

**HERMITE-HADAMARD TYPE INEQUALITIES FOR PRODUCT
OF HARMONICALLY CONVEX FUNCTIONS VIA
RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS**

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ABSTRACT. In this paper, some Hermite-Hadamard type inequalities for products of two harmonically convex functions via Riemann-Liouville fractional integrals are established. Our results about harmonically convex functions are analogous generalizations for some other results proved by Pachpatte for convex functions.

1. INTRODUCTION

In recent years, very large number of studies of error estimations have been done for Hermite-Hadamard type inequalities. It is known that Hermite-hadamard integral inequality was built on a convex function. In time, Hermite-Hadamard inequality is developed other kinds of convex functions. For some results which generalize, improve, and extend the Hermite-Hadamard inequality see [2, 7, 10, 16, 18] and references therein.

Hermite-Hadamard type inequalities for products of two convex functions are interesting problem and firstly developed by Pachpatte in [14]. In [15], Pachpatte also established Hermite-hadamard type inequalities involving two log-convex functions. In [11], Kirmacı et. al. proved several Hermite-Hadamard type inequalities for products of two convex and s -convex functions. In [17], Sarıkaya et. al. proved some Hermite-Hadamard type inequalities for products of two h -convex functions. In [1], Bakula et. al. established Hermite-Hadamard type inequalities for products of two m -convex and (α, m) -convex functions. In [4, 6], Chen and Wu obtained some Hermite-Hadamard type inequalities for products of two convex and harmonically s -convex functions. In [19], Yin and Qi established some Hermite-Hadamard type inequalities for products of two convex functions. In [5], Chen obtained some new Hermite-Hadamard type inequalities for products of two convex functions via Riemann-Liouville fractional integrals and in [3] he extended this problem to m -convex and (α, m) -convex functions.

In this work, we establish Hermite-Hadamard type inequalities for products of two harmonically convex functions via Hadamard fractional integrals. Our results are analogous generalization for some results in [14].

2. PRELIMINARIES

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \quad (2.1)$$

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is well known in the literature as Hermite-Hadamard's inequality [8].

In [14], Pachpette established following two Hermite-Hadamard type inequalities for products of convex functions as follows:

Theorem 1. *Let f and g be real-valued, non-negative and convex functions on $[a, b]$. Then*

$$\frac{1}{b-a} \int_a^b f(x)g(x) dx \leq \frac{1}{3}M(a, b) + \frac{1}{6}N(a, b) \quad (2.2)$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)g(x) dx + \frac{1}{6}M(a, b) + \frac{1}{3}N(a, b) \quad (2.3)$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

In [10], İşcan gave definition of harmonically convex functions as follows:

Definition 1. *Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \rightarrow \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x) \quad (2.4)$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (2.4) is reversed, then f is said to be harmonically concave.

We will now give definitions of the right-hand side and left-hand side Riemann-Liouville fractional integrals which are used throughout this paper.

Definition 2. [13]. *Let $f \in L[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ and

$$J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x).$$

In [9], İşcan and Wu represented Hermite-Hadamard's inequalities for harmonically convex functions in fractional integral forms as follows:

Theorem 2. *Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a}\right)^\alpha \left\{ \begin{array}{l} J_{1/a-}^\alpha (f \circ h)(1/b) \\ + J_{1/b+}^\alpha (f \circ h)(1/a) \end{array} \right\} \\ &\leq \frac{f(a) + f(b)}{2} \end{aligned} \quad (2.5)$$

with $\alpha > 0$ and $h(x) = 1/x$.

In [12], Kunt and İşcan represented, Hermite-Hadamard inequality in fractional integral forms for harmonically convex functions as follows:

Theorem 3. *Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^\alpha \left\{ \begin{array}{l} J_{\frac{a+b}{2ab}^+}^\alpha (f \circ h)(1/a) \\ + J_{\frac{a+b}{2ab}^-}^\alpha (f \circ h)(1/b) \end{array} \right\} \\ &\leq \frac{f(a) + f(b)}{2} \end{aligned} \quad (2.6)$$

with $\alpha > 0$ and $h(x) = 1/x$, $x \in [\frac{1}{b}, \frac{1}{a}]$.

3. GENERAL RESULTS

Theorem 4. *Let f and $g : [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be non-negative and harmonically convex functions with $a < b$ and $f \in L[a, b]$, then the following inequality for fractional integrals hold:*

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a}\right)^\alpha \left\{ \begin{array}{l} J_{1/a^-}^\alpha (f \circ h)(1/b) (g \circ h)(1/b) \\ + J_{1/b^+}^\alpha (f \circ h)(1/a) (g \circ h)(1/a) \end{array} \right\} \\ &\leq \left(\frac{\alpha}{\alpha+2} - \frac{\alpha}{\alpha+1} + \frac{1}{2} \right) M(a, b) + \frac{\alpha}{(\alpha+2)(\alpha+1)} N(a, b) \end{aligned} \quad (3.1)$$

where $\alpha > 0$, $h(x) = 1/x$, $x \in [\frac{1}{b}, \frac{1}{a}]$, $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Proof. Since f and g are non-negative and harmonically convex functions on $[a, b]$, we have for all $t \in [0, 1]$

$$f\left(\frac{ab}{tb + (1-t)a}\right) \leq tf(a) + (1-t)f(b), \quad (3.2)$$

and

$$g\left(\frac{ab}{tb + (1-t)a}\right) \leq tg(a) + (1-t)g(b). \quad (3.3)$$

From products of (3.2) and (3.3), we have

$$\begin{aligned} f\left(\frac{ab}{tb + (1-t)a}\right)g\left(\frac{ab}{tb + (1-t)a}\right) &\leq t^2 f(a)g(a) + (1-t)^2 f(b)g(b) \\ &\quad + t(1-t)[f(a)g(b) + f(b)g(a)]. \end{aligned} \quad (3.4)$$

Similarly (3.4), we have

$$\begin{aligned} f\left(\frac{ab}{ta + (1-t)b}\right)g\left(\frac{ab}{ta + (1-t)b}\right) &\leq (1-t)^2 f(a)g(a) + t^2 f(b)g(b) \\ &\quad + t(1-t)[f(a)g(b) + f(b)g(a)]. \end{aligned} \quad (3.5)$$

The sum of (3.4) and (3.5), we have

$$\begin{aligned} &f\left(\frac{ab}{tb + (1-t)a}\right)g\left(\frac{ab}{tb + (1-t)a}\right) + f\left(\frac{ab}{ta + (1-t)b}\right)g\left(\frac{ab}{ta + (1-t)b}\right) \\ &\leq (2t^2 - 2t + 1) M(a, b) + 2t(1-t) N(a, b) \end{aligned} \quad (3.6)$$

Multiplying both sides of (3.6) by $t^{\alpha-1}\frac{\alpha}{2}$, then integrating the obtained inequality with respect to t over $[0, 1]$, we have

$$\begin{aligned}
& \frac{\alpha}{2} \left\{ \int_0^1 t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)a}\right) g\left(\frac{ab}{tb+(1-t)a}\right) dt \right. \\
& \quad \left. + \int_0^1 t^{\alpha-1} f\left(\frac{ab}{ta+(1-t)b}\right) g\left(\frac{ab}{ta+(1-t)b}\right) dt \right\} \\
= & \frac{\alpha}{2} \left\{ \left(\frac{ab}{a-b}\right)^\alpha \int_{\frac{1}{b}}^{\frac{1}{a}} \left(u - \frac{1}{b}\right)^{\alpha-1} f\left(\frac{1}{u}\right) g\left(\frac{1}{u}\right) du \right. \\
& \quad \left. + \left(\frac{ab}{a-b}\right)^\alpha \int_{\frac{1}{b}}^{\frac{1}{a}} \left(\frac{1}{a} - v\right)^{\alpha-1} f\left(\frac{1}{v}\right) g\left(\frac{1}{v}\right) dv \right\} \\
= & \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a}\right)^\alpha \left\{ \begin{array}{l} J_{1/a-}^\alpha (f \circ h)(1/b) (g \circ h)(1/b) \\ + J_{1/b+}^\alpha (f \circ h)(1/a) (g \circ h)(1/a) \end{array} \right\} \\
\leq & \frac{\alpha}{2} \left[M(a, b) \int_0^1 t^{\alpha-1} (2t^2 - 2t + 1) dt + N(a, b) \int_0^1 t^{\alpha-1} 2t(1-t) dt \right] \\
= & \left(\frac{\alpha}{\alpha+2} - \frac{\alpha}{\alpha+1} + \frac{1}{2} \right) M(a, b) + \frac{\alpha}{(\alpha+2)(\alpha+1)} N(a, b)
\end{aligned}$$

and this completes the proof. \square

Remark 1. *Theorem 4 is an analogous generalization of (2.2) for harmonically convex functions.*

Corollary 1. *In Theorem 4, if we take $g : [a, b] \rightarrow \mathbb{R}$ as $g(x) = 1$ for all $x \in [a, b]$, then we have*

$$\frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a}\right)^\alpha \left\{ \begin{array}{l} J_{1/a-}^\alpha (f \circ h)(1/b) \\ + J_{1/b+}^\alpha (f \circ h)(1/a) \end{array} \right\} \leq \frac{f(a) + f(b)}{2}$$

which is the right hand side of (2.5).

Corollary 2. *In Theorem 4, if we take $\alpha = 1$, then we have*

$$\frac{ab}{b-a} \int_a^b f(x) g(x) \frac{dx}{x^2} \leq \frac{1}{3} M(a, b) + \frac{1}{6} N(a, b)$$

for harmonically convex functions.

Theorem 5. *Let f and $g : [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be non-negative and harmonically convex functions with $a < b$ and $f \in L[a, b]$, then the following inequality for fractional integrals hold:*

$$\begin{aligned}
& 2f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a}\right)^\alpha \\
& \quad \times \left\{ \begin{array}{l} J_{1/a-}^\alpha (f \circ h)(1/b) (g \circ h)(1/b) \\ + J_{1/b+}^\alpha (f \circ h)(1/a) (g \circ h)(1/a) \end{array} \right\} \\
& \quad + \frac{\alpha}{(\alpha+2)(\alpha+1)} M(a, b) + \left(\frac{\alpha}{\alpha+2} - \frac{\alpha}{\alpha+1} + \frac{1}{2} \right) N(a, b) \quad (3.7)
\end{aligned}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Proof. Since f and g are non-negative and harmonically convex functions on $[a, b]$, we have for all $x, y \in [a, b]$ (with $t = \frac{1}{2}$ in the inequality (2.4))

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{f(x) + f(y)}{2} \quad (3.8)$$

and

$$g\left(\frac{2xy}{x+y}\right) \leq \frac{g(x) + g(y)}{2}. \quad (3.9)$$

In (3.8) and (3.9) choosing $x = \frac{ab}{tb+(1-t)a}$ and $y = \frac{ab}{ta+(1-t)b}$, we have for all $t \in [0, 1]$

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{f\left(\frac{ab}{tb+(1-t)a}\right) + f\left(\frac{ab}{ta+(1-t)b}\right)}{2} \quad (3.10)$$

and

$$g\left(\frac{2ab}{a+b}\right) \leq \frac{g\left(\frac{ab}{tb+(1-t)a}\right) + g\left(\frac{ab}{ta+(1-t)b}\right)}{2} \quad (3.11)$$

By product of(3.10) and (3.11)

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right) &\leq \frac{f\left(\frac{ab}{tb+(1-t)a}\right) + f\left(\frac{ab}{ta+(1-t)b}\right)}{2} \frac{g\left(\frac{ab}{tb+(1-t)a}\right) + g\left(\frac{ab}{ta+(1-t)b}\right)}{2} \\ &= \frac{1}{4} \left\{ f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right) \right. \\ &\quad + f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right) \\ &\quad + f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{ta+(1-t)b}\right) \\ &\quad \left. + f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right) \right\} \\ &\leq \frac{1}{4} \left\{ f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right) \right. \\ &\quad + f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right) \\ &\quad + [tf(a) + (1-t)f(b)][tg(b) + (1-t)g(a)] \\ &\quad \left. + [tf(b) + (1-t)f(a)][tg(a) + (1-t)g(b)] \right\} \\ &= \frac{1}{4} \left\{ f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right) \right. \\ &\quad + f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right) \\ &\quad + 2t(1-t)[f(a)g(a) + f(b)g(b)] \\ &\quad \left. + (2t^2 - 2t + 1)[f(a)g(b) + f(b)g(a)] \right\} \quad (3.12) \end{aligned}$$

Multiplying both sides of (3.12) by $2\alpha t^{\alpha-1}$, then integrating the obtained inequality with respect to t over $[0, 1]$, we have

$$\begin{aligned} 2f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a}\right)^\alpha \\ &\quad \times \left\{ \begin{aligned} &J_{1/a-}^\alpha (f \circ h)(1/b)(g \circ h)(1/b) \\ &+ J_{1/b+}^\alpha (f \circ h)(1/a)(g \circ h)(1/a) \end{aligned} \right\} \\ &\quad + \frac{\alpha}{(\alpha+2)(\alpha+1)} M(a,b) + \left(\frac{\alpha}{\alpha+2} - \frac{\alpha}{\alpha+1} + \frac{1}{2}\right) N(a,b) \end{aligned}$$

and this completes the proof. \square

Remark 2. *Theorem 5 is an analogous generalization of (2.3) for harmonically convex functions.*

Corollary 3. *In Theorem 5, if we take $g : [a, b] \rightarrow \mathbb{R}$ as $g(x) = 1$ for all $x \in [a, b]$, then we have*

$$2f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a}\right)^\alpha \left\{ J_{1/a-}^\alpha (f \circ h)(1/b) + J_{1/b+}^\alpha (f \circ h)(1/a) \right\} + \frac{f(a) + f(b)}{2}.$$

Corollary 4. *In Theorem 5, if we take $\alpha = 1$, then we have*

$$2f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b f(x) g(x) \frac{dx}{x^2} + \frac{1}{6}M(a, b) + \frac{1}{3}N(a, b)$$

for harmonically convex functions.

Theorem 6. *Let f and $g : [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be non-negative and harmonically convex functions with $a < b$ and $f \in L[a, b]$, then the following inequality for fractional integrals hold:*

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^\alpha \left\{ J_{\frac{a+b}{2ab}+}^\alpha (f \circ h)(1/a) (g \circ h)(1/a) \right. \\ & \left. + J_{\frac{a+b}{2ab}-}^\alpha (f \circ h)(1/b) (g \circ h)(1/b) \right\} \\ & \leq \left(\frac{\alpha}{4(\alpha+2)} - \frac{\alpha}{2(\alpha+1)} + \frac{1}{2} \right) M(a, b) \\ & \quad + \frac{\alpha^2 + 3\alpha}{4(\alpha+2)(\alpha+1)} N(a, b) \end{aligned} \quad (3.13)$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Proof. Since f and g are non-negative and harmonically convex functions on $[a, b]$, multiplying both sides of (3.6) by $t^{\alpha-1} \frac{\alpha}{2^{1-\alpha}}$, then integrating the obtained inequality with respect to t over $[0, \frac{1}{2}]$, we have

$$\begin{aligned} & \frac{\alpha}{2^{1-\alpha}} \left\{ \int_0^{\frac{1}{2}} t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)a}\right) g\left(\frac{ab}{tb+(1-t)a}\right) dt \right. \\ & \left. + \int_0^{\frac{1}{2}} t^{\alpha-1} f\left(\frac{ab}{ta+(1-t)b}\right) g\left(\frac{ab}{ta+(1-t)b}\right) dt \right\} \\ & = \frac{\alpha}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^\alpha \left\{ \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left(u - \frac{1}{b}\right)^{\alpha-1} f\left(\frac{1}{u}\right) g\left(\frac{1}{u}\right) du \right. \\ & \left. + \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left(\frac{1}{a} - v\right)^{\alpha-1} f\left(\frac{1}{v}\right) g\left(\frac{1}{v}\right) dv \right\} \\ & = \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^\alpha \left\{ J_{\frac{a+b}{2ab}+}^\alpha (f \circ h)(1/a) (g \circ h)(1/a) \right. \\ & \left. + J_{\frac{a+b}{2ab}-}^\alpha (f \circ h)(1/b) (g \circ h)(1/b) \right\} \\ & \leq \frac{\alpha}{2^{1-\alpha}} \left[M(a, b) \int_0^{\frac{1}{2}} t^{\alpha-1} (2t^2 - 2t + 1) dt + N(a, b) \int_0^{\frac{1}{2}} t^{\alpha-1} 2t(1-t) dt \right] \\ & = \left(\frac{\alpha}{4(\alpha+2)} - \frac{\alpha}{2(\alpha+1)} + \frac{1}{2} \right) M(a, b) + \frac{\alpha^2 + 3\alpha}{4(\alpha+2)(\alpha+1)} N(a, b) \end{aligned}$$

and this completes the proof. \square

Remark 3. Theorem 6 is an other analogous generalization of (2.2) for harmonically convex functions.

Corollary 5. In Theorem 6, if we take $g : [a, b] \rightarrow \mathbb{R}$ as $g(x) = 1$ for all $x \in [a, b]$, then we have

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha \left\{ J_{\frac{a+b}{2ab}+}^\alpha (f \circ h)(1/a) + J_{\frac{a+b}{2ab}-}^\alpha (f \circ h)(1/b) \right\} \\ & \leq \frac{f(a) + f(b)}{2} \end{aligned}$$

which is the right hand side of (2.6).

Corollary 6. In Theorem 6, if we take $\alpha = 1$, then we have

$$\frac{ab}{b-a} \int_a^b f(x) g(x) \frac{dx}{x^2} \leq \frac{1}{3} M(a, b) + \frac{1}{6} N(a, b)$$

for harmonically convex functions.

Theorem 7. Let f and $g : [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be non-negative and harmonically convex functions with $a < b$ and $f \in L[a, b]$, then the following inequality for fractional integrals hold:

$$\begin{aligned} 2f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) & \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha \\ & \times \left\{ \begin{aligned} & J_{\frac{a+b}{2ab}+}^\alpha (f \circ h)(1/a) (g \circ h)(1/a) \\ & + J_{\frac{a+b}{2ab}-}^\alpha (f \circ h)(1/b) (g \circ h)(1/b) \end{aligned} \right\} \\ & + \frac{\alpha^2 + 3\alpha}{4(\alpha+2)(\alpha+1)} M(a, b) \\ & + \left(\frac{\alpha}{4(\alpha+2)} - \frac{\alpha}{2(\alpha+1)} + \frac{1}{2} \right) N(a, b) \end{aligned}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Proof. Multiplying both sides of (3.12) by $2^{1+\alpha}\alpha t^{\alpha-1}$, then integrating the obtained inequality with respect to t over $[0, \frac{1}{2}]$, we have desired result. \square

Remark 4. Theorem 7 is an other analogous generalization of (2.3) for harmonically convex functions.

Corollary 7. In Theorem 7, if we take $g : [a, b] \rightarrow \mathbb{R}$ as $g(x) = 1$ for all $x \in [a, b]$, then we have

$$\begin{aligned} 2f\left(\frac{2ab}{a+b}\right) & \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a} \right)^\alpha \left\{ J_{\frac{a+b}{2ab}+}^\alpha (f \circ h)(1/a) + J_{\frac{a+b}{2ab}-}^\alpha (f \circ h)(1/b) \right\} \\ & + \frac{f(a) + f(b)}{2}. \end{aligned}$$

Corollary 8. In Theorem 7, if we take $\alpha = 1$, then we have

$$2f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b f(x) g(x) \frac{dx}{x^2} + \frac{1}{6} M(a, b) + \frac{1}{3} N(a, b)$$

for harmonically convex functions.

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