Received 03/10/15, Revised 07/06/16

REFINEMENTS AND REVERSES OF HÖLDER-MCCARTHY OPERATOR INEQUALITY

S. S. DRAGOMIR[1,2]

Abstract. In this paper we obtain some new refinements and reverses of Hölder-McCarthy inequality for positive operators on Hilbert spaces.

1. Introduction

Let A be a nonnegative operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$, namely $\langle Ax, x \rangle \geq 0$ for any $x \in H$. We write this as $A \geq 0$.

By the use of the spectral resolution of A and the Hölder inequality, C. A. McCarthy [14] proved that

$$\langle Ax, x \rangle^p \leq \langle A^p x, x \rangle, \ p \in (1, \infty) \tag{1.1}$$

and

$$\langle A^p x, x \rangle \leq \langle Ax, x \rangle^p, \ p \in (0, 1) \tag{1.2}$$

for any $x \in H$ with $\|x\| = 1$.

Let A be a selfadjoint operator on H with

$$mI \leq A \leq MI, \tag{1.3}$$

where I is the *identity operator* and m, M are real numbers with $m < M$.

In [6, Theorem 3] Fujii et al. obtained the following interesting ratio inequality that provides a reverse of the *Hölder-McCarthy inequality* (1.1) for an operator A that satisfy the condition (1.3) with $m > 0$

$$\langle A^p x, x \rangle \leq \left\{ \frac{1}{p^{1/p} q^{1/q}} \frac{M^p - m^p}{(M-m)^{1/p} (mM^p - Mm^p)^{1/q}} \right\}^p \langle Ax, x \rangle^p, \tag{1.4}$$

for any $x \in H$ with $\|x\| = 1$, where $q = p/(p-1)$, $p > 1$.

If A satisfies the condition (1.3) with $m \geq 0$, then we also have the additive reverse of (1.1) that has been obtained by the author in 2008, see [3]

$$\begin{aligned} 0 &\leq \langle A^p x, x \rangle - \langle Ax, x \rangle^p \\ &\leq p \begin{cases} \frac{1}{2}(M-m)\left[\left\|A^{p-1}x\right\|^2 - \left\langle A^{p-1}x, x\right\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}\left(M^{p-1} - m^{p-1}\right)\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \end{cases} \\ &\leq \frac{1}{4} p (M-m)\left(M^{p-1} - m^{p-1}\right), \end{aligned} \tag{1.5}$$

1991 *Mathematics Subject Classification.* 26D15; 26D10, 47A63, 47A30, 15A60.

Key words and phrases. Young's inequality, Hölder-McCarthy operator inequality, Arithmetic mean-Geometric mean inequality.

for any $x \in H$ with $\|x\| = 1$, where $p > 1$.

We also have [3]

$$0 \leq \langle A^p x, x \rangle - \langle Ax, x \rangle^p \tag{1.6}$$
$$\leq p \begin{cases} \frac{1}{4} \frac{(M-m)\left(M^{p-1}-m^{p-1}\right)}{M^{p/2}m^{p/2}} \langle Ax, x \rangle \left\langle A^{p-1}x, x \right\rangle, \text{ (for } m > 0), \\ \left(\sqrt{M} - \sqrt{m}\right) \left(M^{(p-1)/2} - m^{(p-1)/2}\right) \left[\langle Ax, x \rangle \left\langle A^{p-1}x, x \right\rangle\right]^{\frac{1}{2}}, \end{cases}$$
$$\leq p \begin{cases} \frac{1}{4} (M-m) \left(M^{p-1} - m^{p-1}\right) \left(\frac{M}{m}\right)^{p/2}, \text{ (for } m > 0), \\ \left(\sqrt{M} - \sqrt{m}\right) \left(M^{(p-1)/2} - m^{(p-1)/2}\right) M^{p/2}, \end{cases}$$

for any $x \in H$ with $\|x\| = 1$, where $p > 1$.

For various related inequalities, see [5]-[9] and [12]-[13].

By the use of some new reverses and refinements of Young's inequality from [1]-[2], [10]-[11], [15] and [17] we obtain in this paper some new refinements and reverses of Hölder-McCarthy operator inequalities (1.1) and (1.2).

2. Some Refinements and Reverse Results

We have:

Theorem 1. *Let A be a positive operator, namely $\langle Ax, x \rangle > 0$ for any $x \in H$, $x \neq 0$ and $p \in (0,1) \setminus \left\{\frac{1}{2}\right\}$. Then for any $x \in H$ with $\|x\| = 1$, we have*

$$\begin{aligned} &2r \langle Ax, x \rangle^{p-\frac{1}{2}} \left(\langle Ax, x \rangle^{1/2} - \left\langle A^{1/2}x, x \right\rangle\right) \\ &\leq \langle Ax, x \rangle^p - \langle A^p x, x \rangle \\ &\leq 2R \langle Ax, x \rangle^{p-\frac{1}{2}} \left(\langle Ax, x \rangle^{1/2} - \left\langle A^{1/2}x, x \right\rangle\right), \end{aligned} \tag{2.1}$$

where $r := \min\{1-p, p\}$ and $R := \max\{1-p, p\}$.

Proof. We use the following double inequality obtained by Kittaneh and Manasrah [10], [11] that provide a refinement and an additive reverse for Young's inequality as follows:

$$r\left(\sqrt{a} - \sqrt{b}\right)^2 \leq (1-p)\,a + pb - a^{1-p}b^p \leq R\left(\sqrt{a} - \sqrt{b}\right)^2 \tag{2.2}$$

where a, $b \geq 0$, $p \in [0,1]$, $r = \min\{1-p, p\}$ and $R = \max\{1-p, p\}$.

This is equivalent to

$$r\left(a - 2\sqrt{a}\sqrt{b} + b\right) \leq (1-p)\,a + pb - a^{1-p}b^p \leq R\left(a - 2\sqrt{a}\sqrt{b} + b\right) \tag{2.3}$$

for any a, $b \geq 0$, $p \in [0,1]$.

Fix $a \geq 0$ and by using the continuous functional calculus, we have by replacing b with the operator $A \geq 0$ that

$$\begin{aligned} r\left(aI - 2\sqrt{a}A^{1/2} + A\right) &\leq (1-p)\,aI + pA - a^{1-p}A^p \\ &\leq R\left(aI - 2\sqrt{a}A^{1/2} + A\right). \end{aligned} \tag{2.4}$$

Therefore, by (2.4) we have

$$r\left(a-2\sqrt{a}\left\langle A^{1/2}x,x\right\rangle+\langle Ax,x\rangle\right)\leq(1-p)\,a+p\,\langle Ax,x\rangle-a^{1-p}\,\langle A^{p}x,x\rangle \leq R\left(a-2\sqrt{a}\left\langle A^{1/2}x,x\right\rangle+\langle Ax,x\rangle\right) \tag{2.5}$$

for any $x\in H$ with $\|x\|=1$ and any $a\geq 0$.

Now, if we take $a=\langle Ax,x\rangle$ with $x\in H$ and $\|x\|=1$, then by (2.5) we get

$$2r\,\langle Ax,x\rangle^{1/2}\left(\langle Ax,x\rangle^{1/2}-\left\langle A^{1/2}x,x\right\rangle\right)\leq\langle Ax,x\rangle^{1-p}\left(\langle Ax,x\rangle^{p}-\langle A^{p}x,x\rangle\right) \leq 2R\,\langle Ax,x\rangle^{1/2}\left(\langle Ax,x\rangle^{1/2}-\left\langle A^{1/2}x,x\right\rangle\right)$$

and, by dividing with $\langle Ax,x\rangle^{1-p}>0$, we get the desired result (2.1). □

Remark 1. *Let A be a positive operator with $0<mI\leq A\leq MI$.*

(i) If $p\in\left(\frac{1}{2},1\right)$, then by (2.1) we have

$$\begin{aligned}&2\,(1-p)\,m^{p-\frac{1}{2}}\left(\langle Ax,x\rangle^{1/2}-\left\langle A^{1/2}x,x\right\rangle\right)\\&\leq 2\,(1-p)\,\langle Ax,x\rangle^{p-\frac{1}{2}}\left(\langle Ax,x\rangle^{1/2}-\left\langle A^{1/2}x,x\right\rangle\right)\\&\leq\langle Ax,x\rangle^{p}-\langle A^{p}x,x\rangle\\&\leq 2p\,\langle Ax,x\rangle^{p-\frac{1}{2}}\left(\langle Ax,x\rangle^{1/2}-\left\langle A^{1/2}x,x\right\rangle\right)\\&\leq 2pM^{p-\frac{1}{2}}\left(\langle Ax,x\rangle^{1/2}-\left\langle A^{1/2}x,x\right\rangle\right)\end{aligned}\tag{2.6}$$

for any $x\in H$ with $\|x\|=1$.

(ii) If $p\in\left(0,\frac{1}{2}\right)$, then by (2.1) we have

$$\begin{aligned}&2pM^{p-\frac{1}{2}}\left(\langle Ax,x\rangle^{1/2}-\left\langle A^{1/2}x,x\right\rangle\right)\\&\leq 2p\,\langle Ax,x\rangle^{p-\frac{1}{2}}\left(\langle Ax,x\rangle^{1/2}-\left\langle A^{1/2}x,x\right\rangle\right)\\&\leq\langle Ax,x\rangle^{p}-\langle A^{p}x,x\rangle\\&\leq 2\,(1-p)\,\langle Ax,x\rangle^{p-\frac{1}{2}}\left(\langle Ax,x\rangle^{1/2}-\left\langle A^{1/2}x,x\right\rangle\right)\\&\leq 2\,(1-p)\,m^{p-\frac{1}{2}}\left(\langle Ax,x\rangle^{1/2}-\left\langle A^{1/2}x,x\right\rangle\right)\end{aligned}\tag{2.7}$$

for any $x\in H$ with $\|x\|=1$.

We recall that *Specht's ratio* is defined by

$$S\,(h):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h\in(0,1)\cup(1,\infty)\,,\\[2ex] 1 & \text{if } h=1.\end{cases}$$

It is well known that $\lim_{h\to 1}S\,(h)=1$, $S\,(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq 1$. The function S is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

Tominaga [17] had proved a reverse Young inequality with the Specht's ratio [16] as follows:

$$\left(a^{1-\nu}b^{\nu}\leq\right)(1-\nu)\,a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\tag{2.8}$$

for any a, $b > 0$ and $\nu \in [0,1]$.

Theorem 2. *Let A be a positive operator with $0 < mI \leq A \leq MI$.*

(i) If $p \in (0,1)$, then for any $x \in H$ with $\|x\| = 1$, we have

$$\langle Ax, x\rangle^p \leq S\left(\frac{M}{m}\right)\langle A^p x, x\rangle. \tag{2.9}$$

(ii) If $p \in (1,\infty)$, then for any $x \in H$ with $\|x\| = 1$, we have

$$\langle A^p x, x\rangle \leq S^p\left(\left(\frac{M}{m}\right)^p\right)\langle Ax, x\rangle^p. \tag{2.10}$$

Proof. (i) Assume that $p \in (0,1)$. Let a, $b \in [m, M] \subset (0,\infty)$, then $\frac{m}{M} \leq \frac{a}{b} \leq \frac{M}{m}$ with $\frac{m}{M} < 1 < \frac{M}{m}$. If $\frac{a}{b} \in \left[\frac{m}{M}, 1\right)$ then $S\left(\frac{a}{b}\right) \leq S\left(\frac{m}{M}\right) = S\left(\frac{M}{m}\right)$. If $\frac{a}{b} \in \left(1, \frac{M}{m}\right]$ then also $S\left(\frac{a}{b}\right) \leq S\left(\frac{M}{m}\right)$. Therefore for any a, $b \in [m, M]$ we have by (2.8) that

$$(1-p)\,a + pb \leq S\left(\frac{M}{m}\right)a^{1-p}b^p. \tag{2.11}$$

Using the functional calculus, we have from (2.11) that

$$(1-p)\,aI + pA \leq S\left(\frac{M}{m}\right)a^{1-p}A^p$$

for any $a \in [m, M]$ and $p \in (0,1)$, that is equivalent to

$$(1-p)\,a + p\langle Ax, x\rangle \leq S\left(\frac{M}{m}\right)a^{1-p}\langle A^p x, x\rangle \tag{2.12}$$

for any $x \in H$ with $\|x\| = 1$.

If $x \in H$ with $\|x\| = 1$ then $\langle Ax, x\rangle \in [m, M]$ and by taking $a = \langle Ax, x\rangle$ in (2.12) we get

$$\langle Ax, x\rangle \leq S\left(\frac{M}{m}\right)\langle Ax, x\rangle^{1-p}\langle A^p x, x\rangle$$

and by dividing with $\langle Ax, x\rangle^{1-p} > 0$ we deduce the desired result (2.9).

(ii) Assume that $p \in (1,\infty)$. Then $m^p I \leq A^p \leq M^p I$ and by applying (2.9) we have

$$\langle A^p x, x\rangle^{\frac{1}{p}} \leq S\left(\frac{M^p}{m^p}\right)\langle Ax, x\rangle \tag{2.13}$$

for any $x \in H$ with $\|x\| = 1$.

By taking the power $p > 1$ in (2.13) we get the desired result (2.10). □

Remark 2. *If we take $p = \frac{1}{2}$ in (2.9) then we get*

$$\langle Ax, x\rangle^{1/2} \leq S\left(\frac{M}{m}\right)\left\langle A^{1/2}x, x\right\rangle \tag{2.14}$$

for any $x \in H$ with $\|x\| = 1$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{2.15}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$(1-\nu)a+\nu b \le K^R\left(\frac{a}{b}\right)a^{1-\nu}b^\nu, \tag{2.16}$$

where a, $b>0$, $\nu\in[0,1]$ and $R=\max\{1-\nu,\nu\}$.

This inequality has been obtained by Liao et al. in [15].

Theorem 3. *Let A be a positive operator with $0<mI\le A\le MI$.*

(i) If $p\in(0,1)$, then for any $x\in H$ with $\|x\|=1$, we have

$$\langle Ax,x\rangle^p \le K^R\left(\frac{M}{m}\right)\langle A^p x,x\rangle, \tag{2.17}$$

where $R=\max\{1-p,p\}$.

(ii) If $p\in(1,\infty)$, then for any $x\in H$ with $\|x\|=1$, we have

$$\langle A^p x,x\rangle \le K^T\left(\left(\frac{M}{m}\right)^p\right)\langle Ax,x\rangle^p, \tag{2.18}$$

where $T=\max\{p-1,1\}$.

Proof. (i) Assume that $p\in(0,1)$ and put $R=\max\{1-p,p\}$. Let $a,b\in[m,M]\subset(0,\infty)$, then $\frac{m}{M}\le\frac{a}{b}\le\frac{M}{m}$ with $\frac{m}{M}<1<\frac{M}{m}$. If $\frac{a}{b}\in\left[\frac{m}{M},1\right)$ then $K^R\left(\frac{a}{b}\right)\le K^R\left(\frac{m}{M}\right)=K^R\left(\frac{M}{m}\right)$. If $\frac{a}{b}\in\left(1,\frac{M}{m}\right]$ then also $K^R\left(\frac{a}{b}\right)\le K^R\left(\frac{M}{m}\right)$. Therefore for any $a,b\in[m,M]$ we have by (2.16) that

$$(1-p)a+pb\le K^R\left(\frac{M}{m}\right)a^{1-p}b^p. \tag{2.19}$$

Now, on making use of a similar argument to the one from (i) in Theorem 2, we get (2.17).

(ii) Let $p\in(1,\infty)$. Then $\frac{1}{p}\in(0,1)$ and $\max\left\{1-\frac{1}{p},\frac{1}{p}\right\}=\frac{1}{p}\max\{p-1,1\}=\frac{1}{p}T$. Since $m^pI\le A^p\le M^pI$ and by applying (2.19) we have

$$\langle A^p x,x\rangle^{\frac{1}{p}}\le K^{\frac{1}{p}T}\left(\frac{M^p}{m^p}\right)\langle Ax,x\rangle \tag{2.20}$$

for any $x\in H$ with $\|x\|=1$.

Now, by taking the power $p>1$ in (2.20) we get the desired result (2.18). □

Remark 3. *If we take $p=\frac{1}{2}$ in (2.17), then we get*

$$\langle Ax,x\rangle^{1/2}\le K^{1/2}\left(\frac{M}{m}\right)\left\langle A^{1/2}x,x\right\rangle, \tag{2.21}$$

for any $x\in H$ with $\|x\|=1$.

3. Further Results

In the recent paper [1] we obtained the following reverses of Young's inequality as well:

$$0\le(1-\nu)a+\nu b-a^{1-\nu}b^\nu\le\nu(1-\nu)(a-b)(\ln a-\ln b) \tag{3.1}$$

and

$$1\le\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^\nu}\le\exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{3.2}$$

where $a,\ b>0,\ \nu\in[0,1]$.

It has been shown in [1] that there is no ordering for the upper bounds of the quantity $(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}$ as provided by the inequalities (2.2) and (3.1). The same conclusion is true for the upper bounds of the quantity $\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}$ incorporated in the inequalities (2.8), (2.16) and (3.2).

Theorem 4. *Let A be a positive operator and $p\in(0,1)$. Then for any $x\in H$ with $\|x\|=1$, we have*

$$(0\leq)\left\langle Ax,x\right\rangle^{p}-\left\langle A^{p}x,x\right\rangle \leq p\,(1-p)\left\langle Ax,x\right\rangle^{p-1}\left(\left\langle A\ln Ax,x\right\rangle-\left\langle Ax,x\right\rangle\left\langle \ln Ax,x\right\rangle\right). \tag{3.3}$$

In particular, we have

$$(0\leq)\left\langle Ax,x\right\rangle^{1/2}-\left\langle A^{1/2}x,x\right\rangle \leq\frac{1}{4}\left\langle Ax,x\right\rangle^{-1/2}\left(\left\langle A\ln Ax,x\right\rangle-\left\langle Ax,x\right\rangle\left\langle \ln Ax,x\right\rangle\right). \tag{3.4}$$

Proof. The inequality (3.1) can be written as

$$(1-p)\,a+pb-a^{1-p}b^{p}\leq p\,(1-p)\,(b\ln b+a\ln a-a\ln b-b\ln a) \tag{3.5}$$

for any $a,\ b>0$ and $p\in(0,1)$.

Using the functional calculus for continuous functions of selfadjoint operators, we have

$$(1-p)\,aI+pA-a^{1-p}A^{p}\leq p\,(1-p)\,(a\ln aI+A\ln A-a\ln A-\ln aA) \tag{3.6}$$

for any $a>0$ and $p\in(0,1)$.

Therefore, by (3.6) we have

$$(1-p)\,a+p\left\langle Ax,x\right\rangle-a^{1-p}\left\langle A^{p}x,x\right\rangle \leq p\,(1-p)\left(a\ln a+\left\langle A\ln Ax,x\right\rangle-a\left\langle \ln Ax,x\right\rangle-\ln a\left\langle Ax,x\right\rangle\right) \tag{3.7}$$

for any $a>0,\ p\in(0,1)$ and $x\in H$ with $\|x\|=1$.

Now, if we take in (3.7) $a=\left\langle Ax,x\right\rangle>0$ for $x\in H$ with $\|x\|=1$, then we get

$$\begin{aligned}&(1-p)\left\langle Ax,x\right\rangle+p\left\langle Ax,x\right\rangle-\left\langle Ax,x\right\rangle^{1-p}\left\langle A^{p}x,x\right\rangle\\ &\leq p\,(1-p)\left(\left\langle Ax,x\right\rangle\ln\left\langle Ax,x\right\rangle+\left\langle A\ln Ax,x\right\rangle-\left\langle Ax,x\right\rangle\left\langle \ln Ax,x\right\rangle-\left\langle Ax,x\right\rangle\ln\left\langle Ax,x\right\rangle\right),\end{aligned}$$

which is equivalent to

$$\left\langle Ax,x\right\rangle-\left\langle Ax,x\right\rangle^{1-p}\left\langle A^{p}x,x\right\rangle\leq p\,(1-p)\left(\left\langle A\ln Ax,x\right\rangle-\left\langle Ax,x\right\rangle\left\langle \ln Ax,x\right\rangle\right)$$

and the inequality (3.3) is proved. □

Remark 4. *The following Grüss' type inequality for two functions of selfadjoint operators is known (see for instance* [4, p. 79]*):*

Let $f,g:[m,M]\to\mathbb{R}$ be two continuous functions and A a selfadjoint operator with $\mathrm{Sp}\,(A)\subseteq[m,M]$. Then we have

$$\begin{aligned}&\left|\left\langle f\,(A)\,g\,(A)\,x,x\right\rangle-\left\langle f\,(A)\,x,x\right\rangle\left\langle g\,(A)\,x,x\right\rangle\right|\\ &\leq\frac{1}{2}\left(\max_{t\in[m,M]}f\,(t)-\min_{t\in[m,M]}f\,(t)\right)\left(\left\|g\,(A)\,x\right\|^{2}-\left\langle g\,(A)\,x,x\right\rangle^{2}\right)^{1/2}\\ &\leq\frac{1}{4}\left(\max_{t\in[m,M]}f\,(t)-\min_{t\in[m,M]}f\,(t)\right)\left(\max_{t\in[m,M]}g\,(t)-\min_{t\in[m,M]}g\,(t)\right)\end{aligned} \tag{3.8}$$

for any $x \in H$ *with* $\|x\| = 1$.

Let A *be a positive operator with* $0 < mI \leq A \leq MI$. *By applying the inequality (3.8) we obtain*

$$
\begin{aligned}
(3.9)\qquad 0 &\leq \langle A \ln Ax, x\rangle - \langle Ax, x\rangle \langle \ln Ax, x\rangle \\
&\leq \frac{1}{2} \begin{cases} (\ln M - \ln m) \left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)^{1/2} \\ \\ (M - m) \left(\|\ln Ax\|^2 - \langle \ln Ax, x\rangle^2\right)^{1/2} \end{cases} \\
&\leq \frac{1}{4} (M - m) (\ln M - \ln m)
\end{aligned}
$$

for any $x \in H$ *with* $\|x\| = 1$.

From (3.3) we then obtain

$$
\begin{aligned}
(3.10)\qquad (0 \leq)\ & \langle Ax, x\rangle - \langle Ax, x\rangle^{1-p} \langle A^p x, x\rangle \\
&\leq \frac{1}{2} p (1 - p) \begin{cases} (\ln M - \ln m) \left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)^{1/2} \\ \\ (M - m) \left(\|\ln Ax\|^2 - \langle \ln Ax, x\rangle^2\right)^{1/2} \end{cases} \\
&\leq \frac{1}{4} p (1 - p) (M - m) (\ln M - \ln m)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.11)\qquad (0 \leq)\ & \langle Ax, x\rangle^p - \langle A^p x, x\rangle \\
&\leq \frac{1}{2} p (1 - p)\, m^{p-1} \begin{cases} (\ln M - \ln m) \left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)^{1/2} \\ \\ (M - m) \left(\|\ln Ax\|^2 - \langle \ln Ax, x\rangle^2\right)^{1/2} \end{cases} \\
&\leq \frac{1}{4} p (1 - p)\, m^{p-1} (M - m) (\ln M - \ln m)
\end{aligned}
$$

for any $x \in H$ *with* $\|x\| = 1$.

We also have:

Theorem 5. *Let* A *be a positive operator with* $0 < mI \leq A \leq MI$.

(i) If $p \in (0, 1)$, *then for any* $x \in H$ *with* $\|x\| = 1$, *we have*

$$
(3.12)\qquad \langle Ax, x\rangle^p \leq \exp\left[4p (1 - p) \left(K\left(\frac{M}{m}\right) - 1\right)\right] \langle A^p x, x\rangle .
$$

(ii) If $p \in (1, \infty)$, *then for any* $x \in H$ *with* $\|x\| = 1$, *we have*

$$
(3.13)\qquad \langle A^p x, x\rangle \leq \exp\left[4\left(1 - \frac{1}{p}\right)\left(K\left(\left(\frac{M}{m}\right)^p\right) - 1\right)\right] \langle Ax, x\rangle^p .
$$

Proof. (i) For any $a, b \in [m, M]$ we have from (3.2) that

$$
(1 - p)\, a + pb \leq a^{1-p} b^p \exp\left[4p (1 - p) \left(K\left(\frac{M}{m}\right) - 1\right)\right],
$$

where a, $b > 0$, $p \in [0, 1]$.

Now, on making use of a similar argument to the one from (i) in Theorem 2, we get (3.12).

(ii) Let $p \in (1,\infty)$. Then $\frac{1}{p} \in (0,1)$ and since $m^p I \leq A^p \leq M^p I$ we have by (3.12) that

$$\langle A^p x, x\rangle^{\frac{1}{p}} \leq \exp\left[\frac{4}{p}\left(1-\frac{1}{p}\right)\left(K\left(\left(\frac{M}{m}\right)^p\right)-1\right)\right]\langle Ax, x\rangle, \tag{3.14}$$

for any $x \in H$ with $\|x\| = 1$.

Taking the power $p > 1$ in (3.14) yields the desired result (3.13). □

In [2] we obtained the following Young related inequalities:

For any $a, b > 0$ and $\nu \in [0,1]$ we have

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \min\{a,b\} &\leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \\ &\leq \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \max\{a,b\}\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max^2\{a,b\}}\right] &\leq \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min^2\{a,b\}}\right].\end{aligned} \tag{3.16}$$

For a positive operator A and vector $x \in H$ with $\|x\| = 1$, define

$$\ell(A;x) := \left\langle (\ln A)^2 x, x\right\rangle - 2\langle \ln Ax, x\rangle \ln\langle Ax, x\rangle + (\ln\langle Ax, x\rangle)^2 \geq 0.$$

Theorem 6. *Let A be a positive operator with $0 < mI \leq A \leq MI$. We have for $p \in (0,1)$ and for any $x \in H$ with $\|x\| = 1$ that*

$$\begin{aligned}\frac{1}{2}p(1-p)m\ell(A;x) &\leq \langle Ax, x\rangle - \langle Ax, x\rangle^{1-p}\langle A^p x, x\rangle \\ &\leq \frac{1}{2}p(1-p)M\ell(A;x)\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\frac{1}{2}p(1-p)mM^{p-1}\ell(A;x) &\leq \langle Ax, x\rangle^p - \langle A^p x, x\rangle \\ &\leq \frac{1}{2}p(1-p)Mm^{p-1}\ell(A;x).\end{aligned} \tag{3.18}$$

Proof. From the inequality (3.15) we have

$$\begin{aligned}\frac{1}{2}p(1-p)(\ln a - \ln b)^2 m &\leq (1-p)a + pb - a^{1-p}b^p \\ &\leq \frac{1}{2}p(1-p)(\ln a - \ln b)^2 M\end{aligned}$$

that is equivalent to

$$\begin{aligned}&\frac{1}{2}p(1-p)\left[(\ln b)^2-2\ln b\ln a+(\ln a)^2\right]m\\&\leq(1-p)a+pb-a^{1-p}b^p\\&\leq\frac{1}{2}p(1-p)\left[(\ln b)^2-2\ln b\ln a+(\ln a)^2\right]M\end{aligned}$$

for any $a,b\in[m,M]$.

Making use of an argument similar to that in the proof of Theorem 1 we get (3.17).

From (3.17) we have

$$\begin{aligned}\frac{1}{2}p(1-p)m\langle Ax,x\rangle^{p-1}\ell(A;x)&\leq\langle Ax,x\rangle^p-\langle A^px,x\rangle\\&\leq\frac{1}{2}p(1-p)\langle Ax,x\rangle^{p-1}M\ell(A;x)\end{aligned}\tag{3.19}$$

and since $M^{p-1}\leq\langle Ax,x\rangle^{p-1}\leq m^{p-1}$ for any $x\in H$ with $\|x\|=1$, hence (3.18) follows by (3.19). □

Remark 5. *If we take in Theorem 6* $p=\frac{1}{2}$, *then we get the inequalities*

$$\frac{1}{8}m\ell(A;x)\leq\langle Ax,x\rangle-\langle Ax,x\rangle^{-1/2}\left\langle A^{1/2}x,x\right\rangle\leq\frac{1}{8}M\ell(A;x)\tag{3.20}$$

and

$$\frac{1}{8}M^{-1/2}\ell(A;x)\leq\langle Ax,x\rangle^{1/2}-\left\langle A^{1/2}x,x\right\rangle\leq\frac{1}{8}Mm^{-1/2}\ell(A;x)\tag{3.21}$$

for any $x\in H$ *with* $\|x\|=1$.

Finally, we have:

Theorem 7. *Let* A *be a positive operator with* $0<mI\leq A\leq MI$.

(i) If $p\in(0,1)$, *then for any* $x\in H$ *with* $\|x\|=1$, *we have*

$$\langle Ax,x\rangle^p\leq\langle A^px,x\rangle\exp\left[\frac{1}{2}p(1-p)\left(\frac{M}{m}-1\right)^2\right].\tag{3.22}$$

(ii) If $p\in(1,\infty)$, *then for any* $x\in H$ *with* $\|x\|=1$, *we have*

$$\langle A^px,x\rangle\leq\langle Ax,x\rangle^p\exp\left[\frac{p-1}{2p}\left(\left(\frac{M}{m}\right)^p-1\right)^2\right].\tag{3.23}$$

Proof. Let $p\in(0,1)$. From (3.16) we have

$$(1-p)a+pb\leq a^{1-p}b^p\exp\left[\frac{1}{2}p(1-p)\frac{(b-a)^2}{\min^2\{a,b\}}\right],\tag{3.24}$$

for any $a,b>0$.

Since

$$\frac{(b-a)^2}{\min^2\{a,b\}}=\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2,$$

then we have

$$(1-p)a+pb\leq a^{1-p}b^p\exp\left[\frac{1}{2}p(1-p)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right],\tag{3.25}$$

for any $a,\, b > 0.$

If $a,\, b \in [m, M] \subset (0, \infty)$ and since

$$0 < \frac{\max\{a, b\}}{\min\{a, b\}} - 1 \leq \frac{M}{m} - 1,$$

hence

$$\left(\frac{\max\{a, b\}}{\min\{a, b\}} - 1\right)^2 \leq \left(\frac{M}{m} - 1\right)^2.$$

Therefore, by (3.25) we get

$$(1-p)\,a + pb \leq a^{1-p}b^p \exp\left[\frac{1}{2}p\,(1-p)\left(\frac{M}{m} - 1\right)^2\right], \tag{3.26}$$

for any $a,\, b \in [m, M]$ and $p \in (0, 1)$.

Now, on making use of a similar argument to the one from (i) in Theorem 2, we get (3.22).

(ii) Let $p \in (1, \infty)$. Then $\frac{1}{p} \in (0, 1)$ and since $m^p I \leq A^p \leq M^p I$ we have by (3.22) that

$$\langle A^p x, x\rangle^{\frac{1}{p}} \leq \langle Ax, x\rangle \exp\left[\frac{1}{2p}\left(1 - \frac{1}{p}\right)\left(\left(\frac{M}{m}\right)^p - 1\right)^2\right], \tag{3.27}$$

for any $x \in H$ with $\|x\| = 1$.

Taking the power $p > 1$ in (3.27) yields the desired result (3.23). □

Remark 6. *If we take $p = \frac{1}{2}$ in (3.22), then we get*

$$\langle Ax, x\rangle^{1/2} \leq \left\langle A^{1/2}x, x\right\rangle \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right], \tag{3.28}$$

for any $x \in H$ with $\|x\| = 1$.

If we take $p = 2$ in (3.23), then we get

$$\left\langle A^2 x, x\right\rangle \leq \langle Ax, x\rangle^2 \exp\left[\frac{1}{4}\left(\left(\frac{M}{m}\right)^2 - 1\right)^2\right], \tag{3.29}$$

for any $x \in H$ with $\|x\| = 1$.

References

[1] S. S. Dragomir, A note on Young's inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 126. [http://rgmia.org/papers/v18/v18a126.pdf].

[2] S. S. Dragomir, A note on new refinements and reverses of Young's inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. . [http://rgmia.org/papers/v1].

[3] S. S. Dragomir, Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces, *J. Inequal. & Appl.*, Volume **2010**, Article ID 496821, 15 pages doi:10.1155/2010/496821. Preprint RGMIA *Res. Rep. Coll.*, **11** (2008), Supliment. Art. 15. [Online http://rgmia.org/papers/v11e/RevJensenOp.pdf].

[4] S. S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type,* Springer Briefs in Mathematics, Springer, 2012.

[5] M. Fujii, S. Izumino and R. Nakamoto, Classes of operators determined by the Heinz-Kato-Furuta inequality and the Hölder-McCarthy inequality. *Nihonkai Math. J.* **5** (1994), no. 1, 61–67.

[6] M. Fuji, S. Izumino, R. Nakamoto and Y. Seo, Operator inequalities related to Cauchy-Schwarz and Hölder-McCarthy inequalities, *Nihonkai Math. J.*, **8** (1997), 117-122.

[7] T. Furuta, Extensions of Hölder-McCarthy and Kantorovich inequalities and their applications. *Proc. Japan Acad. Ser. A Math. Sci.* **73** (1997), no. 3, 38–41.

[8] T. Furuta, Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities. *J. Inequal. Appl.* **2** (1998), no. 2, 137–148.

[9] T. Furuta, The Hölder-McCarthy and the Young inequalities are equivalent for Hilbert space operators. *Amer. Math. Monthly* **108** (2001), no. 1, 68–69.

[10] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.* **361** (2010), 262-269.

[11] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra,* **59** (2011), 1031-1037.

[12] C.-S. Lin and Y. J. Cho, On Hölder-McCarthy-type inequalities with powers. *J. Korean Math. Soc.* **39** (2002), no. 3, 351–361.

[13] C.-S. Lin and Y. J. Cho, On Kantorovich inequality and Hölder-McCarthy inequalities. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **11** (2004), no. 4, 481–490.

[14] C. A. McCarthy, c_p, *Israel J. Math.* **5** (1967), 249–271.

[15] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.

[16] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.*, **74** (1960), pp. 91-98.

[17] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa