n} \frac{(M_i - m_i)^2}{(M_i + m_i)}.$$

□

Corollary 2.4. *Let e be an element in $\mathcal{X}$ with $|e| = 1$ and $x_i \in \mathcal{X}$, $i \in \{1, ..., n\}$ and $\mathcal{A}$ be a unital C^*-algebra. If $M_i \geq 0$ for all $i \in \{1, ..., n\}$, are such that*

$$\left| x_i - \frac{M_i}{2} e \right|^2 \leq M_i^2, \tag{8}$$

then

$$\sum_{i=1}^{n} |x_i|^2 - \operatorname{Re}\left\langle \sum_{i=1}^{n} M_i x_i, e \right\rangle \leq \frac{3}{4} \sum_{i=1}^{n} M_i^2.$$

Proof. From the condition (8) we obtain

$$\left\langle x_i - \frac{M_i}{2} e, x_i - \frac{M_i}{2} e \right\rangle = |x_i|^2 + \left| \frac{M_i}{2} \right|^2 |e| - 2 \operatorname{Re}\left\langle x_i, \frac{M_i}{2} e \right\rangle$$

by assumption we conclude that

$$|x_i|^2 + \left| \frac{M_i}{2} \right|^2 |e| - 2 \operatorname{Re}\left\langle x_i, \frac{M_i}{2} e \right\rangle \leq M_i^2$$

or equivalently

$$|x_i|^2 - \operatorname{Re}\langle M_i x_i, e\rangle \leq \frac{3}{4} M_i^2. \tag{9}$$

If we sum in (9) over i from 1 to n, then we get the desired result. □

The following result will be useful in the sequel.

Lemma 2.5. *Let x, y in $\mathcal{A}$ be arbitrary. Then*

$$|x|^2 + |y|^2 \leq 2 \operatorname{Re} xy^*.$$

Proof. We know

$$(x - y)^* (x - y) \geq 0,$$

from the fact $\operatorname{Re} x^* y = \frac{xy^* + yx^*}{2}$, we obtain

$$x^* x + y^* y - x^* y - y^* x = |x|^2 + |y|^2 - 2 \operatorname{Re} x^* y \geq 0.$$

□

As another consequence of Theorem 2.1 we have the following generalization of the triangle inequality in the framework of Hilbert C^*-modules.

Proposition 2.6. *Let p_i,r_i and x_i for all $i \in \{1, ..., n\}$ be as in the statement of Theorem 2.1 with the additional assumption that $\mathcal{A}$ is commutative. Then*

$$\operatorname{Re}\sum_{i=1}^{n} p_i |x_i| \left|\sum_{j=1}^{n} p_j x_j\right| \leq \left|\sum_{i=1}^{n} p_i x_i\right|^2 + \frac{1}{2}\sum_{i=1}^{n} p_i r_i^2. \tag{10}$$

Proof. From (4) we obviosly have

$$|x_i|^2 + \left|\sum_{j=1}^{n} p_j x_j\right|^2 \leq 2\operatorname{Re}\left\langle x_i, \sum_{j=1}^{n} p_j x_j\right\rangle + r_i^2, \tag{11}$$

for all $i \in \{1, ..., n\}$. By Lemma 2.5 we deduce

$$2\operatorname{Re}|x_i| \left|\sum_{j=1}^{n} p_j x_j\right| \leq |x_i|^2 + \left|\sum_{j=1}^{n} p_j x_j\right|^2$$

for all $i \in \{1, ..., n\}$, which together with (11) produces

$$2\operatorname{Re}|x_i| \left|\sum_{j=1}^{n} p_j x_j\right| \leq 2\operatorname{Re}\left\langle x_i, \sum_{j=1}^{n} p_j x_j\right\rangle + r_i^2 \tag{12}$$

for all $i \in \{1, ..., n\}$. Now if we multiply (12) by $p_i > 0$ and sum over i from 1 to n, we deduce the desired inequality (13). □

Remark 2.7. In particular, if $\mathcal{A}$ be a commutative C^*-algebra, by utilizing the inequality $|x|^2 + |y|^2 \geq 2|x||y|$, we can obtain from (11) the following result:

$$\sum_{i=1}^{n} p_i |x_i| \left|\sum_{j=1}^{n} p_j x_j\right| \leq \left|\sum_{i=1}^{n} p_i x_i\right|^2 + \frac{1}{2}\sum_{i=1}^{n} p_i r_i^2.$$

The following useful result is deduced from Theorem 2.1. It can be proved directly as well.

Proposition 2.8. *Let p_i,r_i and x_i for all $i \in \{1, ..., n\}$ be as in the statement of Theorem 2.1 with the additional assumption that $\mathcal{A}$ is commutative. Then*

$$2\left|\sum_{j=1}^{n} p_j x_j\right|^2 + \sum_{i=1}^{n} p_i r_i^2 \geq \frac{2}{\sqrt{n}}\sum_{i=1}^{n} \sqrt{p_i}\,|x_i| \left|\sum_{i=1}^{n} p_j x_j\right|. \tag{14}$$

Proof. If we multiply (11) by $p_i > 0$ and sum over i from 1 to n, we get

$$\sum_{i=1}^{n} p_i |x_i|^2 + \left|\sum_{j=1}^{n} p_j x_j\right|^2 \leq 2\left|\sum_{j=1}^{n} p_j x_j\right|^2 + \sum_{i=1}^{n} p_i r_i^2, \tag{15}$$

from well known inequality $|x|^2 + |y|^2 \geq 2\,|x|\,|y|$, we obtain

$$\begin{aligned}\sum_{i=1}^{n} p_i |x_i|^2 + \sum_{i=1}^{n} \frac{1}{n}\left|\sum_{i=1}^{n} p_j x_j\right|^2 &= \sum_{i=1}^{n}\left(p_i |x_i|^2 + \frac{1}{n}\left|\sum_{i=1}^{n} p_j x_j\right|^2\right) \\ &\geq \frac{2}{\sqrt{n}} \sum_{i=1}^{n} \sqrt{p_i}\,|x_i| \left|\sum_{i=1}^{n} p_j x_j\right|\end{aligned}$$

for all $i \in \{1, ..., n\}$. Therefore,

$$2\left|\sum_{j=1}^{n} p_j x_j\right|^2 + \sum_{i=1}^{n} p_i r_i^2 \geq \frac{2}{\sqrt{n}} \sum_{i=1}^{n} \sqrt{p_i}\,|x_i| \left|\sum_{i=1}^{n} p_j x_j\right|. \tag{16}$$

□

The following particular case is of interest:

Theorem 2.9. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$ module and $e_1, e_2, ..., e_n$ be a sequence of unit vectors in $\mathcal{X}$ such that $\langle e_i, e_j \rangle = 0$ for $i \neq j \leq n$, and let $x_i \in \mathcal{X}$ for all $i \in \{1, ..., n\}$, and p_i are in real or complex number field such that $\sum_{i=1}^{n} p_i = 1$. If there exist constants positive elements r_i in $\mathcal{A}$ so that*

$$\left|x_i - \sum_{j=1}^{n} p_j \langle e_j, x_j \rangle e_j\right|^2 \leq r_i^2 \tag{17}$$

for all $i \in \{1, ..., n\}$, then

$$\sum_{i=1}^{n} p_i |x_i|^2 - \sum_{j=1}^{n} |p_j \langle e_j, x_j \rangle|^2 \leq \sum_{i=1}^{n} p_i r_i^2. \tag{18}$$

Proof. For the left side of inequality (17) we have

$$\left\langle x_i - \sum_{j=1}^{n} p_j \langle e_j, x_j\rangle e_j, x_i - \sum_{j=1}^{n} p_j \langle e_j, x_j\rangle e_j \right\rangle$$
$$= \langle x_i, x_i\rangle + \left\langle \sum_{i=1}^{n} p_i e_i \langle e_i, x_i\rangle, \sum_{j=1}^{n} p_j e_j \langle e_j, x_j\rangle \right\rangle - 2\sum_{j=1}^{n} |p_j \langle e_j, x_j\rangle|^2$$
$$= \langle x_i, x_i\rangle + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j \langle e_i, x_i\rangle^* \langle e_j, e_j\rangle \langle e_j, x_j\rangle - 2\sum_{j=1}^{n} |p_j \langle e_j, x_j\rangle|^2$$
$$= |x_i|^2 + \sum_{i=1}^{n} p_j^2 \langle e_j, x_j\rangle^* \langle e_j, e_j\rangle \langle e_j, x_j\rangle - 2\sum_{j=1}^{n} |p_j \langle e_j, x_j\rangle|^2$$
$$= |x_i|^2 + \sum_{j=1}^{n} p_j^2 \langle e_j, x_j\rangle^* \langle e_j, x_j\rangle - 2\sum_{j=1}^{n} |p_j \langle e_j, x_j\rangle|^2$$
$$= |x_i|^2 - \sum_{j=1}^{n} |p_j \langle e_j, x_j\rangle|^2.$$

It means

$$|x_i|^2 - \sum_{j=1}^{n} |p_j \langle e_j, x_j\rangle|^2 \le r_i^2. \tag{19}$$

If we multiply (19) by $p_i \ge 0$ and sum over i from 1 to n, we obtain

$$\sum_{i=1}^{n} p_i |x_i|^2 - \sum_{j=1}^{n} |p_j \langle e_j, x_j\rangle|^2 \le \sum_{i=1}^{n} p_i r_i^2.$$

□

Remark 2.10. If we take $p_i = \frac{1}{n}$, $i \in \{1, ..., n\}$ in (18), then we obtain

$$\frac{\sum_{i=1}^{n} |x_i|^2}{n} - \frac{1}{n}\sum_{j=1}^{n} |\langle e_j, x_j\rangle|^2 \le \frac{\sum_{i=1}^{n} r_i^2}{n}.$$

Using the Cauchy-Schwarz inequality, we have the following theorem for Hilbert modules (see [4], for the case of Hilbert space).

Theorem 2.11. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, $\mathcal{A}$ be a commutative C^*-algebra and e be a unit vector in $\mathcal{X}$. If $x, x_i \in \mathcal{X}$ for all $i \in \{1, ..., n\}$ and r_i are positive elements in $\mathcal{A}$ for $i \in \{1, ..., n\}$, such that*

$$|x_i| - \operatorname{Re}\langle e, x_i\rangle \le r_i \tag{20}$$

for each $i \in \{1, ..., n\}$, then

$$\sum_{i=1}^{n} |x_i| - \left|\sum_{i=1}^{n} x_i\right| \le \sum_{i=1}^{n} r_i. \tag{21}$$

Proof. If we sum in (20) over i from 1 to n, then we get

$$\sum_{i=1}^{n} |x_i| \leq \mathrm{Re}\left\langle e, \sum_{i=1}^{n} x_i \right\rangle + \sum_{i=1}^{n} r_i. \tag{22}$$

We also have

$$\begin{aligned} \mathrm{Re}\left\langle e, \sum_{i=1}^{n} x_i \right\rangle &\leq \left| \mathrm{Re}\left\langle e, \sum_{i=1}^{n} x_i \right\rangle \right| \\ &\leq \left| \left\langle e, \sum_{i=1}^{n} x_i \right\rangle \right| \leq \|e\| \left| \sum_{i=1}^{n} x_i \right| = \left| \sum_{i=1}^{n} x_i \right|. \end{aligned} \tag{23}$$

Making use of (22) and (23), we deduce the desired inequality (21). □

References

[1] L. Arambašić, and R. Rajić, On the C*-valued triangle equality and inequality in Hilbert C^*-modules. Acta Mathematica Hungarica, 119 (4) (2008): 373-380.

[2] F. Dadipour, M. S. Moslehian, J. M. Rassias, and S. E. Takahasi, *Characterization of a generalized triangle inequality in normed spaces*, Nonlinear Analysis: Theory, Methods Applications, 75(2) (2012): 735-741.

[3] S. S. Dragomir, Y. J. Cho, and S. S. Kim, *Some inequalities in inner product spaces related to the generalized triangle inequality*, Applied Mathematics and Computation 217 (18) (2011): 7462-7468.

[4] S. S. Dragomir, *Reverses of the triangle inequality in inner product spaces*, RGMIA Res. Rep. Coll., 7 (2004), Supplement, Article 7.

[5] D. R. Farenick, and J. P. Panayiotis, *A triangle inequality in Hilbert modules over matrix algebras*, Linear algebra and its applications 341.1 (2002): 57-67.

[6] I. Kaplansky, *Modules over operator algebras*, American Journal of Mathematics (1953): 839-858.

[7] E. C. Lance, *Hilbert C^*-modules: a toolkit for operator algebraists*, Vol. 210. Cambridge University Press, 1995.

[8] M. S. Moslehian, and R. Rajić, *Generalizations of Bohr's inequality in Hilbert C^*-modules*, Linear and multilinear algebra, 58 (3) (2010): 323-331.

[9] R. Rajić, *Characterization of the norm triangle equality in Pre-Hilbert C^*-modules and applications*, Journal of Mathematical Inequalities 3.3 (2009): 347-355.

[10] K. Shrawan, B. Leeb, and J. Millson, *The generalized triangle inequalities for rank 3 symmetric spaces of noncompact type*, Contemporary Mathematics 332 (2003): 171-196.

[1]Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran
Email: hrmoradi@mshdiau.ac.ir
[2]Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran
Email: math.erfanian@gmail.com
[3]Mathematics, College of Engineering and Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia
Email: sever.dragomir@vu.edu.au