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SUPERMEASURES ASSOCIATED TO SOME CLASSICAL INEQUALITIES

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Abstract. In this paper we give some examples of supermeasures that are naturally associated to classical inequalities such as Jensen's inequality, Hölder's inequality, Minkowski's inequality, Cauchy-Bunyakovsky-Schwarz's inequality, Čebyšev's inequality, Hermite-Hadamard's inequalities and the definition of convexity property. As a consequence of monotonic nondecreasing property of these supermeasures, some refinements of the above inequalities are also obtained.

1. Introduction

Let Ω be a nonempty set. A subset $\mathcal{A}$ of the power set 2^{Ω} is called an *algebra* if the following conditions are satisfied:

(i) Ω is in $\mathcal{A}$;
(ii) $\mathcal{A}$ is closed under complementation, namely, if $A \in \mathcal{A}$ then $\Omega \setminus A \in \mathcal{A}$;
(iii) $\mathcal{A}$ is closed under union, i.e. if $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$.

By applying *de Morgan*'s laws it follows that $\mathcal{A}$ is closed under intersection, namely if $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$. It also follows that the empty set $\emptyset$ belongs to $\mathcal{A}$. Elements of the algebra are called measurable sets. An ordered pair $(\Omega, \mathcal{A})$, where Ω is a set and $\mathcal{A}$ is a algebra over Ω, is called a *measurable space*.

The function $\mu : \mathcal{A} \to [0, \infty)$ is called a *measure* [*submeasure* (*supermeasure*)] on $\mathcal{A}$ if

(a) For all $A \in \mathcal{A}$ we have $\mu(A) \geq 0$ (nonnegativity);
(aa) We have $\mu(\emptyset) = 0$ (null empty set);
(aaa) For any $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$ we have

$$\mu(A \cup B) = [\leq (\geq)]\, \mu(A) + \mu(B), \tag{1.1}$$

i.e., μ is *additive* [*subadditive* (*superadditive*)] on $\mathcal{A}$.

For μ as above we denote by

$$\mathcal{A}_{\mu} := \{A \in \mathcal{A} \ |\mu(A) > 0\}.$$

If $\mathcal{A}_{\mu} = \mathcal{A} \setminus \{\emptyset\}$ then we say that μ is *positive* on $\mathcal{A}$.

Let $A, B \in \mathcal{A}$ with $A \subset B$, then $B = A \cup (B \setminus A)$, $A \cap (B \setminus A) = \emptyset$ and $B \setminus A \in \mathcal{A}$. If μ is additive (superadditive) then

$$\mu(B) = \mu(A \cup (B \setminus A)) = (\geq)\, \mu(A) + \mu(B \setminus A) \geq \mu(A)$$

showing that μ is *monotonic nondecreasing* on $\mathcal{A}$.

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In this paper we give some examples of supermeasures that can be naturally associated to classical inequalities such as Jensen's inequality, Hölder's inequality, Minkowski's inequality, Cauchy-Bunyakovsky-Schwarz's inequality, Čebyšev's inequality, Hermite-Hadamard's inequalities and the definition of convexity property. As a consequence of monotonic nondecreasing property of these supermeasures, some refinements of the above inequalities are also obtained.

2. The Case of Jensen's Inequality

Let $(\Omega, \mathcal{A}, \nu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure ν on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a ν-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for ν -a.e. (almost every) $x\in\Omega$, consider the *Lebesgue space*

$$L_w(\Omega,\nu) := \{f:\Omega\to\mathbb{R},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w(x)\left|f(x)\right| d\nu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\nu$ instead of $\int_\Omega w(x)\, d\nu(x)$.

Let also

$$\mathcal{A}_\nu := \{A\in\mathcal{A}\ |\mu(A) > 0\}.$$

For a ν-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)>0$ for ν -a.e. $x\in\Omega$, we consider the functional $J(\cdot,w;\Phi,f):\mathcal{A}_\nu\to[0,\infty)$ defined by

$$J(A,w;\Phi,f) := \int_A w(\Phi\circ f)\, d\nu - \Phi\left(\frac{\int_A wf d\nu}{\int_A w d\nu}\right)\int_A w d\nu \geq 0, \tag{2.1}$$

where $\Phi:[m,M]\to\mathbb{R}$ is a continuous convex function on the interval of real numbers $[m,M]$, $f:\Omega\to[m,M]$ is ν-measurable and such that $f,\Phi\circ f\in L_w(\Omega,\nu)$.

Theorem 1. *Let $\Phi:[m,M]\to\mathbb{R}$ be a continuous convex function on the interval of real numbers $[m,M]$, $f:\Omega\to[m,M]$ is ν-measurable and such that $f,\Phi\circ f\in L_w(\Omega,\nu)$. Then the functional $J(\cdot,w;\Phi,f)$ defined by (2.2) is a supermeasure on $\mathcal{A}_\nu$.*

Proof. Let $A,B\in\mathcal{A}_\nu$ with $A\cap B=\emptyset$. Observe that

$$\begin{aligned} &J(A\cup B,w;\Phi,f) \\ &= \int_A w(\Phi\circ f)\, d\nu + \int_B w(\Phi\circ f)\, d\nu \\ &- \Phi\left(\frac{\int_A wf d\nu + \int_B wf d\nu}{\int_A w d\nu + \int_B w d\nu}\right)\left(\int_A w d\nu + \int_B w d\nu\right) \\ &= \int_A w(\Phi\circ f)\, d\nu + \int_B w(\Phi\circ f)\, d\nu \\ &- \Phi\left(\frac{\int_A w d\nu \frac{\int_A wf d\nu}{\int_A w d\nu} + \int_B w d\nu \frac{\int_B wf d\nu}{\int_B w d\nu}}{\int_A w d\nu + \int_B w d\nu}\right)\left(\int_A w d\nu + \int_B w d\nu\right) \\ &=: L. \end{aligned} \tag{2.2}$$

By convexity of the function $\Phi : [m, M] \to \mathbb{R}$ and since

$$\frac{\int_A wfd\nu}{\int_A wd\nu}, \frac{\int_B wfd\nu}{\int_B wd\nu} \in [m, M]$$

we have

$$\Phi\left(\frac{\int_A wd\nu \frac{\int_A wfd\nu}{\int_A wd\nu} + \int_B wd\nu \frac{\int_B wfd\nu}{\int_B wd\nu}}{\int_A wd\nu + \int_B wd\nu}\right) \leq \frac{\int_A wd\nu \Phi\left(\frac{\int_A wfd\nu}{\int_A wd\nu}\right) + \int_B wd\nu \Phi\left(\frac{\int_B wfd\nu}{\int_B wd\nu}\right)}{\int_A wd\nu + \int_B wd\nu}.$$

Therefore by (2.2) we have

$$\begin{aligned} L &\geq \int_A w(\Phi \circ f) d\nu + \int_B w(\Phi \circ f) d\nu \\ &\quad - \frac{\int_A wd\nu \Phi\left(\frac{\int_A wfd\nu}{\int_A wd\nu}\right) + \int_B wd\nu \Phi\left(\frac{\int_B wfd\nu}{\int_B wd\nu}\right)}{\int_A wd\nu + \int_B wd\nu} \left(\int_A wd\nu + \int_B wd\nu\right) \\ &= \int_A w(\Phi \circ f) d\nu - \int_A wd\nu \Phi\left(\frac{\int_A wfd\nu}{\int_A wd\nu}\right) \\ &\quad + \int_B w(\Phi \circ f) d\nu - \int_B wd\nu \Phi\left(\frac{\int_B wfd\nu}{\int_B wd\nu}\right) \\ &= J(A, w; \Phi, f) + J(B, w; \Phi, f). \end{aligned} \tag{2.3}$$

Making use of (2.2) and (2.3) we conclude that

$$J(A \cup B, w; \Phi, f) \geq J(A, w; \Phi, f) + J(B, w; \Phi, f)$$

for any $A, B \in \mathcal{A}_\nu$ with $A \cap B = \emptyset$, which shows that $J(\cdot, w; \Phi, f)$ is a supermeasure on $\mathcal{A}$. □

For some Jensen's inequality related functionals and their properties see [1], [2], [4], [12], [19], [6], [7] and [10].

Let $\Phi : [m, M] \to \mathbb{R}$ be a continuous convex function on the interval of real numbers $[m, M]$, $x = (x_i)_{i \in \mathbb{N}}$ a sequence of real numbers with $x_i \in [m, M]$, $i \in \mathbb{N}$, and $w = (w_i)_{i \in \mathbb{N}}$ a sequence of positive real numbers.

Let $\Omega = \mathbb{N}$ and $\mathcal{P}_f(\mathbb{N})$ be the algebra of finite parts of natural numbers $\mathbb{N}$. By the monotonicity property of supermeasure on $\mathcal{P}_f(\mathbb{N})$ we have from the above results that the sequence

$$J_n(w; \Phi, x) := \left(\sum_{i=0}^n w_i\right)^{-1} \left[\frac{\sum_{i=0}^n w_i \Phi(x_i)}{\sum_{i=0}^n w_i} - \Phi\left(\frac{\sum_{i=0}^n w_i x_i}{\sum_{i=0}^n w_i}\right)\right],$$

is *monotonic nondecreasing*, namely

$$\begin{aligned} &\left(\sum_{i=0}^{n+1} w_i\right)^{-1} \left[\frac{\sum_{i=0}^{n+1} w_i \Phi(x_i)}{\sum_{i=0}^{n+1} w_i} - \Phi\left(\frac{\sum_{i=0}^{n+1} w_i x_i}{\sum_{i=0}^{n+1} w_i}\right)\right] \\ &\geq \left(\sum_{i=0}^n w_i\right)^{-1} \left[\frac{\sum_{i=0}^n w_i \Phi(x_i)}{\sum_{i=0}^n w_i} - \Phi\left(\frac{\sum_{i=0}^n w_i x_i}{\sum_{i=0}^n w_i}\right)\right] \end{aligned} \tag{2.4}$$

for any $n \in \mathbb{N}$ and

$$J_n(w;\Phi,x) \geq \max_{0\leq i\neq j\leq n}\left\{(w_i+w_j)^{-1}\left[\frac{w_i\Phi(x_i)+w_j\Phi(x_j)}{w_i+w_j}-\Phi\left(\frac{w_ix_i+w_jx_j}{w_i+w_j}\right)\right]\right\}. \tag{2.5}$$

We also have for $n \geq 1$ that

$$\begin{aligned}&\left(\sum_{i=0}^{2n} w_i\right)^{-1}\left[\frac{\sum_{i=0}^{2n} w_i\Phi(x_i)}{\sum_{i=0}^{2n} w_i}-\Phi\left(\frac{\sum_{i=0}^{2n} w_ix_i}{\sum_{i=0}^{2n} w_i}\right)\right]\\ &\geq\left(\sum_{i=0}^{n} w_{2i}\right)^{-1}\left[\frac{\sum_{i=0}^{n} w_{2i}\Phi(x_{2i})}{\sum_{i=0}^{n} w_{2i}}-\Phi\left(\frac{\sum_{i=0}^{n} w_{2i}x_{2i}}{\sum_{i=0}^{n} w_{2i}}\right)\right]\\ &+\left(\sum_{i=0}^{n-1} w_{2i+1}\right)^{-1}\left[\frac{\sum_{i=0}^{n-1} w_{2i+1}\Phi(x_{2i+1})}{\sum_{i=0}^{n-1} w_{2i+1}}-\Phi\left(\frac{\sum_{i=0}^{n-1} w_{2i+1}x_{2i+1}}{\sum_{i=0}^{n-1} w_{2i+1}}\right)\right]\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}&\left(\sum_{i=0}^{2n+1} w_i\right)^{-1}\left[\frac{\sum_{i=0}^{2n+1} w_i\Phi(x_i)}{\sum_{i=0}^{2n} w_i}-\Phi\left(\frac{\sum_{i=0}^{2n+1} w_ix_i}{\sum_{i=0}^{2n} w_i}\right)\right]\\ &\geq\left(\sum_{i=0}^{n} w_{2i}\right)^{-1}\left[\frac{\sum_{i=0}^{n} w_{2i}\Phi(x_{2i})}{\sum_{i=0}^{n} w_{2i}}-\Phi\left(\frac{\sum_{i=0}^{n} w_{2i}x_{2i}}{\sum_{i=0}^{n} w_{2i}}\right)\right]\\ &+\left(\sum_{i=0}^{n} w_{2i+1}\right)^{-1}\left[\frac{\sum_{i=0}^{n} w_{2i+1}\Phi(x_{2i+1})}{\sum_{i=0}^{n} w_{2i+1}}-\Phi\left(\frac{\sum_{i=0}^{n} w_{2i+1}x_{2i+1}}{\sum_{i=0}^{n} w_{2i+1}}\right)\right].\end{aligned} \tag{2.7}$$

3. The Case of Hölder's Inequality

Let $(\Omega,\mathcal{A},\nu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure ν on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a ν-measurable function $w:\Omega\to\mathbb{C}$, with $w(x)\geq 0$ for ν -a.e. (almost every) $x\in\Omega$, consider the Lebesgue space

$$L_w^\alpha(\Omega,\nu) := \{f:\Omega\to\mathbb{C},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w|f|^\alpha d\nu<\infty\},$$

for $\alpha\geq 1$.

The following inequality is well known in the literature as *Hölder's inequality*

$$\left|\int_\Omega wfgd\nu\right| \leq \left(\int_\Omega w|f|^\alpha d\nu\right)^{1/\alpha}\left(\int_\Omega w|g|^\beta d\nu\right)^{1/\beta} \tag{3.1}$$

where $\alpha,\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and $f\in L_w^\alpha(\Omega,\nu)$, $g\in L_w^\beta(\Omega,\nu)$.

We consider the functional $H_{\alpha,\beta}(\cdot,w;f,g):\mathcal{A}_\nu\to[0,\infty)$ defined by

$$H_{\alpha,\beta}(A,w;f,g) = \left(\int_A w|f|^\alpha d\nu\right)^{1/\alpha}\left(\int_A w|g|^\beta d\nu\right)^{1/\beta} - \left|\int_A wfgd\nu\right|. \tag{3.2}$$

We have:

Theorem 2. *Let $f\in L_w^\alpha(\Omega,\nu)$, $g\in L_w^\beta(\Omega,\nu)$ where $\alpha,\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$. Then the functional $H_{\alpha,\beta}(\cdot,w;f,g):\mathcal{A}_\nu\to[0,\infty)$ defined by (3.2) is a supermeasure.*

Proof. Let $A, B \in \mathcal{A}_\nu$ with $A \cap B = \emptyset$. Observe that

$$
\begin{aligned}
(3.3)\qquad & H_{\alpha,\beta}(A \cup B, w; f, g) \\
&= \left(\int_{A\cup B} w\,|f|^\alpha\,d\nu\right)^{1/\alpha}\left(\int_{A\cup B} w\,|g|^\beta\,d\nu\right)^{1/\beta} - \left|\int_{A\cup B} wfgd\nu\right| \\
&= \left(\int_A w\,|f|^\alpha\,d\nu + \int_B w\,|f|^\alpha\,d\nu\right)^{1/\alpha}\left(\int_A w\,|g|^\beta\,d\nu + \int_B w\,|g|^\beta\,d\nu\right)^{1/\beta} \\
&- \left|\int_A wfgd\nu + \int_B wfgd\nu\right| \\
&= \left(\left[\left(\int_A w\,|f|^\alpha\,d\nu\right)^{1/\alpha}\right]^\alpha + \left[\left(\int_B w\,|f|^\alpha\,d\nu\right)^{1/\alpha}\right]^\alpha\right)^{1/\alpha} \\
&\times \left(\left[\left(\int_A w\,|g|^\beta\,d\nu\right)^{1/\beta}\right]^\beta + \left[\left(\int_B w\,|g|^\beta\,d\nu\right)^{1/\beta}\right]^\beta\right)^{1/\beta} \\
&- \left|\int_A wfgd\nu + \int_B wfgd\nu\right| \\
&:= U.
\end{aligned}
$$

By the elementary inequality

$$(a^\alpha + b^\alpha)^{1/\alpha}\left(c^\beta + d^\beta\right)^{1/\beta} \geq ac + bd,$$

where $a, b, c, d \geq 0$ and $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and the triangle inequality for modulus we have

$$
\begin{aligned}
(3.4)\qquad U &\geq \left(\int_A w\,|f|^\alpha\,d\nu\right)^{1/\alpha}\left(\int_A w\,|f|^\beta\,d\nu\right)^{1/\beta} \\
&+ \left(\int_B w\,|f|^\alpha\,d\nu\right)^{1/\alpha}\left(\int_B w\,|f|^\beta\,d\nu\right)^{1/\beta} - \left|\int_A wfgd\nu\right| - \left|\int_B wfgd\nu\right| \\
&= H_{\alpha,\beta}(A, w; f, g) + H_{\alpha,\beta}(B, w; f, g).
\end{aligned}
$$

By (3.3) and (3.4) we get the desired result. □

For some Hölder's inequality related functionals and their properties see [3], [5] and [20].

Let $\Omega = \mathbb{N}$ and $\mathcal{P}_f(\mathbb{N})$ be the algebra of finite parts of natural numbers $\mathbb{N}$. By the monotonicity property of supermeasure on $\mathcal{P}_f(\mathbb{N})$ we have from the above results that the sequence

$$(3.5)\qquad H_{n,\alpha,\beta}(w; x, y) := \left(\sum_{i=0}^{n} w_i\,|x_i|^\alpha\right)^{1/\alpha}\left(\sum_{i=0}^{n} w_i\,|y_i|^\beta\right)^{1/\beta} - \left|\sum_{i=0}^{n} w_i x_i y_i\right|,$$

is monotonic nondecreasing and

$$
\begin{aligned}
(3.6)\qquad & H_{n,\alpha,\beta}(w; x, y) \\
&\geq \max_{0\leq i\neq j\leq n}\left\{\left[(w_i\,|x_i|^\alpha + w_j\,|x_j|^\alpha)^{1/\alpha}\left(w_i\,|y_i|^\beta + w_j\,|y_j|^\beta\right)^{1/\beta}\right.\right. \\
&\left.\left. - |w_i x_i y_i + w_j x_j y_j|\right]\right\}.
\end{aligned}
$$

We also have for $n \geq 1$ that

$$\left(\sum_{i=0}^{2n} w_i |x_i|^\alpha\right)^{1/\alpha} \left(\sum_{i=0}^{2n} w_i |y_i|^\beta\right)^{1/\beta} - \left|\sum_{i=0}^{2n} w_i x_i y_i\right| \tag{3.7}$$
$$\geq \left(\sum_{i=0}^{n} w_{2i} |x_{2i}|^\alpha\right)^{1/\alpha} \left(\sum_{i=0}^{n} w_{2i} |y_{2i}|^\beta\right)^{1/\beta} - \left|\sum_{i=0}^{n} w_{2i} x_{2i} y_{2i}\right|$$
$$+ \left(\sum_{i=0}^{n-1} w_{2i+1} |x_{2i+1}|^\alpha\right)^{1/\alpha} \left(\sum_{i=0}^{n-1} w_{2i+1} |y_{2i+1}|^\beta\right)^{1/\beta} - \left|\sum_{i=0}^{n-1} w_{2i+1} x_{2i+1} y_{2i+1}\right|.$$

4. The Case of Minkowski's Inequality

Let $(\Omega, \mathcal{A}, \nu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure ν on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a ν-measurable function $w : \Omega \to \mathbb{C}$, with $w(x) \geq 0$ for ν -a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w^r(\Omega, \nu) := \{f : \Omega \to \mathbb{C},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w |f|^r \, d\nu < \infty\},$$

for $r \geq 1$.

The following inequality is well known in the literature as Minkowski's inequality

$$\left(\int_\Omega w |f+g|^r \, d\nu\right)^{1/r} \leq \left(\int_\Omega w |f|^r \, d\nu\right)^{1/r} + \left(\int_\Omega w |g|^r \, d\nu\right)^{1/r} \tag{4.1}$$

for any $f, g \in L_w^r(\Omega, \nu)$.

Consider the functional $M_r(\cdot, w; f, g) : \mathcal{A}_\nu \to [0, \infty)$ defined by

$$M_r(A, w; f, g) \tag{4.2}$$
$$:= \left[\left(\int_A w |f|^r \, d\nu\right)^{1/r} + \left(\int_A w |g|^r \, d\nu\right)^{1/r}\right]^r - \int_A w |f+g|^r \, d\nu.$$

Theorem 3. *Let $f, g \in L_w^r(\Omega, \nu)$ for $r \geq 1$. Then the functional $M_r(\cdot, w; f, g) : \mathcal{A}_\nu \to [0, \infty)$ defined by (4.2) is a supermeasure.*

Proof. Let $A, B \in \mathcal{A}_\nu$ with $A \cap B = \emptyset$. Observe that

$$\begin{aligned}
(4.3)\quad & M_r(A \cup B, w; f, g) \\
&= \left[\left(\int_{A\cup B} w\,|f|^r\,d\nu\right)^{1/r} + \left(\int_{A\cup B} w\,|g|^r\,d\nu\right)^{1/r}\right]^r - \int_{A\cup B} w\,|f+g|^r\,d\nu \\
&= \left[\left(\int_A w\,|f|^r\,d\nu + \int_B w\,|f|^r\,d\nu\right)^{1/r} + \left(\int_A w\,|g|^r\,d\nu + \int_B w\,|g|^r\,d\nu\right)^{1/r}\right]^r \\
&- \int_A w\,|f+g|^r\,d\nu - \int_B w\,|f+g|^r\,d\nu \\
&= \left[\left(\left[\left(\int_A w\,|f|^r\,d\nu\right)^{1/r}\right]^r + \left[\left(\int_B w\,|f|^r\,d\nu\right)^{1/r}\right]^r\right)^{1/r}\right. \\
&\left. + \left(\left[\left(\int_A w\,|g|^r\,d\nu\right)^{1/r}\right]^r + \left[\left(\int_B w\,|g|^r\,d\nu\right)^{1/r}\right]^r\right)^{1/r}\right]^r \\
&- \int_A w\,|f+g|^r\,d\nu - \int_B w\,|f+g|^r\,d\nu \\
&=: V.
\end{aligned}$$

By the elementary inequality

$$(a^r + b^r)^{1/r} + (c^r + d^r)^{1/r} \geq [(a+c)^r + (b+d)^r]^{1/r}$$

that holds for $a, b, c, d \geq 0$ and $r \geq 1$, we have

$$\left[(a^r + b^r)^{1/r} + (c^r + d^r)^{1/r}\right]^r \geq (a+c)^r + (b+d)^r. \tag{4.4}$$

Applying the inequality (4.4) we have

$$\begin{aligned}
(4.5)\quad & \left[\left(\left[\left(\int_A w\,|f|^r\,d\nu\right)^{1/r}\right]^r + \left[\left(\int_B w\,|f|^r\,d\nu\right)^{1/r}\right]^r\right)^{1/r}\right. \\
&\left. + \left(\left[\left(\int_A w\,|g|^r\,d\nu\right)^{1/r}\right]^r + \left[\left(\int_B w\,|g|^r\,d\nu\right)^{1/r}\right]^r\right)^{1/r}\right]^r \\
&\geq \left[\left(\int_A w\,|f|^r\,d\nu\right)^{1/r} + \left(\int_A w\,|g|^r\,d\nu\right)^{1/r}\right]^r \\
&+ \left[\left(\int_B w\,|f|^r\,d\nu\right)^{1/r} + \left(\int_B w\,|g|^r\,d\nu\right)^{1/r}\right]^r
\end{aligned}$$

for $A, B \in \mathcal{A}_\nu$ with $A \cap B = \emptyset$.

On making use of (4.5) we then have

$$\begin{aligned} V &\geq \left[\left(\int_A w|f|^r d\nu\right)^{1/r} + \left(\int_A w|g|^r d\nu\right)^{1/r}\right]^r - \int_A w|f+g|^r d\nu \\ &+ \left[\left(\int_B w|f|^r d\nu\right)^{1/r} + \left(\int_B w|g|^r d\nu\right)^{1/r}\right]^r - \int_B w|f+g|^r d\nu \\ &= M_r(A,w;f,g) + M_r(B,w;f,g), \end{aligned}$$

for $A,B \in \mathcal{A}_\nu$ with $A \cap B = \emptyset$.

This completes the proof. □

For some Minkowski's inequality related functionals and their properties see [3], [5] and [20].

Let $x = (x_i)_{i\in\mathbb{N}}$ and $x = (y_i)_{i\in\mathbb{N}}$ be sequences of complex numbers and $w = (w_i)_{i\in\mathbb{N}}$ a sequence of positive real numbers. Let $\Omega = \mathbb{N}$ and $\mathcal{P}_f(\mathbb{N})$ be the algebra of finite parts of natural numbers $\mathbb{N}$. By the monotonicity property of supermeasure on $\mathcal{P}_f(\mathbb{N})$ we have from the above results that the sequence

$$\begin{aligned} &M_{n,r}(w;x,y) \\ &:= \left[\left(\sum_{i=0}^n w_i|x_i|^r\right)^{1/r} + \left(\sum_{i=0}^n w_i|y_i|^r\right)^{1/r}\right]^r - \sum_{i=0}^n w_i|x_i+y_i|^r \end{aligned} \tag{4.6}$$

is monotonic nondecreasing and

$$\begin{aligned} &M_{n,r}(w;x,y) \\ &\geq \max_{0\leq i\neq j\leq n}\left\{\left(\left[(w_i|x_i|^r + w_j|x_j|^r)^{1/r} + (w_i|y_i|^r + w_j|y_j|^r)^{1/r}\right]^r\right.\right. \\ &\left.\left. -w_i|x_i+y_i|^r - w_j|x_j+y_j|^r\right)\right\}. \end{aligned} \tag{4.7}$$

We have the inequality

$$\begin{aligned} &\left[\left(\sum_{i=0}^{2n} w_i|x_i|^r\right)^{1/r} + \left(\sum_{i=0}^{2n} w_i|y_i|^r\right)^{1/r}\right]^r - \sum_{i=0}^{2n} w_i|x_i+y_i|^r \\ &\geq \left[\left(\sum_{i=0}^{n} w_{2i}|x_{2i}|^r\right)^{1/r} + \left(\sum_{i=0}^{n} w_{2i}|y_{2i}|^r\right)^{1/r}\right]^r - \sum_{i=0}^{n} w_{2i}|x_{2i}+y_{2i}|^r \\ &\quad + \left[\left(\sum_{i=0}^{n-1} w_{2i+1}|x_{2i+1}|^r\right)^{1/r} + \left(\sum_{i=0}^{n-1} w_{2i+1}|y_{2i+1}|^r\right)^{1/r}\right]^r \\ &\quad - \sum_{i=0}^{n-1} w_{2i+1}|x_{2i+1}+y_{2i+1}|^r. \end{aligned} \tag{4.8}$$

5. The Case of Cauchy-Bunyakovsky-Schwarz's Inequality

Let $(\Omega, \mathcal{A}, \nu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure ν on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$.

For a ν-measurable function $w : \Omega \to \mathbb{C}$, with $w(x) \geq 0$ for ν -a.e. (almost every) $x \in \Omega$, consider the Hilbert space

$$L_w^2(\Omega, \nu) := \{f : \Omega \to \mathbb{C},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w |f|^2 d\nu < \infty\}.$$

The following inequality is well known in the literature as Cauchy-Bunyakovsky-Schwarz's (CBS) inequality

$$\left|\int_\Omega wfgd\nu\right| \leq \left(\int_\Omega w |f|^2 d\nu\right)^{1/2} \left(\int_\Omega w |g|^2 d\nu\right)^{1/2} \tag{5.1}$$

where $f \in L_w^2(\Omega, \nu)$, $g \in L_w^2(\Omega, \nu)$.

We consider the functional $H(\cdot, w; f, g) : \mathcal{A}_\nu \to [0, \infty)$ defined by

$$H(A, w; f, g) = \left(\int_A w |f|^2 d\nu\right)^{1/2} \left(\int_A w |g|^2 d\nu\right)^{1/2} - \left|\int_A wfgd\nu\right|. \tag{5.2}$$

Taking into account that $H(A, w; f, g) = H_{\alpha,\beta}(A, w; f, g)$ for $\alpha = \beta = 2$, see (3.2), we have:

Theorem 4. *Let $f \in L_w^2(\Omega, \nu)$, $g \in L_w^2(\Omega, \nu)$, then the functional $H(\cdot, w; f, g) : \mathcal{A}_\nu \to [0, \infty)$ defined by (5.2) is a supermeasure.*

Now, consider the functional $L(\cdot, w; f, g) : \mathcal{A}_\nu \to [0, \infty)$ defined by

$$L(A, w; f, g) = \int_A w |f|^2 d\nu \int_A w |g|^2 d\nu - \left|\int_A wfgd\nu\right|^2. \tag{5.3}$$

Theorem 5. *Let $f \in L_w^2(\Omega, \nu)$, $g \in L_w^2(\Omega, \nu)$. Then for any $A, B \in \mathcal{A}_\nu$ with $A \cap B = \emptyset$ we have*

$$\begin{aligned} &L(A \cup B, w; f, g) \\ &\geq L(A, w; f, g) + L(B, w; f, g) \\ &+ \left(\det\begin{bmatrix} \left(\int_A w |f|^2 d\nu\right)^{1/2} & \left(\int_A w |g|^2 d\nu\right)^{1/2} \\ \left(\int_B w |f|^2 d\nu\right)^{1/2} & \left(\int_B w |g|^2 d\nu\right)^{1/2} \end{bmatrix}\right)^2. \end{aligned} \tag{5.4}$$

Proof. Let $A, B \in \mathcal{A}_\nu$ with $A \cap B = \emptyset$. Then we have

$$\begin{aligned}
(5.5)\qquad & L(A \cup B, w; f, g) \\
&= \int_{A\cup B} w|f|^2 d\nu \int_{A\cup B} w|f|^2 d\nu - \left|\int_{A\cup B} wfgd\nu\right|^2 \\
&= \left(\int_A w|f|^2 d\nu + \int_B w|f|^2 d\nu\right)\left(\int_A w|g|^2 d\nu + \int_B w|g|^2 d\nu\right) \\
&- \left|\int_A wfgd\nu + \int_B wfgd\nu\right|^2 \\
&= \int_A w|f|^2 d\nu \int_A w|g|^2 d\nu + \int_A w|f|^2 d\nu \int_B w|g|^2 d\nu \\
&+ \int_B w|f|^2 d\nu \int_A w|g|^2 d\nu + \int_B w|f|^2 d\nu \int_B w|g|^2 d\nu \\
&- \left|\int_A wfgd\nu + \int_B wfgd\nu\right|^2.
\end{aligned}$$

Observe that

$$\begin{aligned}
(5.6)\qquad & \left|\int_A wfgd\nu + \int_B wfgd\nu\right|^2 \\
&= \left|\int_A wfgd\nu\right|^2 + \left|\int_B wfgd\nu\right|^2 + 2\operatorname{Re}\left(\int_A wfgd\nu \overline{\int_B wfgd\nu}\right) \\
&\le \left|\int_A wfgd\nu\right|^2 + \left|\int_B wfgd\nu\right|^2 + 2\left|\int_A wfgd\nu \overline{\int_B wfgd\nu}\right| \\
&= \left|\int_A wfgd\nu\right|^2 + \left|\int_B wfgd\nu\right|^2 + 2\left|\int_A wfgd\nu\right|\left|\int_B wfgd\nu\right| \\
&\le \left|\int_A wfgd\nu\right|^2 + \left|\int_B wfgd\nu\right|^2 \\
&+ 2\left(\int_A w|f|^2 d\nu\right)^{1/2}\left(\int_A w|g|^2 d\nu\right)^{1/2} \\
&\times \left(\int_B w|f|^2 d\nu\right)^{1/2}\left(\int_B w|g|^2 d\nu\right)^{1/2},
\end{aligned}$$

where for the last inequality we used the (CBS) inequality twice.

Making use of (5.5) and (5.6) we get

$$\begin{aligned}
L(A \cup B, w; f, g) &\ge L(A, w; f, g) + L(B, w; f, g) \\
&+ \int_A w|f|^2 d\nu \int_B w|g|^2 d\nu + \int_B w|f|^2 d\nu \int_A w|g|^2 d\nu \\
&- 2\left(\int_A w|f|^2 d\nu\right)^{1/2}\left(\int_A w|g|^2 d\nu\right)^{1/2} \\
&\times \left(\int_B w|f|^2 d\nu\right)^{1/2}\left(\int_B w|g|^2 d\nu\right)^{1/2}
\end{aligned}$$

and the inequality (5.4) is proved. □

Corollary 1. *Let $f \in L_w^2(\Omega,\nu)$, $g \in L_w^2(\Omega,\nu)$. The functional $L(\cdot,w;f,g) : \mathcal{A}_\nu \to [0,\infty)$ defined by (5.3) is a supermeasure.*

Let $f \in L_w^2(\Omega,\nu)$, $g \in L_w^2(\Omega,\nu)$. We can also consider the functional $Q(\cdot,w;f,g) : \mathcal{A}_\nu \to [0,\infty)$ defined by

$$Q(A,w;f,g) = \left[\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu - \left|\int_A wfgd\nu\right|^2\right]^{1/2} = \sqrt{L(A,w;f,g)}. \tag{5.7}$$

Theorem 6. *Let $f \in L_w^2(\Omega,\nu)$, $g \in L_w^2(\Omega,\nu)$, then the functional $H(\cdot,w;f,g) : \mathcal{A}_\nu \to [0,\infty)$ defined by (5.7) is a supermeasure.*

Proof. Let $A, B \in \mathcal{A}_\nu$ with $A \cap B = \emptyset$. Then we have

$$\begin{aligned}
&Q^2(A\cup B,w;f,g) \\
&= \int_{A\cup B} w|f|^2 d\nu \int_{A\cup B} w|f|^2 d\nu - \left|\int_{A\cup B} wfgd\nu\right|^2 \\
&= \left(\int_A w|f|^2 d\nu + \int_B w|f|^2 d\nu\right)\left(\int_A w|g|^2 d\nu + \int_B w|g|^2 d\nu\right) \\
&- \left|\int_A wfgd\nu + \int_B wfgd\nu\right|^2 \\
&= \int_A w|f|^2 d\nu \int_A w|g|^2 d\nu + \int_A w|f|^2 d\nu \int_B w|g|^2 d\nu \\
&+ \int_B w|f|^2 d\nu \int_A w|g|^2 d\nu + \int_B w|f|^2 d\nu \int_B w|g|^2 d\nu \\
&- \left|\int_A wfgd\nu\right|^2 - \left|\int_B wfgd\nu\right|^2 - 2\operatorname{Re}\left(\int_A wfgd\nu \int_B \overline{wfg}d\nu\right) \\
&= Q^2(A,w;f,g) + Q^2(B,w;f,g) \\
&+ \int_A w|f|^2 d\nu \int_B w|g|^2 d\nu + \int_B w|f|^2 d\nu \int_A w|g|^2 d\nu \\
&- 2\operatorname{Re}\left(\int_A wfgd\nu \overline{\int_B wfgd\nu}\right).
\end{aligned} \tag{5.8}$$

On the other hand, by the arithmetic mean-geometric mean inequality we have

$$\begin{aligned}
&\int_A w|f|^2 d\nu \int_B w|g|^2 d\nu + \int_B w|f|^2 d\nu \int_A w|g|^2 d\nu \\
&\geq 2\sqrt{\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu \int_B w|f|^2 d\nu \int_B w|g|^2 d\nu}.
\end{aligned} \tag{5.9}$$

By the CBS integral inequality and the properties of modulus we also have

$$\begin{aligned}(5.10)\quad & \sqrt{\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu \int_B w|f|^2 d\nu \int_B w|g|^2 d\nu} \\ & \geq \left|\int_A wfg d\nu\right| \left|\int_B wfg d\nu\right| = \left|\int_A wfg d\nu\right| \left|\int_B \overline{wfg} d\nu\right| \\ & = \left|\int_A wfg d\nu \int_B \overline{wfg} d\nu\right| \geq \operatorname{Re}\left(\int_A wfg d\nu \int_B \overline{wfg} d\nu\right).\end{aligned}$$

By (5.9) and (5.10) we have

$$\begin{aligned}(5.11)\quad & \int_A w|f|^2 d\nu \int_B w|g|^2 d\nu + \int_B w|f|^2 d\nu \int_A w|g|^2 d\nu \\ & - 2\operatorname{Re}\left(\int_A wfg d\nu \int_B \overline{wfg} d\nu\right) \\ & \geq 2\left(\sqrt{\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu \int_B w|f|^2 d\nu \int_B w|g|^2 d\nu}\right. \\ & \left. - \left|\int_A wfg d\nu\right| \left|\int_B wfg d\nu\right|\right) \\ & \geq 0.\end{aligned}$$

By the elementary inequality

$$(ab - cd)^2 \geq (a^2 - c^2)(b^2 - d^2), \ d, b, c, d \in \mathbb{R}$$

we have

$$\begin{aligned} & \left(\sqrt{\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu}\sqrt{\int_B w|f|^2 d\nu \int_B w|g|^2 d\nu} - \left|\int_A wfg d\nu\right| \left|\int_B wfg d\nu\right|\right)^2 \\ & \geq \left(\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu - \left|\int_A wfg d\nu\right|^2\right) \\ & \times \left(\int_B w|f|^2 d\nu \int_B w|g|^2 d\nu - \left|\int_B wfg d\nu\right|^2\right)\end{aligned}$$

and since the term in the left bracket is nonnegative, by taking the square root we get

$$\begin{aligned}(5.12)\quad & \sqrt{\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu}\sqrt{\int_B w|f|^2 d\nu \int_B w|g|^2 d\nu} \\ & - \left|\int_A wfg d\nu\right| \left|\int_B wfg d\nu\right| \\ & \geq \left(\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu - \left|\int_A wfg d\nu\right|^2\right)^{1/2} \\ & \times \left(\int_B w|f|^2 d\nu \int_B w|g|^2 d\nu - \left|\int_B wfg d\nu\right|^2\right)^{1/2} \\ & = Q(A, w; f, g) Q(B, w; f, g).\end{aligned}$$

Finally, by (5.8), (5.11) and (5.12) we have

$$\begin{aligned} Q^2(A\cup B,w;f,g) &\geq Q^2(A,w;f,g)+Q^2(B,w;f,g)+2Q(A,w;f,g)\,Q(B,w;f,g) \\ &= \left(Q(A,w;f,g)+Q(B,w;f,g)\right)^2 \end{aligned}$$

and the superadditivity of the mapping $Q(\cdot,w;f,g)$ is proved. □

For some CBS's inequality related functionals and their properties see [3], [5], [13], [14] and [15].

Let $x=(x_i)_{i\in\mathbb{N}}$ and $x=(y_i)_{i\in\mathbb{N}}$ be sequences of complex numbers and $w=(w_i)_{i\in\mathbb{N}}$ a sequence of positive numbers. Let $\Omega=\mathbb{N}$ and $\mathcal{P}_f(\mathbb{N})$ be the algebra of finite parts of natural numbers $\mathbb{N}$. By the monotonicity property of supermeasure on $\mathcal{P}_f(\mathbb{N})$ we have from the above results that the sequences

$$\begin{aligned} &H_n(w;x,y) \\ &:= \left(\sum_{i=0}^{n} w_i|x_i|^2\right)^{1/2}\left(\sum_{i=0}^{n} w_i|y_i|^2\right)^{1/2} - \left|\sum_{i=0}^{n} w_i x_i y_i\right| \end{aligned} \tag{5.13}$$

and

$$L_n(w;x,y) := \sum_{i=0}^{n} w_i|x_i|^2 \sum_{i=0}^{n} w_i|y_i|^2 - \left|\sum_{i=0}^{n} w_i x_i y_i\right|^2 \tag{5.14}$$

are monotonic nondecreasing and

$$\begin{aligned} &H_n(w;x,y) \\ &\geq \max_{0\leq i\neq j\leq n}\left[\left(w_i|x_i|^2+w_j|x_j|^2\right)^{1/2}\left(w_i|y_i|^2+w_j|y_j|^2\right)^{1/2} - |w_i x_i y_i + w_j x_j y_j|\right] \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} &H_n(w;x,y) \\ &\geq \max_{0\leq i\neq j\leq n}\left[\left(w_i|x_i|^2+w_j|x_j|^2\right)\left(w_i|y_i|^2+w_j|y_j|^2\right) - |w_i x_i y_i + w_j x_j y_j|^2\right] \end{aligned} \tag{5.16}$$

for any $p\geq 1$.

Finally, the sequence

$$Q_n(w;x,y) := \left[\sum_{i=0}^{n} w_i|x_i|^2 \sum_{i=0}^{n} w_i|y_i|^2 - \left|\sum_{i=0}^{n} w_i x_i y_i\right|^2\right]^{1/2} \tag{5.17}$$

is also monotonic nondecreasing and we have the bound

$$\begin{aligned} &Q_n(w;x,y) \\ &\geq \max_{0\leq i\neq j\leq n}\left[\left(w_i|x_i|^2+w_j|x_j|^2\right)\left(w_i|y_i|^2+w_j|y_j|^2\right) - |w_i x_i y_i + w_j x_j y_j|^2\right]^{1/2}. \end{aligned} \tag{5.18}$$

6. The Case of Čebyšev's Inequality

We say that the pair of measurable functions (f, g) are *synchronous* on Ω if

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \tag{6.1}$$

for ν-a.e. $x, y \in \Omega$. If the inequality reverses in (6.1), the functions are called *asynchronous* on Ω.

If (f, g) are synchronous on Ω and f, g, $fg \in L_w(\Omega, \nu)$ then the following inequality, that is known in the literature as *Čebyšev's Inequality,* holds

$$\int_\Omega w d\nu \int_\Omega wfg d\nu \geq \int_\Omega wf d\nu \int_\Omega wg d\nu, \tag{6.2}$$

where $w(x) \geq 0$ for ν -a.e. (almost every) $x \in \Omega$.

We consider the *Čebyšev functional* $C(\cdot, w; f, g) : \mathcal{A}_\nu \to [0, \infty)$ defined by

$$C(A, w; f, g) := \int_A w d\nu \int_A wfg d\nu - \int_A wf d\nu \int_A wg d\nu. \tag{6.3}$$

The following result is known in the literature as *Korkine's identity:*

$$\begin{aligned} & C(A, w; f, g) \\ & = \frac{1}{2} \int_A \int_A w(x) w(y) (f(x) - f(y)) (g(x) - g(y)) d\nu(x) d\nu(y). \end{aligned} \tag{6.4}$$

The proof is obvious by developing the right side of (6.4) and using Fubini's theorem.

Theorem 7. *Let (f, g) be synchronous on Ω and f, g, $fg \in L_w(\Omega, \nu)$. Then the Čebyšev functional defined by (6.3) is a supermeasure on $\mathcal{A}_\nu$.*

Proof. Let $A, B \in \mathcal{A}_\nu$ with $A \cap B = \emptyset$. Then by (6.4) we have

$$\begin{aligned} & C(A \cup B, w; f, g) \\ & = \frac{1}{2} \int_{A \cup B} \int_{A \cup B} w(x) w(y) (f(x) - f(y)) (g(x) - g(y)) d\nu(x) d\nu(y). \end{aligned}$$

Since

$$(A \cup B) \times (A \cup B) = (A \times A) \cup (B \times A) \cup (A \times B) \cup (B \times A)$$

then

$$\int_{A \cup B} \int_{A \cup B} = \int_A \int_A + \int_B \int_A + \int_A \int_B + \int_B \int_B.$$

Therefore

$$\begin{aligned} & C(A \cup B, w; f, g) \\ & = C(A, w; f, g) + C(B, w; f, g) \\ & + \int_A \int_B w(x) w(y) (f(x) - f(y)) (g(x) - g(y)) d\nu(x) d\nu(y) \end{aligned} \tag{6.5}$$

since by symmetry reasons

$$\begin{aligned} & \int_A \int_B w(x) w(y) (f(x) - f(y)) (g(x) - g(y)) d\nu(x) d\nu(y) \\ & = \int_B \int_A w(x) w(y) (f(x) - f(y)) (g(x) - g(y)) d\nu(x) d\nu(y). \end{aligned}$$

Now, since (f,g) are synchronous on Ω, then

$$\int_A \int_B w(x)\, w(y)\,(f(x)-f(y))\,(g(x)-g(y))\, d\nu(x)\, d\nu(y) \geq 0$$

and by (6.5) we get

$$C(A\cup B, w; f,g) \geq C(A,w;f,g) + C(B,w;f,g)$$

that proves the statement. □

For some Čebyšev's inequality related functionals and their properties see [3] and [16].

Let $x=(x_i)_{i\in\mathbb{N}}$ and $x=(y_i)_{i\in\mathbb{N}}$ be synchronous sequences of real numbers and $w=(w_i)_{i\in\mathbb{N}}$ a sequence of positive real numbers. Let $\Omega=\mathbb{N}$ and $\mathcal{P}_f(\mathbb{N})$ be the algebra of finite parts of natural numbers $\mathbb{N}$. By the monotonicity property of supermeasure on $\mathcal{P}_f(\mathbb{N})$ we have from the above results that the sequence

$$C_n(w;x,y) := \sum_{i=0}^{n} w_i \sum_{i=0}^{n} w_i x_i y_i - \sum_{i=0}^{n} w_i x_i \sum_{i=0}^{n} w_i y_i \geq 0$$

is monotonic nondecreasing and we have the bound

$$C_n(w;x,y) \geq \frac{1}{2} \max_{0\leq i\neq j\leq n} \left\{ w_i w_j (x_i - x_j)(y_i - y_j) \right\}.$$

7. The Case of Hermite-Hadamard Inequalities

Let I be an interval consisting of more than one point and $f: I \to \mathbb{R}$ a convex function. If $a,b\in I$ with $a<b$, then we have the well-known *Hermite-Hadamard inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}. \tag{7.1}$$

For some classical results on Hermite-Hadamard inequality see the monograph on-line [18].

Suppose $f: I\to\mathbb{R}$ and for $f\in L[a,b]$ define the functionals

$$H([a,b];f) := \int_a^b f(t)\,dt - (b-a)\, f\left(\frac{a+b}{2}\right)$$

and

$$L([a,b];f) := \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt.$$

We have the following result concerning the properties of these mappings as functions of interval [17]:

Theorem 8. *Let $f: I\to\mathbb{R}$ be a convex function. Then*

(i) For all $a,b,c\in I$ with $a\leq c\leq b$, we have

$$0 \leq H([a,c];f) + H([c,b];f) \leq H([a,b];f) \tag{7.2}$$

and

$$0 \leq L([a,c];f) + L([c,b];f) \leq L([a,b];f), \tag{7.3}$$

i.e. the functionals $H(\cdot;f)$ and $L(\cdot;f)$ are superadditive as functions of interval;

(ii) For all $[c,d]\subseteq[a,b]\subseteq I$*, we have*

$$0\leq H([c,d];f)\leq H([a,b];f) \tag{7.4}$$

and

$$0\leq L([c,d];f)\leq L([a,b];f), \tag{7.5}$$

i.e. the functionals $H(\cdot;f)$ *and* $L(\cdot;f)$ *are monotonic nondecreasing as functions of interval.*

Proof. (i) Let $c\in[a,b]$ and put $\alpha:=(c-a)/(b-a)$, $\beta:=(b-c)/(b-a)$. We have $\alpha+\beta=1$ with $\alpha,\beta\geq 0$ and by the convexity of f we have with $x=(a+c)/2$, $y=(b+c)/2\in I$ that

$$\begin{aligned}&\frac{c-a}{b-a}f\left(\frac{a+c}{2}\right)+\frac{b-c}{b-a}f\left(\frac{b+c}{2}\right)\\&=\alpha f(x)+\beta f(y)\geq f(\alpha x+\beta y)\\&=f\left(\frac{c-a}{b-a}\cdot\frac{a+c}{2}+\frac{b-c}{b-a}\cdot\frac{b+c}{2}\right)=f\left(\frac{a+b}{2}\right).\end{aligned}$$

Hence

$$\begin{aligned}&H([a,b];f)-H([a,c];f)-H([c,b];f)\\&=(c-a)f\left(\frac{a+c}{2}\right)+(b-c)f\left(\frac{b+c}{2}\right)-(b-a)f\left(\frac{a+b}{2}\right)\geq 0\end{aligned}$$

and the second part of (7.2) is proved.

Further, since f is convex on $[a,b]$, we have for all $c\in[a,b]$ that

$$\det\begin{bmatrix}1&1&1\\a&c&b\\f(a)&f(c)&f(b)\end{bmatrix}\geq 0,$$

that is

$$f(a)(b-c)-f(c)(b-a)+f(b)(c-a)\geq 0.$$

Because

$$\begin{aligned}&L([a,b];f)-L([a,c];f)-L([c,b];f)\\&=\frac{f(a)+f(b)}{2}(b-a)-\frac{f(a)+f(c)}{2}(c-a)-\frac{f(c)+f(b)}{2}(b-c)\\&=\frac{1}{2}[f(a)(b-c)-f(c)(b-a)+f(b)(c-a)]\end{aligned}$$

we have therefore that the second part of (7.3) holds also.

The first parts of (7.2) and (7.3) are obvious by (7.1) inequality.

(ii) Follows by (i) and we omit the details. □

For an arbitrary function $f:I\to\mathbb{R}$ we introduce the mapping

$$S([a,b];f):=(b-a)\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]$$

where $a,b\in I$ and $a<b$.

We have [17]:

Theorem 9. *Let $f : I \to \mathbb{R}$ a convex function. Then*
(i) For all $a, b, c \in I$ with $a \le c \le b$, we have

$$0 \le S([a,c];f) + S([c,b];f) \le S([a,b];f) \tag{7.6}$$

i.e. the functional $S(\cdot;f)$ is superadditive as function of interval;
(ii) For all $[c,d] \subseteq [a,b] \subseteq I$, we have

$$0 \le S([c,d];f) \le S([a,b];f) \tag{7.7}$$

i.e. the functional $H(\cdot;f)$ is monotonic nondecreasing as function of interval.

The proof is immediate from Theorem 8 observing that

$$S([a,b];f) = H([a,b];f) + L([a,b];f).$$

8. The Case of Convex Functions Defined on Intervals

Consider a convex function $f : I \subset \mathbb{R} \to \mathbb{R}$ defined on the interval I of the real line $\mathbb{R}$ and two distinct numbers $a, b \in I$ with $a < b$. We denote by $[a,b]$ the closed segment defined by $\{(1-t)a + tb,\ t \in [0,1]\}$. We also define the functional of interval

$$\Psi_f([a,b];t) := (1-t) f(a) + t f(b) - f((1-t)a + tb) \ge 0 \tag{8.1}$$

where $a, b \in I$ with $a < b$ and $t \in [0,1]$.

We have [9]:

Theorem 10. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I. Then for each $a, b \in I$ with $a < b$ and $c \in [a,b]$ we have*

$$(0 \le)\, \Psi_f([a,c];t) + \Psi_f([c,b];t) \le \Psi_f([a,b];t) \tag{8.2}$$

for each $t \in [0,1]$, i.e., the functional $\Psi_f(\cdot;t)$ is superadditive as a function of interval.

If $[c,d] \subset [a,b]$, then

$$(0 \le)\, \Psi_f([c,d];t) \le \Psi_f([a,b];t) \tag{8.3}$$

for each $t \in [0,1]$, i.e., the functional $\Psi_f(\cdot;t)$ is nondecreasing as a function of interval.

Proof. Let $c = (1-s)a + sb$ with $s \in (0,1)$. For $t \in (0,1)$ we have

$$\Psi_f([c,b];t) = (1-t) f((1-s)a + sb) + t f(b) - f((1-t)[(1-s)a + sb] + tb)$$

and

$$\Psi_f([a,c];t) = (1-t) f(a) + t f((1-s)a + sb) - f((1-t)a + t[(1-s)a + sb])$$

giving that

$$\begin{aligned} &\Psi_f([a,c];t) + \Psi_f([c,b];t) - \Psi_f([a,b];t) \\ &\quad = f((1-s)a + sb) + f((1-t)a + tb) \\ &\quad - f((1-t)(1-s)a + [(1-t)s + t]b) - f((1-ts)a + tsb). \end{aligned} \tag{8.4}$$

Now, for a convex function $\varphi : I \subset \mathbb{R} \to \mathbb{R}$, where I is an interval, and any real numbers t_1, t_2, s_1 and s_2 from I and with the properties that $t_1 \le s_1$ and $t_2 \le s_2$ we have that

$$\frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2} \le \frac{\varphi(s_1) - \varphi(s_2)}{s_1 - s_2}. \tag{8.5}$$

Indeed, since φ is convex on I then for any $a \in I$ the function $\psi : I \setminus \{a\} \to \mathbb{R}$

$$\psi(t) := \frac{\varphi(t) - \varphi(a)}{t - a}$$

is monotonic nondecreasing where is defined. Utilising this property repeatedly we have

$$\begin{aligned}\frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2} &\leq \frac{\varphi(s_1) - \varphi(t_2)}{s_1 - t_2} = \frac{\varphi(t_2) - \varphi(s_1)}{t_2 - s_1} \\ &\leq \frac{\varphi(s_2) - \varphi(s_1)}{s_2 - s_1} = \frac{\varphi(s_1) - \varphi(s_2)}{s_1 - s_2}\end{aligned}$$

which proves the inequality (8.5).

Consider the function $\varphi : [0,1] \to \mathbb{R}$ given by $\varphi(t) := f((1-t)a + tb)$. Since f is convex on I it follows that φ is convex on $[0,1]$. Now, if we consider for given $t, s \in (0,1)$

$$t_1 := ts < s =: s_1 \text{ and } t_2 := t < t + (1-t)s =: s_2,$$

then we have

$$\varphi(t_1) = f((1-ts)a + tsb), \varphi(t_2) = f((1-t)a + tb)$$

giving that

$$\frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2} = \frac{f((1-ts)a + tsb) - f((1-t)a + tb)}{t(s-1)}.$$

Also

$$\varphi(s_1) = f((1-s)a + sb), \varphi(s_2) = f((1-t)(1-s)a + [(1-t)s + t]b)$$

giving that

$$\begin{aligned}&\frac{\varphi(s_1) - \varphi(s_2)}{s_1 - s_2} \\ &= \frac{f((1-s)a + sb) - f((1-t)(1-s)a + [(1-t)s + t]b)}{t(s-1)}.\end{aligned}$$

Utilising the inequality (8.5) and multiplying with $t(s-1) < 0$ we deduce the inequality

$$\begin{aligned}f((1-ts)a + tsb) &- f((1-t)a + tb) \\ &\geq f((1-s)a + sb) - f((1-t)(1-s)a + [(1-t)s + t]b).\end{aligned} \tag{8.6}$$

Finally, by (8.4) and (8.6) we get the desired result (8.2).

Applying repeatedly the superadditivity property we have for $[c,d] \subset [a,b]$ that

$$\Psi_f([a,c];t) + \Psi_f([c,d];t) + \Psi_f([d,b];t) \leq \Psi_f([a,b];t)$$

giving that

$$0 \leq \Psi_f([a,c];t) + \Psi_f([d,b];t) \leq \Psi_f([a,b];t) - \Psi_f([c,d];t)$$

which proves (8.3). □

For $t = \frac{1}{2}$ we consider the functional

$$\Psi_f([a,b]) := \Psi_f\left([a,b]; \frac{1}{2}\right) = \frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right),$$

which obviously inherits the superadditivity and monotonicity properties of the functional $\Psi_f(\cdot,\cdot;t)$. We are able then to state the following

Corollary 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I and $a, b \in I$. Then we have the bounds*

$$\inf_{c\in[a,b]}\left[f\left(\frac{a+c}{2}\right)+f\left(\frac{c+b}{2}\right)-f(c)\right]=f\left(\frac{a+b}{2}\right) \tag{8.7}$$

and

$$\sup_{c,d\in[a,b]}\left[\frac{f(c)+f(d)}{2}-f\left(\frac{c+d}{2}\right)\right]=\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right). \tag{8.8}$$

Proof. By the superadditivity of the functional $\Psi_f(\cdot,\cdot)$ we have for each $c \in [a,b]$ that

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\\&\geq\frac{f(a)+f(c)}{2}-f\left(\frac{a+c}{2}\right)+\frac{f(c)+f(b)}{2}-f\left(\frac{c+b}{2}\right),\end{aligned}$$

which is equivalent to

$$f\left(\frac{a+c}{2}\right)+f\left(\frac{c+b}{2}\right)-f(c)\geq f\left(\frac{a+b}{2}\right). \tag{8.9}$$

Since the equality case in (8.9) is realized for either $c = a$ or $c = b$ we get the desired bound (8.7).

The bound (8.8) is obvious by the monotonicity of the functional $\Psi_f(\cdot)$ as a function of interval. □

Consider now the following functional

$$\Gamma_f([a,b];t):=f(a)+f(b)-f((1-t)a+tb)-f((1-t)b+ta),$$

where, as above, $f : I \subset a \to \mathbb{R}$ is a convex function on the interval I and $a, b \in I$ with $a < b$ while $t \in [0,1]$.

We notice that

$$\Gamma_f([a,b];t)=\Gamma_f([a,b];1-t)$$

and

$$\Gamma_f([a,b];t)=\Psi_f([a,b];t)+\Psi_f([a,b];1-t)\geq 0$$

for any $a, b \in I$ with $a < b$ and $t \in [0,1]$.

Therefore, we can state the following result as well:

Corollary 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I and $t \in [0,1]$. The functional $\Gamma_f(\cdot;t)$ is superadditive and monotonic nondecreasing as a function of interval.*

In particular, if $c \in [a,b]$ then we have the inequality

$$\begin{aligned}&\frac{1}{2}\left[f((1-t)a+tb)+f((1-t)b+ta)\right]\\&\leq\frac{1}{2}\left[f((1-t)a+tc)+f((1-t)c+ta)\right]\\&+\frac{1}{2}\left[f((1-t)c+tb)+f((1-t)b+tc)\right]-f(c)\end{aligned} \tag{8.10}$$

Also, if $c,d\in[a,b]$ then we have the inequality

$$\begin{aligned}(8.11)\qquad & f(a)+f(b)-f((1-t)a+tb)-f((1-t)b+ta)\\ &\geq f(c)+f(d)-f((1-t)c+td)-f((1-t)c+td)\end{aligned}$$

for any $t\in[0,1]$.

Perhaps the most interesting functional we can consider from the above is the following one:

$$\begin{aligned}(8.12)\qquad \Theta_f([a,b]) &:= \frac{f(a)+f(b)}{2}-\int_0^1 f((1-t)a+tb)\,dt\\ &= \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(s)\,ds\geq 0,\end{aligned}$$

which is related to the second Hermite-Hadamard inequality.

We observe that

$$(8.13)\qquad \Theta_f([a,b])=\int_0^1\Psi_f([a,b];t)\,dt=\int_0^1\Psi_f([a,b];1-t)\,dt.$$

Utilising this representation, we can state the following result as well:

Corollary 4. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a convex function on the interval I and $t\in[0,1]$. The functional $\Theta_f(\cdot,\cdot)$ is superadditive and monotonic nondecreasing as a function of interval. Moreover, we have the bounds*

$$(8.14)\qquad \inf_{c\in[a,b]}\left[\frac{1}{c-a}\int_a^c f(s)\,ds+\frac{1}{b-c}\int_c^b f(s)\,ds-f(c)\right]=\frac{1}{b-a}\int_a^b f(s)\,ds$$

and

$$\begin{aligned}(8.15)\qquad &\sup_{c,d\in[a,b]}\left[\frac{f(c)+f(d)}{2}-\frac{1}{c-d}\int_d^c f(s)\,ds\right]\\ &=\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(s)\,ds.\end{aligned}$$

For extension of this section's results in the case of convex functions defined on intervals incorporated in convex sets in linear spaces, see [9].

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