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SOME RESULTS RELATED TO CALLEBAUT'S AND HÖLDER'S INEQUALITIES FOR ISOTONIC FUNCTIONALS WITH APPLICATIONS

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Abstract. In this paper, by the use of two new Young's related inequalities, we establish some new inequalities related to Callebaut's and Hölder's inequalities for isotonic linear functionals. Applications for general Lebesgue integral and discrete counting measure are provided as well.

1. Introduction

The famous *Young inequality* for scalars says that if $a,\ b>0$ and $\nu\in[0,1]$, then

$$a^{1-\nu}b^{\nu}\le(1-\nu)\,a+\nu b \tag{1.1}$$

with equality if and only if $a=b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [26]

$$S(h):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h\in(0,1)\cup(1,\infty)\\[2ex] 1 \text{ if } h=1.\end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to1}S(h)=1,\ S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0,\ h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu}\le(1-\nu)\,a+\nu b\le S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.3}$$

where $a,\ b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$.

The second inequality in (1.3) is due to Tominaga [28] while the first one is due to Furuichi [15].

Kittaneh and Manasrah [18], [19] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^{2}\le(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\le R\left(\sqrt{a}-\sqrt{b}\right)^{2} \tag{1.4}$$

where $a,\ b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

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We also consider the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h>0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\geq 1$ for any $h>0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^r\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \leq (1-\nu)a+\nu b \leq K^R\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{1.6}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [29] while the second by Liao et al. [20].

In [29] the authors also showed that $K^r(h)\geq S(h^r)$ for $h>0$ and $r\in\left[0,\frac{1}{2}\right]$ implying that the lower bound in (1.6) is better than the lower bound from (1.3).

In the recent paper [10] we obtained the following reverses of Young's inequality as well:

$$0\leq (1-\nu)a+\nu b - a^{1-\nu}b^{\nu} \leq \nu(1-\nu)(a-b)(\ln a-\ln b) \tag{1.7}$$

and

$$1\leq \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \leq \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{1.8}$$

where $a,b>0$, $\nu\in[0,1]$.

In [11] we obtained the following Young related inequalities:

Theorem 1. *For any $a,b>0$ and $\nu\in[0,1]$ we have*

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\min\{a,b\} &\leq (1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\\ &\leq \frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\max\{a,b\}\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max^2\{a,b\}}\right] &\leq \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min^2\{a,b\}}\right].\end{aligned} \tag{1.10}$$

For an equivalent form and a different approach in proving the results (1.9) and (1.10) see [1].

The second inequalities in (1.9) and (1.10) are better than the corresponding results obtained by Furuichi and Minculete in [17] where instead of constant $\frac{1}{2}$ they had the constant 1.

In this paper, by the use of inequalities (1.9) and (1.10) we establish some new inequalities related to Callebaut's and Hölder's inequalities for isotonic linear functionals. Applications for general Lebesgue integral and discrete counting measure are provided as well.

2. Callebaut's Related Inequalities

Let L be a *linear class* of real-valued functions $g : E \to \mathbb{R}$ having the properties:

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1$, $t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalised if*
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [3], [22] and [23]). For other inequalities for isotonic functionals see [2], [5]-[21] and [24]-[27].

We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second ($p_k \geq 0, k \in E$).

The functional version of *Callebaut's inequality* states that

$$A^2(fg) \leq A\left(f^{2-\nu} g^{\nu}\right) A\left(f^{\nu} g^{2-\nu}\right) \leq A\left(f^2\right) A\left(g^2\right) \tag{2.1}$$

provided that $f^2, g^2, f^{2-\nu}g^{\nu}, f^{\nu}g^{2-\nu}, fg \in L$ for some $\nu \in [0, 2]$. For the discrete and integral of one real variable versions see [4].

If $a, b \in [k, K] \subset (0, \infty)$, then by (1.9) we have the inequality

$$\begin{aligned} \frac{1}{2} k\nu(1-\nu)(\ln a - \ln b)^2 &\leq (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \\ &\leq \frac{1}{2} K\nu(1-\nu)(\ln a - \ln b)^2 \end{aligned} \tag{2.2}$$

for any $\nu \in [0, 1]$.

We start with the following result.

Theorem 2. *Let $A, B : L \to \mathbb{R}$ be two normalised isotonic functionals. If f, $g : E \to \mathbb{R}$ are such that $f > 0$, $g > 0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}$, $g^2 \ln\left(\frac{f}{g}\right)$, $g^2 \ln^2\left(\frac{f}{g}\right) \in L$ for some $\nu \in [0, 1]$ and*

$$0 < m \leq \frac{f}{g} \leq M < \infty \tag{2.3}$$

for some constants m, M, then

$$2m^{2}\nu(1-\nu)\left(A\left(g^{2}\ln^{2}\left(\frac{f}{g}\right)\right)B\left(g^{2}\right)-2A\left(g^{2}\ln\left(\frac{f}{g}\right)\right)B\left(g^{2}\ln\left(\frac{f}{g}\right)\right)+A\left(g^{2}\right)B\left(g^{2}\ln^{2}\left(\frac{f}{g}\right)\right)\right)$$
$$\leq(1-\nu)A\left(f^{2}\right)B\left(g^{2}\right)+\nu A\left(g^{2}\right)B\left(f^{2}\right)-A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right)$$
$$\leq 2M^{2}\nu(1-\nu)\left(A\left(g^{2}\ln^{2}\left(\frac{f}{g}\right)\right)B\left(g^{2}\right)-2A\left(g^{2}\ln\left(\frac{f}{g}\right)\right)B\left(g^{2}\ln\left(\frac{f}{g}\right)\right)+A\left(g^{2}\right)B\left(g^{2}\ln^{2}\left(\frac{f}{g}\right)\right)\right). \tag{2.4}$$

In particular, we have

$$4m^{2}\nu(1-\nu)\left[A\left(g^{2}\ln^{2}\left(\frac{f}{g}\right)\right)A\left(g^{2}\right)-A^{2}\left(g^{2}\ln\left(\frac{f}{g}\right)\right)\right]$$
$$\leq A\left(f^{2}\right)A\left(g^{2}\right)-A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right)$$
$$\leq 4M^{2}\nu(1-\nu)\left[A\left(g^{2}\ln^{2}\left(\frac{f}{g}\right)\right)A\left(g^{2}\right)-A^{2}\left(g^{2}\ln\left(\frac{f}{g}\right)\right)\right]. \tag{2.5}$$

Proof. For any $x, y\in E$ we have

$$m^{2}\leq\frac{f^{2}(x)}{g^{2}(x)},\ \frac{f^{2}(y)}{g^{2}(y)}\leq M^{2}.$$

If we use the inequalities (2.2) for

$$a=\frac{f^{2}(x)}{g^{2}(x)},\ b=\frac{f^{2}(y)}{g^{2}(y)},$$

then we get

$$\frac{1}{2}m^{2}\nu(1-\nu)\left(\ln\frac{f^{2}(x)}{g^{2}(x)}-\ln\frac{f^{2}(y)}{g^{2}(y)}\right)^{2}$$
$$\leq(1-\nu)\frac{f^{2}(x)}{g^{2}(x)}+\nu\frac{f^{2}(y)}{g^{2}(y)}-\left(\frac{f^{2}(x)}{g^{2}(x)}\right)^{1-\nu}\left(\frac{f^{2}(y)}{g^{2}(y)}\right)^{\nu}$$
$$\leq\frac{1}{2}M^{2}\nu(1-\nu)\left(\ln\frac{f^{2}(x)}{g^{2}(x)}-\ln\frac{f^{2}(y)}{g^{2}(y)}\right)^{2}$$

for any $x, y\in E$.

This is equivalent to

$$2m^{2}\nu(1-\nu)\left(\ln\frac{f(x)}{g(x)}-\ln\frac{f(y)}{g(y)}\right)^{2}$$
$$\leq(1-\nu)\frac{f^{2}(x)}{g^{2}(x)}+\nu\frac{f^{2}(y)}{g^{2}(y)}-\left(\frac{f^{2}(x)}{g^{2}(x)}\right)^{1-\nu}\left(\frac{f^{2}(y)}{g^{2}(y)}\right)^{\nu}$$
$$\leq 2M^{2}\nu(1-\nu)\left(\ln\frac{f(x)}{g(x)}-\ln\frac{f(y)}{g(y)}\right)^{2}$$

and to

$$
\begin{aligned}
(2.6)\qquad & 2m^2\nu(1-\nu)\left(\ln^2\frac{f(x)}{g(x)}-2\ln\frac{f(x)}{g(x)}\ln\frac{f(y)}{g(y)}+\ln^2\frac{f(y)}{g(y)}\right)\\
&\le(1-\nu)\frac{f^2(x)}{g^2(x)}+\nu\frac{f^2(y)}{g^2(y)}-\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}\\
&\le 2M^2\nu(1-\nu)\left(\ln^2\frac{f(x)}{g(x)}-2\ln\frac{f(x)}{g(x)}\ln\frac{f(y)}{g(y)}+\ln^2\frac{f(y)}{g(y)}\right)
\end{aligned}
$$

for any $x,\, y\in E$.

If we multiply (2.6) by $g^2(x)\,g^2(y)>0$ then we get

$$
\begin{aligned}
(2.7)\quad & 2m^2\nu(1-\nu)\\
&\times\left(g^2(y)\,g^2(x)\ln^2\frac{f(x)}{g(x)}-2g^2(x)\ln\frac{f(x)}{g(x)}g^2(y)\ln\frac{f(y)}{g(y)}+g^2(x)\,g^2(y)\ln^2\frac{f(y)}{g(y)}\right)\\
&\le(1-\nu)f^2(x)\,g^2(y)+\nu g^2(x)\,f^2(y)-f^{2(1-\nu)}(x)\,g^{2\nu}(x)\,f^{2\nu}(y)\,g^{2(1-\nu)}(y)\\
&\le 2M^2\nu(1-\nu)\\
&\times\left(g^2(y)\,g^2(x)\ln^2\frac{f(x)}{g(x)}-2g^2(x)\ln\frac{f(x)}{g(x)}g^2(y)\ln\frac{f(y)}{g(y)}+g^2(x)\,g^2(y)\ln^2\frac{f(y)}{g(y)}\right)
\end{aligned}
$$

for any $x,\, y\in E$.

Fix $y\in E$. Then by (2.7) we have in the order of L that

$$
\begin{aligned}
&2m^2\nu(1-\nu)\left(g^2(y)\,g^2\ln^2\frac{f}{g}-2g^2(y)\ln\frac{f(y)}{g(y)}g^2\ln\frac{f}{g}+g^2(y)\ln^2\frac{f(y)}{g(y)}g^2\right)\\
&\le(1-\nu)\,g^2(y)\,f^2+\nu f^2(y)\,g^2-f^{2\nu}(y)\,g^{2(1-\nu)}(y)\,f^{2(1-\nu)}g^{2\nu}\\
&\le 2M^2\nu(1-\nu)\left(g^2(y)\,g^2\ln^2\frac{f}{g}-2g^2(y)\ln\frac{f(y)}{g(y)}g^2\ln\frac{f}{g}+g^2(y)\ln^2\frac{f(y)}{g(y)}g^2\right).
\end{aligned}
$$

If we take the functional A in this inequality, then we get

$$
\begin{aligned}
&2m^2\nu(1-\nu)\\
&\times\left(g^2(y)\,A\left(g^2\ln^2\frac{f}{g}\right)-2g^2(y)\ln\frac{f(y)}{g(y)}A\left(g^2\ln\frac{f}{g}\right)+g^2(y)\ln^2\frac{f(y)}{g(y)}A\left(g^2\right)\right)\\
&\le(1-\nu)\,g^2(y)\,A\left(f^2\right)+\nu f^2(y)\,A\left(g^2\right)-f^{2\nu}(y)\,g^{2(1-\nu)}(y)\,A\left(f^{2(1-\nu)}g^{2\nu}\right)\\
&\le 2M^2\nu(1-\nu)\\
&\times\left(g^2(y)\,A\left(g^2\ln^2\frac{f}{g}\right)-2g^2(y)\ln\frac{f(y)}{g(y)}A\left(g^2\ln\frac{f}{g}\right)+g^2(y)\ln^2\frac{f(y)}{g(y)}A\left(g^2\right)\right),
\end{aligned}
$$

for any $y\in E$.

This inequality can be written in the order of L as

$$\begin{aligned}
(2.8)\quad & 2m^2\nu(1-\nu) \\
& \times\left(A\left(g^2\ln^2\frac{f}{g}\right)g^2-2A\left(g^2\ln\frac{f}{g}\right)g^2\ln\frac{f}{g}+A\left(g^2\right)g^2\ln^2\frac{f}{g}\right) \\
& \le(1-\nu)A\left(f^2\right)g^2+\nu A\left(g^2\right)f^2-A\left(f^{2(1-\nu)}g^{2\nu}\right)f^{2\nu}g^{2(1-\nu)} \\
& \le 2M^2\nu(1-\nu) \\
& \times\left(A\left(g^2\ln^2\frac{f}{g}\right)g^2-2A\left(g^2\ln\frac{f}{g}\right)g^2\ln\frac{f}{g}+A\left(g^2\right)g^2\ln^2\frac{f}{g}\right).
\end{aligned}$$

Now, if we take the functional B in (2.8), then we get the desired result (2.4). □

Corollary 1. *Let A, $B : L \to \mathbb{R}$ be two normalised isotonic functionals. If f, $g : E \to \mathbb{R}$ are such that $f > 0$, $g > 0$, f^2, g^2, fg, $g^2\ln\left(\frac{f}{g}\right)$, $g^2\ln^2\left(\frac{f}{g}\right) \in L$ and the condition (2.3) holds true, then*

$$\begin{aligned}
(2.9)\quad & \frac{1}{2}m^2\left(A\left(g^2\ln^2\left(\frac{f}{g}\right)\right)B\left(g^2\right)\right. \\
& \left.-2A\left(g^2\ln\left(\frac{f}{g}\right)\right)B\left(g^2\ln\left(\frac{f}{g}\right)\right)+A\left(g^2\right)B\left(g^2\ln^2\left(\frac{f}{g}\right)\right)\right) \\
& \le\frac{1}{2}\left[A\left(f^2\right)B\left(g^2\right)+A\left(g^2\right)B\left(f^2\right)\right]-A(fg)B(fg) \\
& \le\frac{1}{2}M^2\left(A\left(g^2\ln^2\left(\frac{f}{g}\right)\right)B\left(g^2\right)\right. \\
& \left.-2A\left(g^2\ln\left(\frac{f}{g}\right)\right)B\left(g^2\ln\left(\frac{f}{g}\right)\right)+A\left(g^2\right)B\left(g^2\ln^2\left(\frac{f}{g}\right)\right)\right).
\end{aligned}$$

In particular, we have

$$\begin{aligned}
(2.10)\quad & m^2\left[A\left(g^2\ln^2\left(\frac{f}{g}\right)\right)A\left(g^2\right)-A^2\left(g^2\ln\left(\frac{f}{g}\right)\right)\right] \\
& \le A\left(f^2\right)A\left(g^2\right)-A^2(fg) \\
& \le M^2\left[A\left(g^2\ln^2\left(\frac{f}{g}\right)\right)A\left(g^2\right)-A^2\left(g^2\ln\left(\frac{f}{g}\right)\right)\right].
\end{aligned}$$

Now, since

$$\frac{(b-a)^2}{\max^2\{a,b\}}=\left(\frac{\min\{a,b\}}{\max\{a,b\}}-1\right)^2,$$

then by (1.10) we have:

$$(2.11)\quad \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\le\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right]$$

for any a, $b > 0$ and $\nu \in [0,1]$.

If a, $b \in [k,K] \subset (0,\infty)$, then by (2.11) we have the inequality

$$(2.12)\quad \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\le\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{K}{k}-1\right)^2\right]$$

for any $\nu \in [0,1]$.

Theorem 3. *Let A, $B : L \to \mathbb{R}$ be two normalised isotonic functionals. If f, $g : E \to \mathbb{R}$ are such that $f > 0$, $g > 0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L$ for some $\nu \in [0,1]$ and the condition (2.3) is valid, then*

$$\begin{aligned}(1-\nu)\, A\left(f^2\right) B\left(g^2\right) + \nu A\left(g^2\right) B\left(f^2\right) \\ \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M^2}{m^2}-1\right)^2\right] A\left(f^{2(1-\nu)}g^{2\nu}\right) B\left(f^{2\nu}g^{2(1-\nu)}\right).\end{aligned} \tag{2.13}$$

In particular, we have

$$\begin{aligned}A\left(f^2\right) A\left(g^2\right) \\ \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M^2}{m^2}-1\right)^2\right] A\left(f^{2(1-\nu)}g^{2\nu}\right) A\left(f^{2\nu}g^{2(1-\nu)}\right).\end{aligned} \tag{2.14}$$

Proof. For any x, $y \in E$ we have

$$m^2 \leq \frac{f^2(x)}{g^2(x)},\ \frac{f^2(y)}{g^2(y)} \leq M^2.$$

If we use the inequality (2.12) for

$$a = \frac{f^2(x)}{g^2(x)},\ b = \frac{f^2(y)}{g^2(y)},$$

then we get

$$\begin{aligned}(1-\nu)\frac{f^2(x)}{g^2(x)} + \nu\frac{f^2(y)}{g^2(y)} \\ \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M^2}{m^2}-1\right)^2\right]\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}\end{aligned} \tag{2.15}$$

for any x, $y \in E$.

Now, if we multiply (2.15) by $g^2(x)\, g^2(y) > 0$ then we get

$$\begin{aligned}(1-\nu)\, f^2(x)\, g^2(y) + \nu g^2(x)\, f^2(y) \\ \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M^2}{m^2}-1\right)^2\right] f^{2(1-\nu)}(x)\, g^{2\nu}(x)\, f^{2\nu}(y)\, g^{2(1-\nu)}(y)\end{aligned} \tag{2.16}$$

for any $x, y \in E$.

On making use of a similar argument as in the proof of Theorem 2 we obtain the desired result (2.13). □

Corollary 2. *Let A, $B : L \to \mathbb{R}$ be two normalised isotonic functionals. If f, $g : E \to \mathbb{R}$ are such that $f > 0$, $g > 0$, f^2, g^2, $fg \in L$ and the condition (2.3) holds true, then*

$$\begin{aligned}(1-\nu)\, A\left(f^2\right) B\left(g^2\right) + \nu A\left(g^2\right) B\left(f^2\right) \\ \leq \exp\left[\frac{1}{8}\left(\frac{M^2}{m^2}-1\right)^2\right] A(fg)\, B(fg),\end{aligned} \tag{2.17}$$

for any $\nu \in [0,1]$.

In particular, we have

$$A\left(f^{2}\right)A\left(g^{2}\right) \over A^{2}\left(fg\right) \leq \exp\left[\frac{1}{8}\left(\frac{M^{2}-m^{2}}{m^{2}}\right)^{2}\right]. \tag{2.18}$$

3. Hölder's Related Inequalities

We have:

Theorem 4. *Let $A: L \to \mathbb{R}$ be a normalised isotonic functional and $p,\ q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $f,\ g: E \to \mathbb{R}$ are such that $fg,\ f^{p},\ g^{q} \in L$ and*

$$0 < m_{1} \leq f \leq M_{1} < \infty,\ \ 0 < m_{2} \leq g \leq M_{2} < \infty, \tag{3.1}$$

for some constants m_1, M_1, m_2, M_2, then by putting

$$M_{p,q} := \max\left\{\left(\frac{M_{1}}{m_{1}}\right)^{p}, \left(\frac{M_{2}}{m_{2}}\right)^{q}\right\}$$

we have

$$\begin{aligned} &\frac{1}{2pq}\frac{1}{M_{p,q}}A\left[\left(p\ln\frac{f}{A\left(f^{p}\right)}-q\ln\frac{g}{A\left(g^{q}\right)}\right)^{2}\right] \\ &\leq 1-\frac{A\left(fg\right)}{\left[A\left(f^{p}\right)\right]^{1/p}\left[A\left(g^{q}\right)\right]^{1/q}} \\ &\leq \frac{1}{2pq}M_{p,q}A\left[\left(p\ln\frac{f}{A\left(f^{p}\right)}-q\ln\frac{g}{A\left(g^{q}\right)}\right)^{2}\right]. \end{aligned} \tag{3.2}$$

Proof. Observe that, by (3.1) we have

$$m_{1}^{p} \leq A\left(f^{p}\right) \leq M_{1}^{p} \text{ and } m_{2}^{q} \leq A\left(g^{q}\right) \leq M_{2}^{q}.$$

Also

$$\left(\frac{m_{1}}{M_{1}}\right)^{p} \leq \frac{f^{p}}{A\left(f^{p}\right)} \leq \left(\frac{M_{1}}{m_{1}}\right)^{p}$$

and

$$\left(\frac{m_{2}}{M_{2}}\right)^{q} \leq \frac{g^{q}}{A\left(g^{q}\right)} \leq \left(\frac{M_{2}}{m_{2}}\right)^{q}$$

giving that

$$m_{p,q} \leq \frac{f^{p}}{A\left(f^{p}\right)}, \frac{g^{q}}{A\left(g^{q}\right)} \leq M_{p,q},$$

where

$$\begin{aligned} m_{p,q} &: = \min\left\{\left(\frac{m_{1}}{M_{1}}\right)^{p}, \left(\frac{m_{2}}{M_{2}}\right)^{q}\right\} = \min\left\{\frac{1}{\left(\frac{M_{1}}{m_{1}}\right)^{p}}, \frac{1}{\left(\frac{M_{2}}{m_{2}}\right)^{q}}\right\} \\ &= \frac{1}{\max\left\{\left(\frac{M_{1}}{m_{1}}\right)^{p}, \left(\frac{M_{2}}{m_{2}}\right)^{q}\right\}} = \frac{1}{M_{p,q}}. \end{aligned}$$

Using the inequality (2.2) for $\nu = \frac{1}{q}$, $a = \frac{f^p}{A(f^p)}$, $b = \frac{g^q}{A(g^q)}$, $k = m_{p,q} = \frac{1}{M_{p,q}}$ and $K = M_{p,q}$ we get

$$\begin{aligned}(3.3)\qquad & \frac{1}{2pq}\frac{1}{M_{p,q}}\left(\ln\frac{f^p}{A(f^p)} - \ln\frac{g^q}{A(g^q)}\right)^2 \\ & \le \frac{1}{p}\frac{f^p}{A(f^p)} + \frac{1}{q}\frac{g^q}{A(g^q)} - \frac{fg}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \\ & \le \frac{1}{2pq}M_{p,q}\left(\ln\frac{f^p}{A(f^p)} - \ln\frac{g^q}{A(g^q)}\right)^2.\end{aligned}$$

If we take the functional A in (3.3), then we get

$$\begin{aligned}& \frac{1}{2pq}\frac{1}{M_{p,q}}A\left[\left(\ln\frac{f^p}{A(f^p)} - \ln\frac{g^q}{A(g^q)}\right)^2\right] \\ & \le \frac{1}{p}\frac{A(f^p)}{A(f^p)} + \frac{1}{q}\frac{A(g^q)}{A(g^q)} - \frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \\ & \le \frac{1}{2pq}M_{p,q}A\left[\left(\ln\frac{f^p}{A(f^p)} - \ln\frac{g^q}{A(g^q)}\right)^2\right]\end{aligned}$$

and the inequality (3.2) is proved. □

Corollary 3. *Let A, $B : L \to \mathbb{R}$ be two normalised isotonic functionals. If f, $g : E \to \mathbb{R}$ are such that $f > 0$, $g > 0$, f^2, g^2, $fg \in L$ and the condition (3.1) holds true, then by putting*

$$M := \max\left\{\frac{M_1}{m_1}, \frac{M_2}{m_2}\right\}$$

we have

$$\begin{aligned}(3.4)\qquad & \frac{2}{pqM^2}A\left[\left(\ln\frac{f}{A(f^2)} - \ln\frac{g}{A(g^2)}\right)^2\right] \\ & \le 1 - \frac{A(fg)}{[A(f^2)]^{1/2}[A(g^2)]^{1/2}} \\ & \le \frac{2M^2}{pq}A\left[\left(\ln\frac{f}{A(f^2)} - \ln\frac{g}{A(g^2)}\right)^2\right].\end{aligned}$$

If a, $b \in [k, K] \subset (0, \infty)$, then by (1.9) we have the following reverse of Young's inequality

$$(3.5)\qquad (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le \frac{1}{2}K\nu(1-\nu)\ln^2\left(\frac{K}{k}\right)$$

for any $\nu \in [0, 1]$.

Theorem 5. *With the assumptions of Theorem 4 we have*

$$(3.6)\qquad 0 \le 1 - \frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \le \frac{2}{pq}M_{p,q}\ln^2(M_{p,q}).$$

Proof. By the use of the inequality (3.5) we have

$$\frac{1}{p}\frac{f^p}{A(f^p)}+\frac{1}{q}\frac{g^q}{A(g^q)}-\frac{fg}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \leq \frac{1}{2pq}M_{p,q}\ln^2\left(M_{p,q}^2\right)$$

and by taking the functional A we get the desired result (3.6). □

Corollary 4. *With the assumptions of Corollary 3, we have*

$$0\leq 1-\frac{A(fg)}{[A(f^2)]^{1/2}[A(g^2)]^{1/2}}\leq 2M^2\ln^2(M). \tag{3.7}$$

The following result also holds:

Theorem 6. *With the assumptions of Theorem 4 we have*

$$\frac{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}{A(fg)}\leq \exp\left[\frac{1}{2pq}\left(M_{p,q}^2-1\right)^2\right]. \tag{3.8}$$

Proof. By the use of the inequality (2.12) for the choices $\nu=\frac{1}{q}$, $a=\frac{f^p}{A(f^p)}$, $b=\frac{g^q}{A(g^q)}$, $k=m_{p,q}=\frac{1}{M_{p,q}}$ and $K=M_{p,q}$ we get

$$\frac{1}{p}\frac{f^p}{A(f^p)}+\frac{1}{q}\frac{g^q}{A(g^q)} \leq \exp\left[\frac{1}{2pq}\left(M_{p,q}^2-1\right)^2\right]\frac{fg}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}. \tag{3.9}$$

If we take the functional A in (3.9), then we get

$$\frac{1}{p}\frac{A(f^p)}{A(f^p)}+\frac{1}{q}\frac{A(g^q)}{A(g^q)} \leq \exp\left[\frac{1}{2pq}\left(M_{p,q}^2-1\right)^2\right]\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}},$$

that is equivalent to the desired result (3.8). □

Corollary 5. *With the assumptions of Corollary 3, we have*

$$\frac{A(f^2)A(g^2)}{A^2(fg)}\leq \exp\left[\frac{1}{4}\left(M^4-1\right)^2\right]. \tag{3.10}$$

4. Integral Inequalities

Let $(\Omega,\mathcal{A},\mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for μ -a.e. (almost every) $x\in\Omega$, consider the *Lebesgue space*

$$L_w(\Omega,\mu):=\{f:\Omega\to\mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|\,w(x)\,d\mu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega wd\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$. The same for other integrals involved below. We assume that $\int_\Omega wd\mu=1$.

Let f, g be μ-measurable functions with the property that there exists the constants M, $m>0$ such that

$$0<m\leq\frac{f}{g}\leq M<\infty\ \mu\text{-almost everywhere (a.e.) on }\Omega. \tag{4.1}$$

If $f^2,\ g^2,\ f^{2(1-\nu)}g^{2\nu},\ f^{2\nu}g^{2(1-\nu)},\ g^2\ln\left(\frac{f}{g}\right),\ g^2\ln^2\left(\frac{f}{g}\right)\in L_w(\Omega,\mu)$ for some $\nu\in[0,1]$, then from (2.5) we have

$$\begin{aligned} &4\nu(1-\nu)m^2\left[\int_\Omega wg^2\ln^2\left(\frac{f}{g}\right)d\mu\int_\Omega wg^2d\mu-\left(\int_\Omega wg^2\ln\left(\frac{f}{g}\right)d\mu\right)^2\right]\\ &\leq\int_\Omega wg^2d\mu\int_\Omega wf^2d\mu-\int_\Omega wf^{2(1-\nu)}g^{2\nu}d\mu\int_\Omega wf^{2\nu}g^{2(1-\nu)}d\mu\\ &\leq 4\nu(1-\nu)M^2\left[\int_\Omega wg^2\ln^2\left(\frac{f}{g}\right)d\mu\int_\Omega wg^2d\mu-\left(\int_\Omega wg^2\ln\left(\frac{f}{g}\right)d\mu\right)^2\right]. \end{aligned} \tag{4.2}$$

If f^2, g^2, $g^2\ln^2\left(\frac{f}{g}\right)$, $g^2\ln\left(\frac{f}{g}\right)\in L_w(\Omega,\mu)$ and the condition (4.1) is satisfied, then we have from (4.2) that

$$\begin{aligned} &m^2\left[\int_\Omega wg^2\ln^2\left(\frac{f}{g}\right)d\mu\int_\Omega wg^2d\mu-\left(\int_\Omega wg^2\ln\left(\frac{f}{g}\right)d\mu\right)^2\right]\\ &\leq\int_\Omega wg^2d\mu\int_\Omega wf^2d\mu-\left(\int_\Omega wfgd\mu\right)^2\\ &\leq M^2\left[\int_\Omega wg^2\ln^2\left(\frac{f}{g}\right)d\mu\int_\Omega wg^2d\mu-\left(\int_\Omega wg^2\ln\left(\frac{f}{g}\right)d\mu\right)^2\right]. \end{aligned} \tag{4.3}$$

If f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L_w(\Omega,\mu)$ for some $\nu\in[0,1]$ and the condition (4.1) is satisfied, then by (2.14) we have

$$\frac{\int_\Omega wg^2d\mu\int_\Omega wf^2d\mu}{\int_\Omega wf^{2(1-\nu)}g^{2\nu}d\mu\int_\Omega wf^{2\nu}g^{2(1-\nu)}d\mu}\leq\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M^2}{m^2}-1\right)^2\right]. \tag{4.4}$$

In particular, if f^2, $g^2\in L_w(\Omega,\mu)$, then from (4.4) we get

$$\frac{\int_\Omega wg^2d\mu\int_\Omega wf^2d\mu}{\left(\int_\Omega wfgd\mu\right)^2}\leq\exp\left[\frac{1}{8}\left(\frac{M^2}{m^2}-1\right)^2\right], \tag{4.5}$$

provided f, g satisfy the condition (4.1).

Let f, g be μ-measurable functions with the property that there exists the constants M_1, M_2, m_1, $m_2>0$ such that

$$0<m_1\leq f\leq M_1<\infty,\ 0<m_2\leq g\leq M_2<\infty,\ \mu\text{-a.e. on }\Omega. \tag{4.6}$$

Then by putting

$$M_{p,q}:=\max\left\{\left(\frac{M_1}{m_1}\right)^p,\left(\frac{M_2}{m_2}\right)^q\right\}$$

where p, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and by assuming that f^p, $g^q \in L_w(\Omega, \mu)$ we have from (3.6) that

$$0 \leq 1 - \frac{\int_\Omega wfgd\mu}{\left(\int_\Omega wf^p d\mu\right)^{1/p} \left(\int_\Omega wg^q d\mu\right)^{1/q}} \leq \frac{2}{pq} M_{p,q} \ln^2 (M_{p,q}) \tag{4.7}$$

and from (3.8) that

$$\frac{\left(\int_\Omega wf^p d\mu\right)^{1/p} \left(\int_\Omega wg^q d\mu\right)^{1/q}}{\int_\Omega wfgd\mu} \leq \exp\left[\frac{1}{2pq}\left(M_{p,q}^2 - 1\right)^2\right]. \tag{4.8}$$

Also, if we denote

$$M := \max\left\{\frac{M_1}{m_1}, \frac{M_2}{m_2}\right\}$$

and assume that f, g with f^2, $g^2 \in L_w(\Omega, \mu)$ satisfy the condition (4.6) then by (3.7) we have

$$0 \leq 1 - \frac{\int_\Omega wfgd\mu}{\left(\int_\Omega wf^2 d\mu\right)^{1/2} \left(\int_\Omega wg^2 d\mu\right)^{1/2}} \leq 2M^2 \ln^2 (M) \tag{4.9}$$

while by (3.10) we have

$$\frac{\int_\Omega wf^2 d\mu \int_\Omega wg^2 d\mu}{\left(\int_\Omega wfgd\mu\right)^2} \leq \exp\left[\frac{1}{4}\left(M^4 - 1\right)^2\right]. \tag{4.10}$$

5. Inequalities for Real Numbers

We consider the n-tuples of positive numbers $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and the probability distribution $p = (p_1, ..., p_n)$, i.e. $p_i \geq 0$ for any $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$.

If there exist the constants m, $M > 0$ such that

$$0 < m \leq \frac{a_i}{b_i} \leq M < \infty \text{ for any } i \in \{1, ..., n\},$$

then by (4.2) for the *counting discrete measure*, we have for any $s \in [0, 1]$ that

$$\begin{aligned} & 4s(1-s)m^2\left[\sum_{i=1}^n p_i b_i^2 \ln^2\left(\frac{a_i}{b_i}\right)\sum_{i=1}^n p_i b_i^2 - \left(\sum_{i=1}^n p_i b_i^2 \ln\left(\frac{a_i}{b_i}\right)\right)^2\right] \\ & \leq \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \sum_{i=1}^n p_i a_i^{2(1-s)} b_i^{2s} \sum_{i=1}^n p_i a_i^{2s} b_i^{2(1-s)} \\ & \leq 4s(1-s)M^2\left[\sum_{i=1}^n p_i b_i^2 \ln^2\left(\frac{a_i}{b_i}\right)\sum_{i=1}^n p_i b_i^2 - \left(\sum_{i=1}^n p_i b_i^2 \ln\left(\frac{a_i}{b_i}\right)\right)^2\right] \end{aligned} \tag{5.1}$$

and, in particular,

$$m^2\left[\sum_{i=1}^{n}p_ib_i^2\ln^2\left(\frac{a_i}{b_i}\right)\sum_{i=1}^{n}p_ib_i^2-\left(\sum_{i=1}^{n}p_ib_i^2\ln\left(\frac{a_i}{b_i}\right)\right)^2\right] \tag{5.2}$$
$$\leq\sum_{i=1}^{n}p_ia_i^2\sum_{i=1}^{n}p_ib_i^2-\left(\sum_{i=1}^{n}p_ia_ib_i\right)^2$$
$$\leq M^2\left[\sum_{i=1}^{n}p_ib_i^2\ln^2\left(\frac{a_i}{b_i}\right)\sum_{i=1}^{n}p_ib_i^2-\left(\sum_{i=1}^{n}p_ib_i^2\ln\left(\frac{a_i}{b_i}\right)\right)^2\right].$$

From (4.4) we have

$$\frac{\sum_{i=1}^{n}p_ia_i^2\sum_{i=1}^{n}p_ib_i^2}{\sum_{i=1}^{n}p_ia_i^{2(1-s)}b_i^{2s}\sum_{i=1}^{n}p_ia_i^{2s}b_i^{2(1-s)}}\leq\exp\left[\frac{1}{2}s\left(1-s\right)\left(\frac{M^2}{m^2}-1\right)^2\right], \tag{5.3}$$

for any $s\in[0,1]$ and, in particular, we have

$$\frac{\sum_{i=1}^{n}p_ia_i^2\sum_{i=1}^{n}p_ib_i^2}{\left(\sum_{i=1}^{n}p_ia_ib_i\right)^2}\leq\exp\left[\frac{1}{8}\left(\frac{M^2}{m^2}-1\right)^2\right]. \tag{5.4}$$

If there exist the constants m_1, M_1, m_2, M_2 such that

$$0<m_1\leq a_i\leq M_1<\infty,\ 0<m_2\leq b_i\leq M_2<\infty \text{ for any } i\in\{1,...,n\} \tag{5.5}$$

and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by putting

$$M_{p,q}:=\max\left\{\left(\frac{M_1}{m_1}\right)^p,\left(\frac{M_2}{m_2}\right)^q\right\}$$

where p, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, we have from (4.7) that

$$0\leq 1-\frac{\sum_{i=1}^{n}p_ia_ib_i}{\left(\sum_{i=1}^{n}p_ia_i^p\right)^{1/p}\left(\sum_{i=1}^{n}p_ib_i^q\right)^{1/q}}\leq\frac{2}{pq}M_{p,q}\ln^2\left(M_{p,q}\right) \tag{5.6}$$

and from (4.7) that

$$\frac{\left(\sum_{i=1}^{n}p_ia_i^p\right)^{1/p}\left(\sum_{i=1}^{n}p_ib_i^q\right)^{1/q}}{\sum_{i=1}^{n}p_ia_ib_i}\leq\exp\left[\frac{1}{2pq}\left(M_{p,q}^2-1\right)^2\right]. \tag{5.7}$$

Also, if we denote

$$M:=\max\left\{\frac{M_1}{m_1},\frac{M_2}{m_2}\right\}$$

and assume that the condition (5.5) is satisfied, then by (4.9) we have

$$0\leq 1-\frac{\sum_{i=1}^{n}p_ia_ib_i}{\left(\sum_{i=1}^{n}p_ia_i^2\right)^{1/2}\left(\sum_{i=1}^{n}p_ib_i^2\right)^{1/2}}\leq 2M^2\ln^2\left(M\right) \tag{5.8}$$

while by (4.10) we have

$$\frac{\sum_{i=1}^{n}p_ia_i^2\sum_{i=1}^{n}p_ib_i^2}{\left(\sum_{i=1}^{n}p_ia_ib_i\right)^2}\leq\exp\left[\frac{1}{4}\left(M^4-1\right)^2\right]. \tag{5.9}$$

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