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# SOME INEQUALITIES FOR THE WEIGHTED CHAOTICALLY GEOMETRIC MEAN

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Abstract. In this paper we obtain some new inequalities for the weighted chaotically geometric mean of two positive operators on a complex Hilbert space.

## 1. Introduction

For positive operators $A$ and $B$ consider the *weighted arithmetic* and *chaotically geometric* means

$$A\nabla_\alpha B := (1-\alpha) A + \alpha B$$

and

$$A\lozenge_\alpha B := \exp[(1-\alpha)\ln A + \alpha \ln B].$$

We recall that *Specht's ratio* is defined by [6]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.1}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

It has been shown in [4] that, if $0 < mI \leq A, B \leq MI$, for some scalars $m < M$ and $h := \frac{M}{m}$, then

$$S^{-1}(h) A\lozenge_\alpha B \leq A\nabla_\alpha B \leq S(h) A\lozenge_\alpha B \tag{1.2}$$

for any $\alpha \in [0,1]$, where $I$ is the identity operator.

With the same assumptions for $A$ and $B$ we also have the additive version obtained in [5]

$$-L(m,M)\ln S(h) I \leq A\nabla_\alpha B - A\lozenge_\alpha B \leq L(m,M)\ln S(h) I, \tag{1.3}$$

where

$$L(m,M) := \begin{cases} \frac{M-m}{\ln M - \ln m}, \text{ if } M \neq m, \\ \\ M, \text{ if } M \neq m \end{cases}$$

is the *logarithmic mean.*

Motivated by these results, we establish in this paper other inequalities involving the chaotically geometric mean.

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## 2. The Results

We have:

**Theorem 1.** *If* $0 < mI \leq A,\ B \leq MI$ *for some scalars* $m < M$, *then*

$$\begin{aligned} &-\frac{2}{\ln M - \ln m}\left|\ln A\nabla_{\alpha}\ln B - \ln G\left(mM\right)I\right| \\ &\leq L\left(m,M\right)\left(\ln A\nabla_{\alpha}\ln B + U\left(m,M\right)I\right) - \frac{1}{2}\left(\sqrt{M}-\sqrt{m}\right)^{2}I - A\Diamond_{\alpha}B \\ &\leq \frac{2}{\ln M - \ln m}\left|\ln A\nabla_{\alpha}\ln B - \ln G\left(mM\right)I\right|, \end{aligned} \tag{2.1}$$

*where*

$$U\left(m,M\right) := \frac{m\ln M - M\ln m}{M-m},\ \ G\left(m,M\right) := \sqrt{mM}$$

*and*

$$\ln A\nabla_{\alpha}\ln B = \left(1-\alpha\right)\ln A + \alpha\ln B.$$

*Proof.* Recall the following result obtained by Dragomir in 2006 [1] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} &n\min_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi\left(x_j\right) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right] \\ &\leq \frac{1}{P_n}\sum_{j=1}^{n}p_j\Phi\left(x_j\right) - \Phi\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\right) \\ &\leq n\max_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi\left(x_j\right) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right], \end{aligned} \tag{2.2}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset $C$ of the linear space $X$, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in $C$ and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n}p_j > 0$.

For $n = 2$, we deduce from (2.2) that

$$\begin{aligned} &2\min\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \\ &\leq \nu\Phi\left(x\right) + \left(1-\nu\right)\Phi\left(y\right) - \Phi\left[\nu x + \left(1-\nu\right)y\right] \\ &\leq 2\max\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{2.3}$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0,1]$.

If we take $\Phi\left(x\right) = \exp\left(x\right)$, then we get from (2.3)

$$\begin{aligned} &2\min\{\nu,1-\nu\}\left[\frac{\exp\left(x\right)+\exp\left(y\right)}{2} - \exp\left(\frac{x+y}{2}\right)\right] \\ &\leq \nu\exp(x) + \left(1-\nu\right)\exp(y) - \exp\left[\nu x + \left(1-\nu\right)y\right] \\ &\leq 2\max\{\nu,1-\nu\}\left[\frac{\exp\left(x\right)+\exp\left(y\right)}{2} - \exp\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{2.4}$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0,1]$.

Let $0 < m < M$ and take $x = \ln M$ and $y = \ln m$ to get

$$\begin{aligned} &2\min\{\nu, 1-\nu\}\left(\frac{m+M}{2} - \sqrt{mM}\right) \\ &\le \nu M + (1-\nu)m - \exp[\nu \ln M + (1-\nu)\ln m] \\ &\le 2\max\{\nu, 1-\nu\}\left(\frac{m+M}{2} - \sqrt{mM}\right) \end{aligned} \tag{2.5}$$

for any $\nu \in [0,1]$.

Let $z \in [\ln m, \ln M]$ and take $\nu \in [0,1]$ such that $\nu \ln M + (1-\nu)\ln m = z$, namely $\nu = \frac{z - \ln m}{\ln M - \ln m}$.

Since

$$\begin{aligned} \min\{\nu, 1-\nu\} &= \frac{1}{2} - \left|\nu - \frac{1}{2}\right| = \frac{1}{2} - \left|\frac{z-\ln m}{\ln M - \ln m} - \frac{1}{2}\right| \\ &= \frac{1}{2} - \frac{1}{\ln M - \ln m}\left|z - \ln G(mM)\right| \end{aligned}$$

and

$$\max\{\nu, 1-\nu\} = \frac{1}{2} + \frac{1}{\ln M - \ln m}\left|z - \ln G(mM)\right|,$$

then by (2.5) we have

$$\begin{aligned} &\frac{1}{2}\left(\sqrt{M} - \sqrt{m}\right)^2\left(1 - \frac{2}{\ln M - \ln m}\left|z - \ln G(mM)\right|\right) \\ &\le \frac{z - \ln m}{\ln M - \ln m}M + \frac{\ln M - z}{\ln M - \ln m}m - \exp z \\ &\le \frac{1}{2}\left(\sqrt{M} - \sqrt{m}\right)^2\left(1 + \frac{2}{\ln M - \ln m}\left|z - \ln G(mM)\right|\right) \end{aligned} \tag{2.6}$$

for any $z \in [\ln m, \ln M]$.

If $X$ is a selfadjoint operator with $\operatorname{Sp}(X) \subset [\ln m, \ln M]$, then by (2.6) we have

$$\begin{aligned} &\frac{1}{2}\left(\sqrt{M} - \sqrt{m}\right)^2\left(I - \frac{2}{\ln M - \ln m}\left|X - \ln G(mM)I\right|\right) \\ &\le \frac{X - \ln mI}{\ln M - \ln m}M + \frac{\ln MI - X}{\ln M - \ln m}m - \exp X \\ &\le \frac{1}{2}\left(\sqrt{M} - \sqrt{m}\right)^2\left(I + \frac{2}{\ln M - \ln m}\left|X - \ln G(mM)I\right|\right). \end{aligned} \tag{2.7}$$

Since

$$\begin{aligned} &\frac{X - \ln mI}{\ln M - \ln m}M + \frac{\ln MI - X}{\ln M - \ln m}m \\ &= \frac{M-m}{\ln M - \ln m}X + \frac{m\ln M - M\ln m}{\ln M - \ln m}I \\ &= L(m,M)\left(X + \frac{m\ln M - M\ln m}{M-m}I\right) \\ &= L(m,M)\left(X + U(m,M)I\right), \end{aligned} \tag{2.8}$$

then by (2.7) and (2.8) we get

$$\begin{aligned}(2.9)\quad &\frac{1}{2}\left(\sqrt{M}-\sqrt{m}\right)^2\left(I-\frac{2}{\ln M-\ln m}\left|X-\ln G(mM)I\right|\right)\\ &\le L(m,M)(X+U(m,M)I)-\exp X\\ &\le \frac{1}{2}\left(\sqrt{M}-\sqrt{m}\right)^2\left(I+\frac{2}{\ln M-\ln m}\left|X-\ln G(mM)I\right|\right),\end{aligned}$$

that is equivalent to

$$\begin{aligned}(2.10)\quad &-\frac{2}{\ln M-\ln m}\left|X-\ln G(mM)I\right|\\ &\le L(m,M)(X+U(m,M)I)-\frac{1}{2}\left(\sqrt{M}-\sqrt{m}\right)^2 I-\exp X\\ &\le \frac{2}{\ln M-\ln m}\left|X-\ln G(mM)I\right|.\end{aligned}$$

This inequality is of interest in itself.

If we take $X=\ln A\nabla_\alpha \ln B$, $\alpha\in[0,1]$, then $\operatorname{Sp}(X)\subset[\ln m,\ln M]$ and by (2.10) we get the desired result (2.1). □

We also have:

**Theorem 2.** *With the assumptions of Theorem 1 we have*

$$\begin{aligned}(2.11)\quad 0&\le L(m,M)(\ln A\nabla_\alpha \ln B+U(m,M)I)-A\Diamond_\alpha B\\ &\le L(m,M)(\ln MI-\ln A\nabla_\alpha\ln B)(\ln A\nabla_\alpha \ln B-\ln mI)\\ &\le \frac{1}{4}(\ln M-\ln m)(M-m)I,\end{aligned}$$

*for any* $\alpha\in[0,1]$.

*Proof.* We use the following inequality for convex functions, see for instance [2]: If the function $f:I\subset\mathbb{R}\to\mathbb{R}$ is a differentiable convex function on $\mathring{I}$, then for any $a,b\in\mathring{I}$ and $\nu\in[0,1]$ we have

$$\begin{aligned}(2.12)\quad 0&\le(1-\nu)f(a)+\nu f(b)-f((1-\nu)a+\nu b)\\ &\le \nu(1-\nu)(b-a)\left[f'(b)-f'(a)\right].\end{aligned}$$

If we write this inequality for the convex function $f(x)=\exp(x)$, then we have

$$\begin{aligned}(2.13)\quad 0&\le(1-\nu)\exp a+\nu\exp b-\exp((1-\nu)a+\nu b)\\ &\le \nu(1-\nu)(b-a)\left[\exp b-\exp a\right],\end{aligned}$$

for any $a,b\in\mathbb{R}$ and $\nu\in[0,1]$.

Let $0<m<M$ and take $b=\ln M$ and $a=\ln m$ to get

$$\begin{aligned}(2.14)\quad 0&\le \nu M+(1-\nu)m-\exp\left[\nu\ln M+(1-\nu)\ln m\right]\\ &\le \nu(1-\nu)(\ln M-\ln m)(M-m).\end{aligned}$$

Let $z\in[\ln m,\ln M]$ and take $\nu\in[0,1]$ such that $\nu\ln M+(1-\nu)\ln m=z$, namely $\nu=\frac{z-\ln m}{\ln M-\ln m}$. Then by (2.14) and upon some simple calculations we get

$$0\le L(m,M)(z+U(m,M))-\exp z\le L(m,M)(\ln M-z)(z-\ln m)$$

for any $z\in[\ln m,\ln M]$.

Since

$$(\ln M - z)(z - \ln m) \le \frac{1}{4}(\ln M - \ln m)^2,$$

then we have

$$0 \le L(m,M)(z + U(m,M)) - \exp z \le L(m,M)(\ln M - z)(z - \ln m) \le \frac{1}{4}(\ln M - \ln m)(M - m) \tag{2.15}$$

for any $z \in [\ln m, \ln M]$.

If $X$ is a selfadjoint operator with $\operatorname{Sp}(X) \subset [\ln m, \ln M]$, then by (2.15) we have

$$\begin{aligned} 0 &\le L(m,M)(X + U(m,M)I) - \exp X \\ &\le L(m,M)(\ln MI - X)(X - \ln mI) \\ &\le \frac{1}{4}(\ln M - \ln m)(M - m)I, \end{aligned} \tag{2.16}$$

which is an inequality of interest in itself as well.

If we take $X = \ln A\nabla_\alpha \ln B$, $\alpha \in [0,1]$, then $\operatorname{Sp}(X) \subset [\ln m, \ln M]$ and by (2.16) we get the desired result (2.11). □

We also have:

**Theorem 3.** *With the assumptions of Theorem 1 we have*

$$\begin{aligned} &\frac{1}{2}m(\ln A\nabla_\alpha \ln B - \ln mI)(\ln MI - \ln A\nabla_\alpha \ln B) \\ &\le L(m,M)(\ln A\nabla_\alpha \ln B + U(m,M)I) - A\Diamond_\alpha B \\ &\le \frac{1}{2}M(\ln A\nabla_\alpha \ln B - \ln mI)(\ln MI - \ln A\nabla_\alpha \ln B), \end{aligned} \tag{2.17}$$

*for any* $\alpha \in [0,1]$.

*Proof.* We use the following result for twice differentiable functions, see for instance [3]:

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$, the interior of $I$. If there exists the constants $d$, $D$ such that

$$d \le f''(t) \le D \text{ for any } t \in \mathring{I}, \tag{2.18}$$

then

$$\frac{1}{2}\nu(1-\nu)d(b-a)^2 \le (1-\nu)f(a) + \nu f(b) - f((1-\nu)a + \nu b) \le \frac{1}{2}\nu(1-\nu)D(b-a)^2 \tag{2.19}$$

for any $a, b \in \mathring{I}$ and $\nu \in [0,1]$.

If we write this inequality for the convex function $f(x) = \exp(x)$, then we have

$$\begin{aligned} &\frac{1}{2}\nu(1-\nu)(b-a)^2 \exp(\min\{a,b\}) \\ &\le (1-\nu)\exp a + \nu \exp b - \exp((1-\nu)a + \nu b) \\ &\le \frac{1}{2}\nu(1-\nu)(b-a)^2 \exp(\max\{a,b\}), \end{aligned} \tag{2.20}$$

for any $a, b \in \mathbb{R}$ and $\nu \in [0,1]$.

Let $0 < m < M$ and take $b = \ln M$ and $a = \ln m$ to get

$$\begin{aligned}(2.21)\quad & \frac{1}{2}\nu(1-\nu)(\ln M - \ln m)^2 m \\ & \le (1-\nu)m + \nu M - \exp((1-\nu)\ln m + \nu \ln M) \\ & \le \frac{1}{2}\nu(1-\nu)(\ln M - \ln m)^2 M,\end{aligned}$$

for any $\nu \in [0,1]$.

Let $z \in [\ln m, \ln M]$. If we take $\nu = \frac{z-\ln m}{\ln M - \ln m} \in [0,1]$, then we get

$$\begin{aligned}(2.22)\quad \frac{1}{2}(z-\ln m)(\ln M - z)m & \le L(m,M)(z + U(m,M)) - \exp z \\ & \le \frac{1}{2}(z-\ln m)(\ln M - z)M,\end{aligned}$$

for any $z \in [\ln m, \ln M]$.

If $X$ is a selfadjoint operator with $\operatorname{Sp}(X) \subset [\ln m, \ln M]$, then by (2.22) we have

$$\begin{aligned}(2.23)\quad \frac{1}{2}m(X - \ln mI)(\ln MI - X) & \le L(m,M)(X + U(m,M)I) - \exp X \\ & \le \frac{1}{2}M(X - \ln mI)(\ln MI - X),\end{aligned}$$

which is an inequality of interest in itself as well.

If we take $X = \ln A \nabla_\alpha \ln B$, $\alpha \in [0,1]$, then $\operatorname{Sp}(X) \subset [\ln m, \ln M]$ and by (2.23) we get the desired result (2.17). □

**Remark 1.** *Since*

$$(\ln A\nabla_\alpha \ln B - \ln mI)(\ln MI - \ln A\nabla_\alpha \ln B) \le \frac{1}{4}(\ln M - \ln m)^2 I,$$

*then from (2.17) we get the following simpler upper bound*

$$(2.24)\quad L(m,M)(\ln A\nabla_\alpha \ln B + U(m,M)I) - A\lozenge_\alpha B \le \frac{1}{8}M(\ln M - \ln m)^2 I.$$

In [7], M. Tominaga obtained the following reverses of *Young's inequality*

$$(2.25)\quad \left(a^{1-\nu}b^\nu \le\right)(1-\nu)a + \nu b \le S\left(\frac{a}{b}\right)a^{1-\nu}b^\nu$$

and

$$(2.26)\quad (0 \le)(1-\nu)a + \nu b - a^{1-\nu}b^\nu \le L(a,b)\ln S\left(\frac{a}{b}\right)$$

for any $a, b > 0$ and $\nu \in [0,1]$.

**Theorem 4.** *With the assumptions of Theorem 1 we have*

$$(2.27)\quad (A\lozenge_\alpha B \le) L(m,M)(\ln A\nabla_\alpha \ln B + U(m,M)I) \le S\left(\frac{M}{m}\right)A\lozenge_\alpha B$$

*and*

$$\begin{aligned}(2.28)\quad & (0 \le) L(m,M)(\ln A\nabla_\alpha \ln B + U(m,M)I) - A\lozenge_\alpha B \\ & \le L(m,M)\ln S\left(\frac{M}{m}\right)I\end{aligned}$$

*for any* $\alpha \in [0,1]$.

*Proof.* For $0 < m < M$ we have by (2.25) that

$$(1-\nu)\, m + \nu M \leq S\left(\frac{m}{M}\right) m^{1-\nu} M^{\nu} = S\left(\frac{M}{m}\right) \exp\left((1-\nu)\ln m + \nu \ln M\right) \tag{2.29}$$

for any $\nu \in [0,1]$.

Similarly,

$$(1-\nu)\, m + \nu M - \exp\left((1-\nu)\ln m + \nu \ln M\right) \leq L(m,M) \ln S\left(\frac{M}{m}\right) \tag{2.30}$$

for any $\nu \in [0,1]$.

Let $z \in [\ln m, \ln M]$. If we take $\nu = \frac{z-\ln m}{\ln M - \ln m} \in [0,1]$ in (2.29) and (2.29) then we get

$$L(m,M)\left(z + U(m,M)\right) \leq S\left(\frac{M}{m}\right) \exp z \tag{2.31}$$

and

$$L(m,M)\left(z + U(m,M)\right) - \exp z \leq L(m,M) \ln S\left(\frac{M}{m}\right) \tag{2.32}$$

for any $z \in [\ln m, \ln M]$.

If $X$ is a selfadjoint operator with $\mathrm{Sp}(X) \subset [\ln m, \ln M]$, then by (2.31) and (2.32) we have

$$L(m,M)\left(X + U(m,M) I\right) \leq S\left(\frac{M}{m}\right) \exp X \tag{2.33}$$

and

$$L(m,M)\left(X + U(m,M) I\right) - \exp X \leq L(m,M) \ln S\left(\frac{M}{m}\right) I, \tag{2.34}$$

which are inequalities of interest in themselves.

If we take $X = \ln A \nabla_{\alpha} \ln B$, $\alpha \in [0,1]$, then $\mathrm{Sp}(X) \subset [\ln m, \ln M]$ and by (2.33) and (2.34) we deduce the desired results (2.27) and (2.28). □

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