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HERMITE-HADAMARD TYPE INEQUALITIES FOR PRODUCT OF GA-CONVEX FUNCTIONS VIA HADAMARD FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, some Hermite-Hadamard type inequalities for products of two GA-convex functions via Hadamard fractional integrals are established. Our results about GA-convex functions are analogous generalizations for some other results proved by Pachpette for convex functions.

1. INTRODUCTION

In recent years, very large number of studies of error estimations have been done for Hermite-Hadamard type inequalities. It is known that Hermite-hadamard integral inequality was built on a convex function. In time, Hermite-Hadamrd inequality is developed other kinds of convex functions. For some results which generalize, improve, and extend the Hermite-Hadamard inequality see [2, 7, 10, 18, 20] and references therein.

Hermite-Hadamard type inequalities for products of two convex functions are interesting problem and firstly developed by Pachpatte in [16]. In [17], Pachpette also established Hermite-hadamard type inequalities involving two log-convex functions. In [11], Kırmacı et. al. proved several Hermite-Hadamard type inequalities for products of two convex and s-convex functions. In [19], Sarıkaya et. al. proved some Hermite-Hadamard type inequalities for products of two h-convex functions. In [1], Bakula et. al. established Hermite-Hadamard type inequalities for products of two m-convex and (α, m)-convex functions. In [4, 6], Chen and Wu obtained some Hermite-Hadamard type inequalities for products of two convex and harmonically s-convex functions. In [21], Yin and Qi established some Hermite-Hadamard type inequalities for products of two convex functions. In [5], Chen obtained some new Hermite-Hadamard type inequalities for products of two convex functions via Riemann-Liouville fractional integrals and in [3] he extended this problem to m-convex and (α, m)-convex functions.

In this work, we establish Hermite-Hadamard type inequalities for products of two GA-convex functions via Hadamard fractional integrals. Our results are analogous generalization for some results in [16].

2. PRELIMINARIES

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{2.1}$$

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is well known in the literature as Hermite-Hadamard's inequality [8].

In [16], Pachpette established following two Hermite-Hadamard type inequalities for products of convex functions as follows:

Theorem 1. *Let f and g be real-valued, non-negative and convex functions on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \le \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b) \tag{2.2}$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{6}M(a,b)+\frac{1}{3}N(a,b) \tag{2.3}$$

where $M(a,b)=f(a)\,g(a)+f(b)\,g(b)$ and $N(a,b)=f(a)\,g(b)+f(b)\,g(a)$.

Definition 1. [14, 15]. *A function $f: I\subseteq(0,\infty)\to\mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if*

$$f(x^t y^{1-t}) \le tf(x)+(1-t)\,f(y)$$

for all $x,y\in I$ and $t\in[0,1]$.

We will now give definitions of the right-hand side and left-hand side Hadamard fractional integrals which are used throughout this paper.

Definition 2. [13]. *Let $f\in L[a,b]$. The right-hand side and left-hand side Hadamard fractional integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha>0$ with $b>a\ge 0$ are defined by*

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\ln\frac{x}{t}\right)^{\alpha-1}f(t)\frac{dt}{t},\ x>a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\ln\frac{t}{x}\right)^{\alpha-1}f(t)\frac{dt}{t},\ x<b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha)=\int_0^{\infty}e^{-t}t^{\alpha-1}dt$.

In [9], İşcan represented Hermite–Hadamard's inequalities for GA-convex functions in fractional integral forms as follows.

Theorem 2. *Let $f: I\subseteq(0,\infty)\to\mathbb{R}$ be a function such that $f\in L[a,b]$ where $a,b\in I$ with $a<b$. If f is a GA-convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\sqrt{ab}\right)\le\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\le\frac{f(a)+f(b)}{2} \tag{2.4}$$

with $\alpha>0$.

In [12], Kunt and İşcan established new Hermite-Hadamard type inequality for GA-convex function in fractional integral forms as follows:

Theorem 3. *Let $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ be a GA-convex function with $a<b$ and $f\in L[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\sqrt{ab}\right)\le\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{\sqrt{ab}-}^{\alpha}f(a)+J_{\sqrt{ab}+}^{\alpha}f(b)\right]\le\frac{f(a)+f(b)}{2}. \tag{2.5}$$

3. General Results

Theorem 4. *Let f and $g : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be non-negative and GA-convex functions with $a < b$ and $f \in L[a,b]$, then the following inequality for fractional integrals hold:*

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)\,g(b)+J_{b-}^{\alpha}f(a)\,g(a)\right] \\ \le\ &\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)M(a,b)+\frac{\alpha}{(\alpha+2)(\alpha+1)}N(a,b) \end{aligned} \tag{3.1}$$

where $\alpha > 0$, $M(a,b) = f(a)\,g(a)+f(b)\,g(b)$ and $N(a,b) = f(a)\,g(b)+f(b)\,g(a)$.

Proof. Since f and g are non-negative and GA-convex functions on $[a,b]$, we have for all $t \in [0,1]$

$$f(a^t b^{1-t}) \le tf(a) + (1-t)\,f(b), \tag{3.2}$$

and

$$g(a^t b^{1-t}) \le tg(a) + (1-t)\,g(b). \tag{3.3}$$

From products of (3.2) and (3.3), we have

$$\begin{aligned} f(a^t b^{1-t})g(a^t b^{1-t}) \ \le\ &t^2 f(a)\,g(a) + (1-t)^2 f(b)\,g(b) \\ &+t(1-t)\left[f(a)\,g(b)+f(b)\,g(a)\right]. \end{aligned} \tag{3.4}$$

Similarly (3.4), we have

$$\begin{aligned} f(a^{1-t} b^{t})g(a^{1-t} b^{t}) \ \le\ &(1-t)^2 f(a)\,g(a) + t^2 f(b)\,g(b) \\ &+t(1-t)\left[f(a)\,g(b)+f(b)\,g(a)\right]. \end{aligned} \tag{3.5}$$

The sum of (3.4) and (3.5), we have

$$\begin{aligned} &f(a^t b^{1-t})g(a^t b^{1-t}) + f(a^{1-t} b^{t})g(a^{1-t} b^{t}) \\ \le\ &\left(2t^2-2t+1\right)M(a,b) + 2t(1-t)\,N(a,b) \end{aligned} \tag{3.6}$$

Multiplying both sides of (3.6) by $t^{\alpha-1}\frac{\alpha}{2}$, then integrating the obtained inequality with respect to t over $[0,1]$, we have

$$\begin{aligned} &\frac{\alpha}{2}\left[\int_0^1 t^{\alpha-1}f(a^t b^{1-t})g(a^t b^{1-t})dt + \int_0^1 t^{\alpha-1}f(a^{1-t} b^{t})g(a^{1-t} b^{t})dt\right] \\ =\ &\frac{\alpha}{2}\left[\int_a^b \left(\frac{\ln\frac{b}{u}}{\ln\frac{b}{a}}\right)^{\alpha-1} f(u)\,g(u)\,\frac{du}{u\ln\frac{b}{a}} + \int_a^b \left(\frac{\ln\frac{v}{a}}{\ln\frac{b}{a}}\right)^{\alpha-1} f(v)\,g(v)\,\frac{du}{v\ln\frac{b}{a}}\right] \\ =\ &\frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[\int_a^b \left(\ln\frac{b}{u}\right)^{\alpha-1} f(u)\,g(u)\,\frac{du}{u} + \int_a^b \left(\ln\frac{v}{a}\right)^{\alpha-1} f(v)\,g(v)\,\frac{du}{v}\right] \\ =\ &\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)\,g(b)+J_{b-}^{\alpha}f(a)\,g(a)\right] \\ \le\ &\frac{\alpha}{2}\left[M(a,b)\int_0^1 t^{\alpha-1}\left(2t^2-2t+1\right)dt + N(a,b)\int_0^1 t^{\alpha-1}2t(1-t)\,dt\right] \\ =\ &\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)M(a,b)+\frac{\alpha}{(\alpha+2)(\alpha+1)}N(a,b) \end{aligned}$$

and this completes the proof. □

Remark 1. *Theorem* 4 *is an analogous generalization of* (2.2) *for GA-convex functions.*

Corollary 1. *In Theorem 4, if we take $g:[a,b]\to\mathbb{R}$ as $g(x)=1$ for all $x\in[a,b]$, then we have*

$$\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2}$$

which is the right hand side of (2.4).

Corollary 2. *In Theorem 4, if we take $\alpha=1$, then we have*

$$\frac{1}{\ln b-\ln a}\int_a^b f(x)\,g(x)\,\frac{dx}{x}\leq\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b)$$

for GA-convex functions.

Theorem 5. *Let f and $g:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ be non-negative and GA-convex functions with $a<b$ and $f\in L[a,b]$, then the following inequality for fractional integrals hold:*

$$2f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right)\leq\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)\,g(b)+J_{b-}^{\alpha}f(a)\,g(a)\right]$$
$$+\frac{\alpha}{(\alpha+2)(\alpha+1)}M(a,b)+\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)N(a,b)\tag{3.7}$$

where $\alpha>0$, $M(a,b)=f(a)\,g(a)+f(b)\,g(b)$ and $N(a,b)=f(a)\,g(b)+f(b)\,g(a)$.

Proof. It is clear for all $t\in[0,1]$

$$\sqrt{ab}=\sqrt{a^tb^{1-t}.a^{1-t}b^t}=\sqrt{a^tb^{1-t}}\sqrt{a^{1-t}b^t}.$$

Since f and g are non-negative and GA-convex functions on $[a,b]$, we have for all $t\in[0,1]$

$$\begin{aligned}
f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) &= f\left(\sqrt{a^tb^{1-t}}\sqrt{a^{1-t}b^t}\right)g\left(\sqrt{a^tb^{1-t}}\sqrt{a^{1-t}b^t}\right)\\
&\leq \frac{1}{4}\left[f\left(a^tb^{1-t}\right)+f\left(a^{1-t}b^t\right)\right]\left[g\left(a^tb^{1-t}\right)+g\left(a^{1-t}b^t\right)\right]\\
&= \frac{1}{4}\left[f\left(a^tb^{1-t}\right)g\left(a^tb^{1-t}\right)+f\left(a^{1-t}b^t\right)g\left(a^{1-t}b^t\right)\right]\\
&\quad+\frac{1}{4}\left[f\left(a^tb^{1-t}\right)g\left(a^{1-t}b^t\right)+f\left(a^{1-t}b^t\right)g\left(a^tb^{1-t}\right)\right]\\
&\leq \frac{1}{4}\left[f\left(a^tb^{1-t}\right)g\left(a^tb^{1-t}\right)+f\left(a^{1-t}b^t\right)g\left(a^{1-t}b^t\right)\right]\\
&\quad+\frac{1}{4}\left[tf(a)+(1-t)f(b)\right]\left[(1-t)g(a)+tg(b)\right]\\
&\quad+\frac{1}{4}\left[(1-t)f(a)+tf(b)\right]\left[tg(a)+(1-t)g(b)\right]\\
&= \frac{1}{4}\left[f\left(a^tb^{1-t}\right)g\left(a^tb^{1-t}\right)+f\left(a^{1-t}b^t\right)g\left(a^{1-t}b^t\right)\right]\\
&\quad+\frac{1}{4}\left\{2t(1-t)\left[f(a)\,g(a)+f(b)\,g(b)\right]\right.\\
&\quad\left.+\left(2t^2-2t+1\right)\left[f(a)\,g(b)+f(b)\,g(a)\right]\right\}
\end{aligned}\tag{3.8}$$

Multiplying both sides of (3.8) by $2\alpha t^{\alpha-1}$, then integrating the obtained inequality with respect to t over $[0,1]$, we have

$$\begin{aligned}
2f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) &\leq \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)\,g(b)+J_{b-}^{\alpha}f(a)\,g(a)\right]\\
&\quad+\frac{\alpha}{(\alpha+2)(\alpha+1)}M(a,b)+\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)N(a,b)
\end{aligned}$$

and this completes the proof. □

Remark 2. *Theorem* 5 *is an analogous generalization of* (2.3) *for GA-convex functions.*

Corollary 3. *In Theorem* 5*, if we take* $g:[a,b]\to\mathbb{R}$ *as* $g(x)=1$ *for all* $x\in[a,b]$*, then we have*

$$2f\left(\sqrt{ab}\right)\leq\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]+\frac{f(a)+f(b)}{2}.$$

Corollary 4. *In Theorem* 5*, if we take* $\alpha=1$*, then we have*

$$2f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right)\leq\frac{1}{\ln b-\ln a}\int_a^b f(x)\,g(x)\,\frac{dx}{x}+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b)$$

for GA-convex functions.

Theorem 6. *Let* f *and* $g:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ *be non-negative and GA-convex functions with* $a<b$ *and* $f\in L[a,b]$*, then the following inequality for fractional integrals hold:*

$$\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{\sqrt{ab}-}^{\alpha}f(a)\,g(a)+J_{\sqrt{ab}+}^{\alpha}f(b)\,g(b)\right]$$
$$\leq\left(\frac{\alpha}{4(\alpha+2)}-\frac{\alpha}{2(\alpha+1)}+\frac{1}{2}\right)M(a,b)+\frac{\alpha^2+3\alpha}{4(\alpha+2)(\alpha+1)}N(a,b)\quad(3.9)$$

where $\alpha>0$*,* $M(a,b)=f(a)\,g(a)+f(b)\,g(b)$ *and* $N(a,b)=f(a)\,g(b)+f(b)\,g(a)$*.*

Proof. Since f and g are non-negative and GA-convex functions on $[a,b]$, multiplying both sides of (3.6) by $t^{\alpha-1}\frac{\alpha}{2^{1-\alpha}}$, then integrating the obtained inequality with respect to t over $\left[0,\frac{1}{2}\right]$, we have

$$\frac{\alpha}{2^{1-\alpha}}\left[\int_0^{\frac{1}{2}}t^{\alpha-1}f(a^tb^{1-t})g(a^tb^{1-t})dt+\int_0^{\frac{1}{2}}t^{\alpha-1}f(a^{1-t}b^t)g(a^{1-t}b^t)dt\right]$$
$$=\frac{\alpha}{2^{1-\alpha}}\left[\int_{\sqrt{ab}}^{b}\left(\frac{\ln\frac{b}{u}}{\ln\frac{b}{a}}\right)^{\alpha-1}f(u)\,g(u)\,\frac{du}{u\ln\frac{b}{a}}+\int_a^{\sqrt{ab}}\left(\frac{\ln\frac{v}{a}}{\ln\frac{b}{a}}\right)^{\alpha-1}f(v)\,g(v)\,\frac{du}{v\ln\frac{b}{a}}\right]$$
$$=\frac{\alpha}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[\int_{\sqrt{ab}}^{b}\left(\ln\frac{b}{u}\right)^{\alpha-1}f(u)\,g(u)\,\frac{du}{u}+\int_a^{\sqrt{ab}}\left(\ln\frac{v}{a}\right)^{\alpha-1}f(v)\,g(v)\,\frac{du}{v}\right]$$
$$=\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{\sqrt{ab}+}^{\alpha}f(b)\,g(b)+J_{\sqrt{ab}-}^{\alpha}f(a)\,g(a)\right]$$
$$\leq\frac{\alpha}{2^{1-\alpha}}\left[M(a,b)\int_0^{\frac{1}{2}}t^{\alpha-1}\left(2t^2-2t+1\right)dt+N(a,b)\int_0^{\frac{1}{2}}t^{\alpha-1}2t(1-t)\,dt\right]$$
$$=\left(\frac{\alpha}{4(\alpha+2)}-\frac{\alpha}{2(\alpha+1)}+\frac{1}{2}\right)M(a,b)+\frac{\alpha^2+3\alpha}{4(\alpha+2)(\alpha+1)}N(a,b)$$

and this completes the proof. □

Remark 3. *Theorem* 6 *is an other analogous generalization of* (2.2) *for GA-convex functions.*

Corollary 5. *In Theorem* 6*, if we take* $g:[a,b]\to\mathbb{R}$ *as* $g(x)=1$ *for all* $x\in[a,b]$*, then we have*

$$\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{\sqrt{ab}-}^{\alpha}f(a)+J_{\sqrt{ab}+}^{\alpha}f(b)\right]\leq\frac{f(a)+f(b)}{2}$$

which is the right hand side of (2.5)*.*

Corollary 6. *In Theorem* 6*, if we take* $\alpha = 1$*, then we have*

$$\frac{1}{\ln b - \ln a} \int_a^b f(x) g(x) \frac{dx}{x} \leq \frac{1}{3} M(a,b) + \frac{1}{6} N(a,b)$$

for GA-convex functions.

Theorem 7. *Let* f *and* $g : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ *be non-negative and GA-convex functions with* $a < b$ *and* $f \in L[a,b]$*, then the following inequality for fractional integrals hold:*

$$2f\left(\sqrt{ab}\right) g\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln \frac{b}{a}\right)^{\alpha}} \left[J^{\alpha}_{\sqrt{ab}-} f(a) g(a) + J^{\alpha}_{\sqrt{ab}+} f(b) g(b)\right]$$
$$+ \frac{\alpha^2 + 3\alpha}{4(\alpha+2)(\alpha+1)} M(a,b) + \left(\frac{\alpha}{4(\alpha+2)} - \frac{\alpha}{2(\alpha+1)} + \frac{1}{2}\right) N(a,b) \quad (3.10)$$

where $\alpha > 0$*,* $M(a,b) = f(a) g(a) + f(b) g(b)$ *and* $N(a,b) = f(a) g(b) + f(b) g(a)$*.*

Proof. Multiplying both sides of (3.8) by $2^{1+\alpha} \alpha t^{\alpha-1}$, then integrating the obtained inequality with respect to t over $\left[0, \frac{1}{2}\right]$, we have desired result. □

Remark 4. *Theorem* 7 *is an other analogous generalization of* (2.3) *for GA-convex functions.*

Corollary 7. *In Theorem* 7*, if we take* $g : [a,b] \to \mathbb{R}$ *as* $g(x) = 1$ *for all* $x \in [a,b]$*, then we have*

$$2f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln \frac{b}{a}\right)^{\alpha}} \left[J^{\alpha}_{\sqrt{ab}-} f(a) + J^{\alpha}_{\sqrt{ab}+} f(b)\right] + \frac{f(a) + f(b)}{2}.$$

Corollary 8. *In Theorem* 7*, if we take* $\alpha = 1$*, then we have*

$$2f\left(\sqrt{ab}\right) g\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a} \int_a^b f(x) g(x) \frac{dx}{x} + \frac{1}{6} M(a,b) + \frac{1}{3} N(a,b)$$

for GA-convex functions.

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