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# COMPOSITE SUB AND SUPERMEASURES WITH APPLICATIONS TO CLASSICAL INEQUALITIES

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Abstract. In this paper we provide some composite sub and supermeasures that are naturally associated to classical inequalities such as Jensen's inequality, Hölder's inequality, Minkowski's inequality, Cauchy-Bunyakovsky-Schwarz's Inequality, Čebyšev's inequality, Hermite-Hadamard's inequalities and the definition of convexity property. Some refinements of the above inequalities are also obtained.

## 1. Introduction

Let $\Omega$ be a nonempty set. A subset $\mathcal{A}$ of the power set $2^{\Omega}$ is called an *algebra* if the following conditions are satisfied:

(i) $\Omega$ is in $\mathcal{A}$;
(ii) $\mathcal{A}$ is closed under complementation, namely, if $A \in \mathcal{A}$ then $\Omega \setminus A \in \mathcal{A}$;
(iii) $\mathcal{A}$ is closed under union, i.e. if $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$.

By applying *de Morgan*'s laws it follows that $\mathcal{A}$ is closed under intersection, namely if $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$. It also follows that the empty set $\emptyset$ belongs to $\mathcal{A}$. Elements of the algebra are called measurable sets. An ordered pair $(\Omega, \mathcal{A})$, where $\Omega$ is a set and $\mathcal{A}$ is a algebra over $\Omega$, is called a *measurable space*.

The function $\mu : \mathcal{A} \to [0, \infty)$ is called a *measure* [*submeasure* (*supermeasure*)] on $\mathcal{A}$ if

(a) For all $A \in \mathcal{A}$ we have $\mu(A) \geq 0$ (nonnegativity);
(aa) We have $\mu(\emptyset) = 0$ (null empty set);
(aaa) For any $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$ we have

$$\mu(A \cup B) = [\leq (\geq)]\, \mu(A) + \mu(B), \tag{1.1}$$

i.e., $\mu$ is *additive* [*subadditive* (*superadditive*)] on $\mathcal{A}$.

For $\mu$ as above we denote by

$$\mathcal{A}_{\mu} := \{A \in \mathcal{A} \ |\mu(A) > 0\}.$$

If $\mathcal{A}_{\mu} = \mathcal{A} \setminus \{\emptyset\}$ then we say that $\mu$ is *positive* on $\mathcal{A}$.

Let $A, B \in \mathcal{A}$ with $A \subset B$, then $B = A \cup (B \setminus A)$, $A \cap (B \setminus A) = \emptyset$ and $B \setminus A \in \mathcal{A}$. If $\mu$ is additive or superadditive on $\mathcal{A}$, then

$$\mu(B) = \mu(A \cup (B \setminus A)) = (\geq)\, \mu(A) + \mu(B \setminus A) \geq \mu(A)$$

showing that $\mu$ is *monotonic nondecreasing* on $\mathcal{A}$.

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In this paper, by the use of some measures and *submeasures* (*supermeasures*) on $\mathcal{A}$ we show that we can construct some natural composite functionals that are in their turn *submeasures* (*supermeasures*) on $\mathcal{A}$. We also provide some examples of supermeasures that can be naturally associated to classical inequalities such as Jensen's inequality, Hölder's inequality, Minkowski's inequality, Cauchy-Bunyakovsky-Schwarz's inequality, Čebyšev's inequality, Hermite-Hadamard's inequalities and the definition of convexity property. As a consequence of monotonic nondecreasing property of these supermeasures, some refinements of the above inequalities are also obtained.

## 2. Some Composite Functionals

We have:

**Theorem 1.** *Let* $(\Omega, \mathcal{A})$ *be a measurable space and* $\mu : \mathcal{A} \to [0, \infty)$ *a positive measure on* $\mathcal{A} \setminus \{\emptyset\}$. *If* $\delta : \mathcal{A} \to [0, \infty)$ *is a supermeasure (submeasure) on* $\mathcal{A}$ *and* $p \geq 1$ $(0 < p < 1)$ *then the functional*

$$\eta_p : \mathcal{A} \to [0, \infty), \eta_p (A) = \delta (A) \mu^{1-\frac{1}{p}} (A) \tag{2.1}$$

*is also a supermeasure (submeasure) on* $\mathcal{A}$.

*Proof.* First, we observe that the following elementary inequality holds:

$$(\alpha + \beta)^p \geq (\leq) \alpha^p + \beta^p \tag{2.2}$$

for any $\alpha, \beta \geq 0$ and $p \geq 1$ $(0 < p < 1)$.

Indeed, if we consider the function

$$f_p : [0, \infty) \to \mathbb{R}, \ f_p (t) = (t + 1)^p - t^p,$$

then we have

$$f_p' (t) = p \left[ (t + 1)^{p-1} - t^{p-1} \right].$$

Observe that for $p > 1$ and $t > 0$ we have that $f_p' (t) > 0$ showing that $f_p$ is strictly increasing on the interval $[0, \infty)$. Now for $t = \frac{\alpha}{\beta}$ $(\beta > 0, \alpha \geq 0)$ we have $f_p (t) > f_p (0)$ giving that

$$\left( \frac{\alpha}{\beta} + 1 \right)^p - \left( \frac{\alpha}{\beta} \right)^p > 1,$$

i.e., the desired inequality (2.2).

For $p \in (0, 1)$ we have that $f_p$ is strictly decreasing on $[0, \infty)$, which proves the second case in (2.2).

Let $A, B \in \mathcal{A} \setminus \{\emptyset\}$ with $A \cap B = \emptyset$.

Now, if $\delta$ is superadditive (subadditive) and $p \geq 1$ $(0 < p < 1)$, then we have by (2.2) that

$$\delta^p (A \cup B) \geq (\leq) [\delta (A) + \delta (B)]^p \geq (\leq) \delta^p (A) + \delta^p (B). \tag{2.3}$$

Utilising (2.3) and the additivity property of $\mu$ we have that

$$\frac{\delta^p(A\cup B)}{\mu(A\cup B)} \geq (\leq) \frac{\delta^p(A)+\delta^p(B)}{\mu(A)+\mu(B)} = \frac{\mu(A)\cdot\frac{\delta^p(A)}{\mu(A)}+\mu(B)\cdot\frac{\delta^p(B)}{\mu(B)}}{\mu(A)+\mu(B)} = \frac{\mu(A)\cdot\left[\frac{\delta(A)}{\mu^{1/p}(A)}\right]^p+\mu(B)\cdot\left[\frac{\delta(B)}{\mu^{1/p}(B)}\right]^p}{\mu(A)+\mu(B)} =: I. \tag{2.4}$$

Since for $p\geq 1$ $(0<p<1)$ the power function $g(t)=t^p$ is convex (concave), then

$$I \geq (\leq) \left[\frac{\mu(A)\cdot\frac{\delta(A)}{\mu^{1/p}(A)}+\mu(B)\cdot\frac{\delta(B)}{\mu^{1/p}(B)}}{\mu(A)+\mu(B)}\right]^p = \left[\frac{\delta(A)\,\mu^{1-1/p}(A)+\delta(B)\,\mu^{1-1/p}(B)}{\mu(A\cup B)}\right]^p. \tag{2.5}$$

By combining (2.4) with (2.5) we get

$$\frac{\delta^p(A\cup B)}{\mu(A\cup B)} \geq (\leq) \left[\frac{\delta(A)\,\mu^{1-1/p}(A)+\delta(B)\,\mu^{1-1/p}(B)}{\mu(A\cup B)}\right]^p,$$

which is equivalent to

$$\frac{\delta(A\cup B)}{\mu^{1/p}(A\cup B)} \geq (\leq) \frac{\delta(A)\,\mu^{1-1/p}(A)+\delta(B)\,\mu^{1-1/p}(B)}{\mu(A\cup B)}$$

i.e., by multiplying with $\mu(A\cup B)$,

$$\eta_p(A\cup B) \geq (\leq)\,\eta_p(A)+\eta_p(B)$$

and the proof is complete. □

**Corollary 1.** *With the assumptions of Theorem 1 for* $(\Omega,\mathcal{A})$, $\mu$ *and if* $\delta:\mathcal{A}\to[0,\infty)$ *is a supermeasure (submeasure) on* $\mathcal{A}$ *while* $p,q\geq 1$ $(0<p,q<1)$ *then the two parameter functional*

$$\eta_{p,q}:\mathcal{A}\to[0,\infty),\, \eta_{p,q}(A)=\delta^q(A)\,\mu^{q\left(1-\frac{1}{p}\right)}(A) \tag{2.6}$$

*is also a supermeasure (submeasure) on* $\mathcal{A}$.

*Proof.* Let $A,B\in\mathcal{A}\setminus\{\emptyset\}$ with $A\cap B=\emptyset$. Observe that $\eta_{p,q}(A)=\left[\eta_p(A)\right]^q$ for $A\in\mathcal{A}$. Therefore, by Theorem 1 and the inequality (2.2) for $q\geq 1$ $(0<q<1)$ we have that

$$\begin{aligned}\eta_{p,q}(A\cup B) &= \left[\eta_p(A\cup B)\right]^q \geq (\leq)\left[\eta_p(A)+\eta_p(B)\right]^q \\ &\geq (\leq)\left[\eta_p(A)\right]^q+\left[\eta_p(B)\right]^q=\eta_{p,q}(A)+\eta_{p,q}(B).\end{aligned}$$

□

**Remark 1.** *If, for* $q=p$ *in the above corollary, we consider the functional* $\psi_p:\mathcal{A}\to[0,\infty)$ *defined by*

$$\psi_p(A):=\delta^p(A)\,\mu^{p-1}(A) \tag{2.7}$$

*for* $p\geq 1$ $(0<p<1)$ *and* $\delta:\mathcal{A}\to[0,\infty)$ *is a supermeasure (submeasure) on* $\mathcal{A}$, *then the functional* $\psi_p$ *is also a supermeasure (submeasure) on* $\mathcal{A}$.

We have:

**Theorem 2.** *Let* $(\Omega, \mathcal{A})$ *a measurable space and* $\mu : \mathcal{A} \to [0, \infty)$ *a positive measure on* $\mathcal{A} \setminus \{\emptyset\}$*. If* $\delta : \mathcal{A} \to (0, \infty)$ *is a supermeasure on* $\mathcal{A} \setminus \{\emptyset\}$ *and* $0 < p < 1$ *then the functional*

$$\varphi_p : \mathcal{A} \setminus \{\emptyset\} \to [0, \infty), \eta_p(A) = \frac{\mu^{1-\frac{1}{p}}(A)}{\delta(A)} \tag{2.8}$$

*is a submeasure on* $\mathcal{A} \setminus \{\emptyset\}$*.*

*Proof.* Let $s := -p \in [-1, 0)$. For $s < 0$ we have the following inequality

$$(\alpha + \beta)^s \le \alpha^s + \beta^s \tag{2.9}$$

for any $\alpha, \beta > 0$.

Indeed, by the convexity of the function $f_s(t) = t^s$ on $(0, \infty)$ with $s < 0$ we have that

$$(\alpha + \beta)^s \le 2^{s-1}(\alpha^s + \beta^s)$$

for any $\alpha, \beta > 0$ and since, obviously, $2^{s-1}(\alpha^s + \beta^s) \le \alpha^s + \beta^s$, then (2.9) holds true.

Let $A$, $B \in \mathcal{A} \setminus \{\emptyset\}$ with $A \cap B = \emptyset$. Taking into account that $\delta$ is superadditive, then by (2.9) we have

$$\delta^s(A \cup B) \le [\delta(A) + \delta(B)]^s \le \delta^s(A) + \delta^s(B) \tag{2.10}$$

for any $A$, $B \in \mathcal{A} \setminus \{\emptyset\}$ with $A \cap B = \emptyset$.

Since $\mu$ is additive, then by (2.9) we have that

$$\begin{aligned} \frac{\delta^s(A \cup B)}{\mu(A \cup B)} &\le \frac{\delta^s(A) + \delta^s(B)}{\mu(A) + \mu(B)} \\ &= \frac{\mu(A) \cdot \left[\frac{\delta(A)}{\mu^{1/s}(A)}\right]^s + \mu(B) \cdot \left[\frac{\delta(B)}{\mu^{1/s}(B)}\right]^s}{\mu(A) + \mu(B)} \\ &= \frac{\mu(A) \cdot \left[\frac{\mu^{1/s}(A)}{\delta(A)}\right]^{-s} + \mu(B) \cdot \left[\frac{\mu^{1/s}(B)}{\delta(B)}\right]^{-s}}{\mu(A) + \mu(B)} =: J. \end{aligned} \tag{2.11}$$

By the concavity of the function $g(t) = t^{-s}$ with $s \in [-1, 0)$ we also have

$$J \le \left[\frac{\mu(A) \cdot \frac{\mu^{1/s}(A)}{\delta(A)} + \mu(B) \cdot \frac{\mu^{1/s}(B)}{\delta(B)}}{\mu(A) + \mu(B)}\right]^{-s}. \tag{2.12}$$

Making use of (2.11) and (2.12) we get

$$\frac{\delta^s(A \cup B)}{\mu(A \cup B)} \le \left[\frac{\mu(A) \cdot \frac{\mu^{1/s}(A)}{\delta(A)} + \mu(B) \cdot \frac{\mu^{1/s}(B)}{\delta(B)}}{\mu(A) + \mu(B)}\right]^{-s}$$

for any $A$, $B \in \mathcal{A}$, which is equivalent to

$$\frac{\delta^{-1}(A \cup B)}{\mu^{-1/s}(A \cup B)} \le \frac{\frac{\mu^{1+1/s}(A)}{\delta(A)} + \frac{\mu^{1+1/s}(B)}{\delta(B)}}{\mu(A) + \mu(B)}$$

and, since $\mu(A\cup B)=\mu(A)+\mu(B)$, with

$$\frac{\mu^{1+1/s}(A+B)}{\delta(A+B)}\leq\frac{\mu^{1+1/s}(A)}{\delta(A)}+\frac{\mu^{1+1/s}(B)}{\delta(B)}$$

for any $A,\ B\in\mathcal{A}\backslash\{\emptyset\}$ with $A\cap B=\emptyset$.

This completes the proof. $\square$

The following result may be stated as well:

**Corollary 2.** *Assume that* $(\Omega,\mathcal{A})$ *and* $\mu$ *are as in Theorem 2. If* $\delta:\mathcal{A}\to(0,\infty)$ *is a supermeasure on* $\mathcal{A}\backslash\{\emptyset\}$ *and* $0<p,q<1$ *then the two parameter functional*

$$\varphi_{p,q}:\mathcal{A}\backslash\{\emptyset\}\to[0,\infty),\ \varphi_{p,q}(A)=\frac{\mu^{q\left(1-\frac{1}{p}\right)}(A)}{\delta^{q}(A)}\tag{2.13}$$

*is a submeasure on* $\mathcal{A}\backslash\{\emptyset\}$.

*Proof.* Observe that $\varphi_{p,q}(A)=\left[\varphi_p(A)\right]^q$ for $A\in\mathcal{A}$. Therefore, by Theorem 2 and the inequality (2.2) for $0<q<1$ we have that

$$\begin{aligned}\varphi_{p,q}(A\cup B)&=\left[\varphi_p(A\cup B)\right]^q\leq\left[\varphi_p(A)+\varphi_p(B)\right]^q\\&\leq\left[\varphi_p(A)\right]^q+\left[\varphi_p(B)\right]^q=\varphi_{p,q}(A)+\varphi_{p,q}(B)\end{aligned}$$

for any $A,\ B\in\mathcal{A}\backslash\{\emptyset\}$ with $A\cap B=\emptyset$ and the statement is proved. $\square$

**Remark 2.** *If we consider the functional for* $0<p<1$ *defined by*

$$\sigma_p(A):=\frac{\mu^{p-1}(A)}{\delta^p(A)},$$

*where* $\mu:\mathcal{A}\to(0,\infty)$ *is a measure on* $\mathcal{A}\backslash\{\emptyset\}$ *and* $\delta:\mathcal{A}\backslash\{\emptyset\}\to(0,\infty)$ *is a supermeasure on* $\mathcal{A}\backslash\{\emptyset\}$, *then the functional* $\sigma_p$ *is a submeasure on* $\mathcal{A}\backslash\{\emptyset\}$.

## 3. Applications for Jensen's Inequality

Let $(\Omega,\mathcal{A},\nu)$ be a measurable space consisting of a set $\Omega$, a $\sigma$-algebra $\mathcal{A}$ of parts of $\Omega$ and a countably additive and positive measure $\nu$ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a $\nu$-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq0$ for $\nu$-a.e. (almost every) $x\in\Omega$, consider the *Lebesgue space*

$$L_w(\Omega,\nu):=\{f:\Omega\to\mathbb{R},\ f\text{ is }\nu\text{-measurable and }\int_\Omega w(x)\left|f(x)\right|d\nu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega wd\nu$ instead of $\int_\Omega w(x)\,d\nu(x)$.

Let also

$$\mathcal{A}_\nu:=\{A\in\mathcal{A}\ |\nu(A)>0\}.$$

For a $\nu$-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)>0$ for $\nu$-a.e. $x\in\Omega$, we consider the functional $J(\cdot,w;\Phi,f):\mathcal{A}_\nu\to[0,\infty)$ defined by

$$J(A,w;\Phi,f):=\int_A w(\Phi\circ f)\,d\nu-\Phi\left(\frac{\int_A wfd\nu}{\int_A wd\nu}\right)\int_A wd\nu\geq0,\tag{3.1}$$

where $\Phi:[m,M]\to\mathbb{R}$ is a continuous convex function on the interval of real numbers $[m,M]$, $f:\Omega\to[m,M]$ is $\nu$-measurable and such that $f,\ \Phi\circ f\in L_w(\Omega,\nu)$.

We use the following result, see [10]:

**Lemma 1.** *Let $\Phi : [m, M] \to \mathbb{R}$ be a continuous convex function on the interval of real numbers $[m, M]$, $f : \Omega \to [m, M]$ is $\nu$-measurable and such that $f$, $\Phi \circ f \in L_w(\Omega, \nu)$. Then the functional $J(\cdot, w; \Phi, f)$ defined by (3.1) is a supermeasure on $\mathcal{A}_\nu$.*

For some Jensen's inequality related functionals and their properties see [1], [2], [4], [13], [20], [6], [7] and [11].

For $p, q \geq 1$ consider the two parameter functional $J_{p,q}(\cdot, w; \Phi, f) : \mathcal{A}_\nu \to [0, \infty)$,

$$J_{p,q}(A, w; \Phi, f) = J^q(A, w; \Phi, f) \left( \int_A w d\nu \right)^{q\left(1-\frac{1}{p}\right)}. \tag{3.2}$$

Observe that

$$J_{p,q}(A, w; \Phi, f) = \left( \int_A w d\nu \right)^{-\frac{q}{p}} \left[ \frac{\int_A w (\Phi \circ f) d\nu}{\int_A w d\nu} - \Phi \left( \frac{\int_A w f d\nu}{\int_A w d\nu} \right) \right]^q. \tag{3.3}$$

In particular, for $q = p$ we have

$$J_p(A, w; \Phi, f) := \left( \int_A w d\nu \right)^{-1} \left[ \frac{\int_A w (\Phi \circ f) d\nu}{\int_A w d\nu} - \Phi \left( \frac{\int_A w f d\nu}{\int_A w d\nu} \right) \right]^p. \tag{3.4}$$

**Theorem 3.** *Let $\Phi : [m, M] \to \mathbb{R}$ be a continuous convex function on the interval of real numbers $[m, M]$, $f : \Omega \to [m, M]$ is $\nu$-measurable and such that $f$, $\Phi \circ f \in L_w(\Omega, \nu)$. Then the functional $J_{p,q}(\cdot, w; \Phi, f)$ defined by (3.2) is a supermeasure on $\mathcal{A}_\nu$ for any $p, q \geq 1$.*

The proof follows by Corollary 1 for $p, q \geq 1$, the measure $\mu : \mathcal{A}_\nu \to (0, \infty)$, $\mu(A) := \int_A w d\nu$ and the supermeasure $\delta : \mathcal{A}_\nu \to [0, \infty)$, $\delta(A) := J(A, w; \Phi, f)$.

If $\Phi : [m, M] \to \mathbb{R}$ is a continuous strictly convex function on $[m, M]$ then Jensen's inequality is strict, namely

$$J(A, w; \Phi, f) := \int_A w (\Phi \circ f) d\nu - \Phi \left( \frac{\int_A w f d\nu}{\int_A w d\nu} \right) \int_A w d\nu > 0$$

for any $A \in \mathcal{A}_\nu$.

In this situation we can also define the two parameters functional $\widetilde{J}_{p,q}(A, w; \Phi, f) : \mathcal{A}_\nu \to [0, \infty)$,

$$\widetilde{J}_{p,q}(A, w; \Phi, f) = \frac{\left( \int_A w d\nu \right)^{q\left(1-\frac{1}{p}\right)}}{J^q(A, w; \Phi, f)} \tag{3.5}$$

where $0 < p, q < 1$.

Observe that

$$\widetilde{J}_{p,q}(A, w; \Phi, f) = \frac{1}{\left( \int_A w d\nu \right)^{\frac{q}{p}} \left[ \frac{\int_A w (\Phi \circ f) d\nu}{\int_A w d\nu} - \Phi \left( \frac{\int_A w f d\nu}{\int_A w d\nu} \right) \right]^q}$$

and

$$\widetilde{J}_p(A, w; \Phi, f) := \frac{1}{\left[ \frac{\int_A w (\Phi \circ f) d\nu}{\int_A w d\nu} - \Phi \left( \frac{\int_A w f d\nu}{\int_A w d\nu} \right) \right]^p \int_A w d\nu}.$$

**Theorem 4.** *Let $\Phi : [m, M] \to \mathbb{R}$ be a continuous strictly convex function on the interval of real numbers $[m, M]$, $f : \Omega \to [m, M]$ is $\nu$-measurable and such that $f$, $\Phi \circ f \in L_w(\Omega, \nu)$. Then the functional $\widetilde{J}_{p,q}(\cdot, w; \Phi, f)$ defined by (3.5) is a submeasure on $\mathcal{A}_\nu$ for any $0 < p, q < 1$.*

The proof follows by Corollary 2 for $0<p,q<1$, the measure $\mu:\mathcal{A}_\nu\rightarrow(0,\infty)$, $\mu(A):=\int_A wd\nu$ and the supermeasure $\delta:\mathcal{A}_\nu\rightarrow[0,\infty)$, $\delta(A):=J(A,w;\Phi,f)$.

Let $\Phi:[m,M]\rightarrow\mathbb{R}$ be a continuous convex function on the interval of real numbers $[m,M]$, $x=(x_i)_{i\in\mathbb{N}}$ a sequence of real numbers with $x_i\in[m,M]$, $i\in\mathbb{N}$, and $w=(w_i)_{i\in\mathbb{N}}$ a sequence of positive real numbers.

Let $\Omega=\mathbb{N}$ and $\mathcal{P}_f(\mathbb{N})$ be the algebra of finite parts of natural numbers $\mathbb{N}$. By the monotonicity property of supermeasure on $\mathcal{P}_f(\mathbb{N})$ we have from the above results that the sequence

$$J_{n,p}(w;\Phi,x):=\left(\sum_{i=0}^{n}w_i\right)^{-\frac{1}{p}}\left[\frac{\sum_{i=0}^{n}w_i\Phi(x_i)}{\sum_{i=0}^{n}w_i}-\Phi\left(\frac{\sum_{i=0}^{n}w_ix_i}{\sum_{i=0}^{n}w_i}\right)\right],$$

where $p\geq1$ is monotonic nondecreasing, namely

$$\begin{aligned}(3.6)\qquad &\left(\sum_{i=0}^{n+1}w_i\right)^{-\frac{1}{p}}\left[\frac{\sum_{i=0}^{n+1}w_i\Phi(x_i)}{\sum_{i=0}^{n+1}w_i}-\Phi\left(\frac{\sum_{i=0}^{n+1}w_ix_i}{\sum_{i=0}^{n+1}w_i}\right)\right]\\ &\geq\left(\sum_{i=0}^{n}w_i\right)^{-\frac{1}{p}}\left[\frac{\sum_{i=0}^{n}w_i\Phi(x_i)}{\sum_{i=0}^{n}w_i}-\Phi\left(\frac{\sum_{i=0}^{n}w_ix_i}{\sum_{i=0}^{n}w_i}\right)\right]\end{aligned}$$

for any $n\in\mathbb{N}$ and

$$\begin{aligned}(3.7)\qquad &J_{n,p}(w;\Phi,x)\\ &\geq\max_{0\leq i\neq j\leq n}\left\{(w_i+w_j)^{-\frac{1}{p}}\left[\frac{w_i\Phi(x_i)+w_j\Phi(x_j)}{w_i+w_j}-\Phi\left(\frac{w_ix_i+w_jx_j}{w_i+w_j}\right)\right]\right\}.\end{aligned}$$

We also have for $n\geq1$ that

$$\begin{aligned}(3.8)\qquad &\left(\sum_{i=0}^{2n}w_i\right)^{-\frac{q}{p}}\left[\frac{\sum_{i=0}^{2n}w_i\Phi(x_i)}{\sum_{i=0}^{2n}w_i}-\Phi\left(\frac{\sum_{i=0}^{2n}w_ix_i}{\sum_{i=0}^{2n}w_i}\right)\right]^q\\ &\geq\left(\sum_{i=0}^{n}w_{2i}\right)^{-\frac{q}{p}}\left[\frac{\sum_{i=0}^{n}w_{2i}\Phi(x_{2i})}{\sum_{i=0}^{n}w_{2i}}-\Phi\left(\frac{\sum_{i=0}^{n}w_{2i}x_{2i}}{\sum_{i=0}^{n}w_{2i}}\right)\right]^q\\ &+\left(\sum_{i=0}^{n-1}w_{2i+1}\right)^{-\frac{q}{p}}\left[\frac{\sum_{i=0}^{n-1}w_{2i+1}\Phi(x_{2i+1})}{\sum_{i=0}^{n-1}w_{2i+1}}-\Phi\left(\frac{\sum_{i=0}^{n-1}w_{2i+1}x_{2i+1}}{\sum_{i=0}^{n-1}w_{2i+1}}\right)\right]^q\end{aligned}$$

and

$$\begin{aligned}(3.9)\qquad &\left(\sum_{i=0}^{2n+1}w_i\right)^{-\frac{q}{p}}\left[\frac{\sum_{i=0}^{2n+1}w_i\Phi(x_i)}{\sum_{i=0}^{2n}w_i}-\Phi\left(\frac{\sum_{i=0}^{2n+1}w_ix_i}{\sum_{i=0}^{2n}w_i}\right)\right]^q\\ &\geq\left(\sum_{i=0}^{n}w_{2i}\right)^{-\frac{q}{p}}\left[\frac{\sum_{i=0}^{n}w_{2i}\Phi(x_{2i})}{\sum_{i=0}^{n}w_{2i}}-\Phi\left(\frac{\sum_{i=0}^{n}w_{2i}x_{2i}}{\sum_{i=0}^{n}w_{2i}}\right)\right]^q\\ &+\left(\sum_{i=0}^{n}w_{2i+1}\right)^{-\frac{q}{p}}\left[\frac{\sum_{i=0}^{n}w_{2i+1}\Phi(x_{2i+1})}{\sum_{i=0}^{n}w_{2i+1}}-\Phi\left(\frac{\sum_{i=0}^{n}w_{2i+1}x_{2i+1}}{\sum_{i=0}^{n}w_{2i+1}}\right)\right]^q.\end{aligned}$$

## 4. Applications for Hölder's Inequality

Let $(\Omega, \mathcal{A}, \nu)$ be a measurable space consisting of a set $\Omega$, a $\sigma$ -algebra $\mathcal{A}$ of parts of $\Omega$ and a countably additive and positive measure $\nu$ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a $\nu$-measurable function $w : \Omega \to \mathbb{C}$, with $w(x) \geq 0$ for $\nu$ -a.e. (almost every) $x \in \Omega$, consider the $\alpha$-Lebesgue space

$$L_w^\alpha(\Omega, \nu) := \{f : \Omega \to \mathbb{C},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w|f|^\alpha d\nu < \infty\},$$

for $\alpha \geq 1$.

The following inequality is well known in the literature as *Hölder's inequality*

$$\left|\int_\Omega wfgd\nu\right| \leq \left(\int_\Omega w|f|^\alpha d\nu\right)^{1/\alpha} \left(\int_\Omega w|g|^\beta d\nu\right)^{1/\beta} \tag{4.1}$$

where $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $f \in L_w^\alpha(\Omega, \nu)$, $g \in L_w^\beta(\Omega, \nu)$.

We consider the functional $H_{\alpha,\beta}(\cdot, w; f, g) : \mathcal{A}_\nu \to [0, \infty)$ defined by

$$H_{\alpha,\beta}(A, w; f, g) = \left(\int_A w|f|^\alpha d\nu\right)^{1/\alpha} \left(\int_A w|g|^\beta d\nu\right)^{1/\beta} - \left|\int_A wfgd\nu\right|. \tag{4.2}$$

We use the following result, see [10]:

**Lemma 2.** *Let $f \in L_w^\alpha(\Omega, \nu)$, $g \in L_w^\beta(\Omega, \nu)$ where $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then the functional $H_{\alpha,\beta}(\cdot, w; f, g) : \mathcal{A}_\nu \to [0, \infty)$ defined by (4.2) is a supermeasure.*

For some Hölder's inequality related functionals and their properties see [3], [5] and [21].

Consider the two parameter functional $H_{\alpha,\beta,p,q}(\cdot, w; f, g) : \mathcal{A} \to [0, \infty)$ defined by

$$\begin{aligned} &H_{\alpha,\beta,p,q}(A, w; f, g) \\ &:= \left[\left(\frac{\int_A w|f|^\alpha d\nu}{\int_A wd\nu}\right)^{1/\alpha} \left(\frac{\int_A w|g|^\beta d\nu}{\int_A wd\nu}\right)^{1/\beta} - \left|\frac{\int_A wfgd\nu}{\int_A wd\nu}\right|\right]^q \\ &\times \left(\int_A wd\nu\right)^{q\left(2-\frac{1}{p}\right)}, \end{aligned} \tag{4.3}$$

where $p, q \geq 1$.

Also, define

$$\begin{aligned} &H_{\alpha,\beta,p}(A, w; f, g) \\ &:= \left[\left(\frac{\int_A w|f|^\alpha d\nu}{\int_A wd\nu}\right)^{1/\alpha} \left(\frac{\int_A w|g|^\beta d\nu}{\int_A wd\nu}\right)^{1/\beta} - \left|\frac{\int_A wfgd\nu}{\int_A wd\nu}\right|\right]^p \\ &\times \left(\int_A wd\nu\right)^{2p-1}, \end{aligned} \tag{4.4}$$

where $p \geq 1$.

**Theorem 5.** *Let $f \in L_w^\alpha(\Omega, \nu)$, $g \in L_w^\beta(\Omega, \nu)$ where $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then the functional $H_{\alpha,\beta,p,q}(\cdot, w; f, g) : \mathcal{A}_\nu \to [0, \infty)$ with $p, q \geq 1$, defined by (4.3) is a supermeasure.*

The proof follows by Corollary 1 for $p,q \geq 1$, the measure $\mu : \mathcal{A}_\nu \to (0,\infty)$, $\mu(A) := \int_A w d\nu$ and the supermeasure $\delta : \mathcal{A}_\nu \to [0,\infty)$, $\delta(A) := H_{\alpha,\beta}(A,w;f,g)$.

Let $x = (x_i)_{i\in\mathbb{N}}$ and $x = (y_i)_{i\in\mathbb{N}}$ be sequences of complex numbers and $w = (w_i)_{i\in\mathbb{N}}$ a sequence of positive real numbers. Assume that $\alpha,\beta > 1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Let $\Omega = \mathbb{N}$ and $\mathcal{P}_f(\mathbb{N})$ be the algebra of finite parts of natural numbers $\mathbb{N}$. By the monotonicity property of supermeasure on $\mathcal{P}_f(\mathbb{N})$ we have from the above results that the sequence

$$H_{n,\alpha,\beta,p}(w;x,y) := \left[\left(\sum_{i=0}^{n} w_i |x_i|^\alpha\right)^{1/\alpha}\left(\sum_{i=0}^{n} w_i |y_i|^\beta\right)^{1/\beta} - \left|\sum_{i=0}^{n} w_i x_i y_i\right|\right] \times \left(\sum_{i=0}^{n} w_i\right)^{1-\frac{1}{p}}, \tag{4.5}$$

is monotonic nondecreasing and

$$\begin{aligned} &H_{n,\alpha,\beta,p}(w;x,y) \\ &\geq \max_{0\leq i\neq j\leq n}\left\{\left[(w_i|x_i|^\alpha + w_j|x_j|^\alpha)^{1/\alpha}\left(w_i|y_i|^\beta + w_j|y_j|^\beta\right)^{1/\beta}\right.\right. \\ &\left.\left. - |w_ix_iy_i + w_jx_jy_j|\right](w_i+w_j)^{1-\frac{1}{p}}\right\}, \end{aligned} \tag{4.6}$$

for any $p \geq 1$.

We also have for $n \geq 1$ that

$$\begin{aligned} &\left[\left(\sum_{i=0}^{2n} w_i |x_i|^\alpha\right)^{1/\alpha}\left(\sum_{i=0}^{2n} w_i |y_i|^\beta\right)^{1/\beta} - \left|\sum_{i=0}^{2n} w_i x_i y_i\right|\right]^q \left(\sum_{i=0}^{2n} w_i\right)^{q\left(1-\frac{1}{p}\right)} \\ &\geq \left[\left(\sum_{i=0}^{n} w_{2i} |x_{2i}|^\alpha\right)^{1/\alpha}\left(\sum_{i=0}^{n} w_{2i} |y_{2i}|^\beta\right)^{1/\beta} - \left|\sum_{i=0}^{n} w_{2i} x_{2i} y_{2i}\right|\right]^q \\ &\qquad \times \left(\sum_{i=0}^{n} w_{2i}\right)^{q\left(1-\frac{1}{p}\right)} \\ &+ \left[\left(\sum_{i=0}^{n-1} w_{2i+1} |x_{2i+1}|^\alpha\right)^{1/\alpha}\left(\sum_{i=0}^{n-1} w_{2i+1} |y_{2i+1}|^\beta\right)^{1/\beta} - \left|\sum_{i=0}^{n-1} w_{2i+1} x_{2i+1} y_{2i+1}\right|\right]^q \\ &\qquad \times \left(\sum_{i=0}^{n-1} w_{2i+1}\right)^{q\left(1-\frac{1}{p}\right)} \end{aligned} \tag{4.7}$$

for any $p,q \geq 1$.

## 5. Applications for Minkowski's Inequality

Consider the $r$-Lebesgue space

$$L_w^r(\Omega,\nu) := \{f : \Omega \to \mathbb{C},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w|f|^r d\nu < \infty\},$$

for $r \geq 1$.

The following inequality is well known in the literature as *Minkowski's inequality*

$$\left(\int_{\Omega} w\left|f+g\right|^{r} d\nu\right)^{1/r} \leq \left(\int_{\Omega} w\left|f\right|^{r} d\nu\right)^{1/r} + \left(\int_{\Omega} w\left|g\right|^{r} d\nu\right)^{1/r} \tag{5.1}$$

for any $f, g \in L_{w}^{r}(\Omega, \nu)$.

Consider the functional $M_{r}(\cdot, w; f, g) : \mathcal{A}_{\nu} \to [0, \infty)$ defined by

$$M_{r}(A, w; f, g) := \left[\left(\int_{A} w\left|f\right|^{r} d\nu\right)^{1/r} + \left(\int_{A} w\left|g\right|^{r} d\nu\right)^{1/r}\right]^{r} - \int_{A} w\left|f+g\right|^{r} d\nu. \tag{5.2}$$

We use the following result, see [10]:

**Lemma 3.** *Let $f, g \in L_{w}^{r}(\Omega, \nu)$ for $r \geq 1$. Then the functional $M_{r}(\cdot, w; f, g) : \mathcal{A}_{\nu} \to [0, \infty)$ defined by (5.2) is a supermeasure.*

For some Minkowski's inequality related functionals and their properties see [3], [5] and [21].

For $p, q, r \geq 1$ consider the two parameter functional $M_{p,q,r}(\cdot, w; \Phi, f) : \mathcal{A}_{\nu} \to [0, \infty)$,

$$M_{p,q,r}(A, w; f, g) = M_{r}^{q}(A, w; f, g)\left(\int_{A} w d\nu\right)^{q\left(1-\frac{1}{p}\right)}. \tag{5.3}$$

Observe that

$$\begin{aligned} & M_{p,q,r}(A, w; f, g) \\ & = \left(\left[\left(\frac{\int_{\Omega} w\left|f\right|^{r} d\nu}{\int_{A} w d\nu}\right)^{1/r} + \left(\frac{\int_{\Omega} w\left|g\right|^{r} d\nu}{\int_{A} w d\nu}\right)^{1/r}\right]^{r} - \frac{\int_{\Omega} w\left|f+g\right|^{r} d\nu}{\int_{A} w d\nu}\right)^{q} \\ & \times \left(\int_{A} w d\nu\right)^{q\left(2-\frac{1}{p}\right)} \end{aligned} \tag{5.4}$$

In particular, for $q = p$ we have

$$\begin{aligned} & M_{p,r}(A, w; f, g) \\ & = \left(\left[\left(\frac{\int_{A} w\left|f\right|^{r} d\nu}{\int_{A} w d\nu}\right)^{1/r} + \left(\frac{\int_{A} w\left|g\right|^{r} d\nu}{\int_{A} w d\nu}\right)^{1/r}\right]^{r} - \frac{\int_{A} w\left|f+g\right|^{r} d\nu}{\int_{A} w d\nu}\right)^{p} \\ & \times \left(\int_{A} w d\nu\right)^{2p-1}. \end{aligned} \tag{5.5}$$

**Theorem 6.** *Let $f, g \in L_{w}^{r}(\Omega, \nu)$ for $r \geq 1$ and $p, q \geq 1$. Then the functional $M_{p,q,r}(\cdot, w; f, g) : \mathcal{A}_{\nu} \to [0, \infty)$ defined by (5.3) is a supermeasure.*

The proof follows by Corollary 1 for $p, q \geq 1$.

Let $x = (x_{i})_{i \in \mathbb{N}}$ and $x = (y_{i})_{i \in \mathbb{N}}$ be sequences of complex numbers and $w = (w_{i})_{i \in \mathbb{N}}$ a sequence of positive real numbers. Let $\Omega = \mathbb{N}$ and $\mathcal{P}_{f}(\mathbb{N})$ be the algebra of finite parts of natural numbers $\mathbb{N}$. By the monotonicity property of supermeasure on $\mathcal{P}_f(\mathbb{N})$ we have from the above results that the sequence

$$\begin{aligned} (5.6)\qquad & M_{n,p,r}(w;x,y) \\ & := \left(\left[\left(\sum_{i=0}^{n} w_i |x_i|^r\right)^{1/r} + \left(\sum_{i=0}^{n} w_i |y_i|^r\right)^{1/r}\right]^r - \sum_{i=0}^{n} w_i |x_i+y_i|^r\right) \\ & \times \left(\sum_{i=0}^{n} w_i\right)^{1-\frac{1}{p}} \end{aligned}$$

is monotonic nondecreasing and

$$\begin{aligned} (5.7)\qquad & M_{n,p,r}(w;x,y) \\ & \geq \max_{0\leq i\neq j\leq n} \left\{ \left( \left[ (w_i|x_i|^r + w_j|x_j|^r)^{1/r} + (w_i|y_i|^r + w_j|y_j|^r)^{1/r} \right]^r \right.\right. \\ & \left.\left. - w_i|x_i+y_i|^r - w_j|x_j+y_j|^r \right)(w_i+w_j)^{1-\frac{1}{p}} \right\} \end{aligned}$$

for any $p \geq 1$.

We have the inequality

$$\begin{aligned} (5.8)\qquad & \left(\left[\left(\sum_{i=0}^{2n} w_i |x_i|^r\right)^{1/r} + \left(\sum_{i=0}^{2n} w_i |y_i|^r\right)^{1/r}\right]^r - \sum_{i=0}^{2n} w_i |x_i+y_i|^r\right) \\ & \qquad \times \left(\sum_{i=0}^{2n} w_i\right)^{1-\frac{1}{p}} \\ \geq & \left(\left[\left(\sum_{i=0}^{n} w_{2i} |x_{2i}|^r\right)^{1/r} + \left(\sum_{i=0}^{n} w_{2i} |y_{2i}|^r\right)^{1/r}\right]^r - \sum_{i=0}^{n} w_{2i} |x_{2i}+y_{2i}|^r\right) \\ & \qquad \times \left(\sum_{i=0}^{n} w_{2i}\right)^{1-\frac{1}{p}} \\ & + \left(\left[\left(\sum_{i=0}^{n-1} w_{2i+1} |x_{2i+1}|^r\right)^{1/r} + \left(\sum_{i=0}^{n-1} w_{2i+1} |y_{2i+1}|^r\right)^{1/r}\right]^r \right. \\ & \left. \qquad - \sum_{i=0}^{n-1} w_{2i+1} |x_{2i+1}+y_{2i+1}|^r\right) \times \left(\sum_{i=0}^{n-1} w_{2i+1}\right)^{1-\frac{1}{p}} \end{aligned}$$

for any $p \geq 1$.

## 6. Applications for Cauchy-Bunyakovsky-Schwarz's Inequality

Consider the Hilbert space

$$L_w^2(\Omega,\nu) := \{f : \Omega \to \mathbb{C},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w|f|^2 d\nu < \infty\}.$$

The following inequality is well known in the literature as Cauchy-Bunyakovsky-Schwarz's (CBS) inequality

$$\left|\int_{\Omega} wfgd\nu\right| \leq \left(\int_{\Omega} w\left|f\right|^{2} d\nu\right)^{1/2}\left(\int_{\Omega} w\left|g\right|^{2} d\nu\right)^{1/2} \tag{6.1}$$

where $f\in L_{w}^{2}\left(\Omega,\nu\right),\ g\in L_{w}^{2}\left(\Omega,\nu\right).$

We consider the functional $H\left(\cdot,w;f,g\right):\mathcal{A}_{\nu}\rightarrow[0,\infty)$ defined by

$$H\left(A,w;f,g\right)=\left(\int_{A} w\left|f\right|^{2} d\nu\right)^{1/2}\left(\int_{A} w\left|g\right|^{2} d\nu\right)^{1/2}-\left|\int_{A} wfgd\nu\right|. \tag{6.2}$$

Taking into account that $H\left(A,w;f,g\right)=H_{\alpha,\beta}\left(A,w;f,g\right)$ for $\alpha=\beta=2$, see (4.2), we have:

**Lemma 4.** *Let $f\in L_{w}^{2}\left(\Omega,\nu\right),\ g\in L_{w}^{2}\left(\Omega,\nu\right)$, then the functional $H\left(\cdot,w;f,g\right):\mathcal{A}_{\nu}\rightarrow[0,\infty)$ defined by (6.2) is a supermeasure.*

and

**Theorem 7.** *Let $f\in L_{w}^{2}\left(\Omega,\nu\right),\ g\in L_{w}^{2}\left(\Omega,\nu\right)$, then the functional $H_{p,q}\left(\cdot,w;f,g\right):\mathcal{A}_{\nu}\rightarrow[0,\infty)$ defined by*

$$\begin{aligned} &H_{p,q}\left(A,w;f,g\right)\\ &:=\left[\left(\int_{A} w\left|f\right|^{2} d\nu\right)^{1/2}\left(\int_{A} w\left|g\right|^{2} d\nu\right)^{1/2}-\left|\int_{A} wfgd\nu\right|\right]^{q}\\ &\times\left(\int_{A} wd\nu\right)^{q\left(1-\frac{1}{p}\right)}\\ &=\left[\left(\frac{\int_{A} w\left|f\right|^{2} d\nu}{\int_{A} wd\nu}\right)^{1/2}\left(\frac{\int_{A} w\left|g\right|^{2} d\nu}{\int_{A} wd\nu}\right)^{1/2}-\left|\frac{\int_{A} wfgd\nu}{\int_{A} wd\nu}\right|\right]^{q}\\ &\times\left(\int_{A} wd\nu\right)^{q\left(2-\frac{1}{p}\right)}, \end{aligned} \tag{6.3}$$

*with $p,q\geq 1$, is a supermeasure.*

Now, consider the functional $L\left(\cdot,w;f,g\right):\mathcal{A}_{\nu}\rightarrow[0,\infty)$ defined by

$$L\left(A,w;f,g\right)=\int_{A} w\left|f\right|^{2} d\nu\int_{A} w\left|g\right|^{2} d\nu-\left|\int_{A} wfgd\nu\right|^{2}. \tag{6.4}$$

We need the following result, see [10]:

**Lemma 5.** *Let $f\in L_{w}^{2}\left(\Omega,\nu\right),\ g\in L_{w}^{2}\left(\Omega,\nu\right)$. Then for any $A,B\in\mathcal{A}_{\nu}$ with $A\cap B=\emptyset$ we have*

$$\begin{aligned} &L\left(A\cup B,w;f,g\right)\\ &\geq L\left(A,w;f,g\right)+L\left(B,w;f,g\right)\\ &+\left(\det\left[\begin{array}{cc} \left(\int_{A} w\left|f\right|^{2} d\nu\right)^{1/2} & \left(\int_{A} w\left|g\right|^{2} d\nu\right)^{1/2}\\ \left(\int_{B} w\left|f\right|^{2} d\nu\right)^{1/2} & \left(\int_{B} w\left|g\right|^{2} d\nu\right)^{1/2} \end{array}\right]\right)^{2}. \end{aligned} \tag{6.5}$$

**Corollary 3.** *Let $f \in L_w^2(\Omega,\nu)$, $g \in L_w^2(\Omega,\nu)$. The functional $L(\cdot,w;f,g) : \mathcal{A}_\nu \to [0,\infty)$ defined by (6.4) is a supermeasure.*

Consider the functional $L_{p,q}(\cdot,w;f,g) : \mathcal{A}_\nu \to [0,\infty)$ defined by

$$\begin{aligned}(6.6)\qquad & L_{p,q}(A,w;f,g) \\ &:= \left[\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu - \left|\int_A wfgd\nu\right|^2\right]^q \left(\int_A wd\nu\right)^{q\left(1-\frac{1}{p}\right)} \\ &= \left[\frac{\int_A w|f|^2 d\nu}{\int_A wd\nu}\frac{\int_A w|g|^2 d\nu}{\int_A wd\nu} - \left|\frac{\int_A wfgd\nu}{\int_A wd\nu}\right|^2\right]^q \left(\int_A wd\nu\right)^{q\left(3-\frac{1}{p}\right)},\end{aligned}$$

where $p,q \geq 1$.

We have:

**Theorem 8.** *Let $f \in L_w^2(\Omega,\nu)$, $g \in L_w^2(\Omega,\nu)$, then the functional $L_{p,q}(\cdot,w;f,g) : \mathcal{A}_\nu \to [0,\infty)$, with $p,q \geq 1$, defined by (6.6) is a supermeasure.*

Let $f \in L_w^2(\Omega,\nu)$, $g \in L_w^2(\Omega,\nu)$. We can also consider the functional $Q(\cdot,w;f,g) : \mathcal{A}_\nu \to [0,\infty)$ defined by

$$\begin{aligned}(6.7)\qquad Q(A,w;f,g) &= \left[\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu - \left|\int_A wfgd\nu\right|^2\right]^{1/2} \\ &= \sqrt{L(A,w;f,g)}.\end{aligned}$$

We also need the following result, see [10]:

**Lemma 6.** *Let $f \in L_w^2(\Omega,\nu)$, $g \in L_w^2(\Omega,\nu)$, then the functional $H(\cdot,w;f,g) : \mathcal{A}_\nu \to [0,\infty)$ defined by (6.7) is a supermeasure.*

Consider the functional $Q_{p,q}(\cdot,w;f,g) : \mathcal{A}_\nu \to [0,\infty)$ defined by

$$\begin{aligned}(6.8)\qquad & Q_{p,q}(A,w;f,g) \\ &:= \left[\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu - \left|\int_A wfgd\nu\right|^2\right]^{\frac{q}{2}} \left(\int_A wd\nu\right)^{q\left(1-\frac{1}{p}\right)} \\ &= \left[\frac{\int_A w|f|^2 d\nu}{\int_A wd\nu}\frac{\int_A w|g|^2 d\nu}{\int_A wd\nu} - \left|\frac{\int_A wfgd\nu}{\int_A wd\nu}\right|^2\right]^{\frac{q}{2}} \left(\int_A wd\nu\right)^{q\left(2-\frac{1}{p}\right)},\end{aligned}$$

$p,q \geq 1$.

We have:

**Theorem 9.** *Let $f \in L_w^2(\Omega,\nu)$, $g \in L_w^2(\Omega,\nu)$, then the functional $Q_{p,q}(\cdot,w;f,g) : \mathcal{A}_\nu \to [0,\infty)$, with $p,q \geq 1$, defined by (6.8) is a supermeasure.*

For some CBS's inequality related functionals and their properties see [3], [5], [14], [15] and [16].

Let $x = (x_i)_{i\in\mathbb{N}}$ and $x = (y_i)_{i\in\mathbb{N}}$ be sequences of complex numbers and $w = (w_i)_{i\in\mathbb{N}}$ a sequence of positive real numbers. Let $\Omega = \mathbb{N}$ and $\mathcal{P}_f(\mathbb{N})$ be the algebra of finite parts of natural numbers $\mathbb{N}$. By the monotonicity property of supermeasure on $\mathcal{P}_f(\mathbb{N})$ we have from the above results that the sequences

$$H_{n,p}(w;x,y) := \left[\left(\sum_{i=0}^{n} w_i |x_i|^2\right)^{1/2}\left(\sum_{i=0}^{n} w_i |y_i|^2\right)^{1/2} - \left|\sum_{i=0}^{n} w_i x_i y_i\right|\right]\left(\sum_{i=0}^{n} w_i\right)^{1-\frac{1}{p}} \tag{6.9}$$

and

$$L_{n,p}(w;x,y) := \left[\sum_{i=0}^{n} w_i |x_i|^2 \sum_{i=0}^{n} w_i |y_i|^2 - \left|\sum_{i=0}^{n} w_i x_i y_i\right|^2\right]\left(\sum_{i=0}^{n} w_i\right)^{1-\frac{1}{p}} \tag{6.10}$$

are monotonic nondecreasing and

$$H_{n,p}(w;x,y) \geq \max_{0\leq i\neq j\leq n}\left\{\left[\left(w_i |x_i|^2 + w_j |x_j|^2\right)^{1/2}\left(w_i |y_i|^2 + w_j |y_j|^2\right)^{1/2} - |w_i x_i y_i + w_j x_j y_j|\right] \times (w_i + w_j)^{1-\frac{1}{p}}\right\} \tag{6.11}$$

and

$$L_{n,p}(w;x,y) \geq \max_{0\leq i\neq j\leq n}\left\{\left[\left(w_i |x_i|^2 + w_j |x_j|^2\right)\left(w_i |y_i|^2 + w_j |y_j|^2\right) - |w_i x_i y_i + w_j x_j y_j|^2\right] \times (w_i + w_j)^{1-\frac{1}{p}}\right\} \tag{6.12}$$

for any $p \geq 1$.

Finally, the sequence

$$Q_{n,p}(w;x,y) := \left[\sum_{i=0}^{n} w_i |x_i|^2 \sum_{i=0}^{n} w_i |y_i|^2 - \left|\sum_{i=0}^{n} w_i x_i y_i\right|^2\right]^{1/2} \times \left(\sum_{i=0}^{n} w_i\right)^{1-\frac{1}{p}} \tag{6.13}$$

is also monotonic nondecreasing and we have the bound

$$Q_{n,p}(w;x,y) \geq \max_{0\leq i\neq j\leq n}\left\{\left[\left(w_i |x_i|^2 + w_j |x_j|^2\right)\left(w_i |y_i|^2 + w_j |y_j|^2\right) - |w_i x_i y_i + w_j x_j y_j|^2\right]^{1/2} \times (w_i + w_j)^{1-\frac{1}{p}}\right\} \tag{6.14}$$

for any $p \geq 1$.

## 7. Applications for Čebyšev's Inequality

We say that the pair of measurable functions $(f,g)$ are *synchronous* on $\Omega$ if

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \tag{7.1}$$

for $\nu$-a.e. $x, y \in \Omega$. If the inequality reverses in (7.1), the functions are called *asynchronous* on $\Omega$.

If $(f, g)$ are synchronous on $\Omega$ and $f,\ g,\ fg \in L_w(\Omega, \nu)$ then the following inequality, that is known in the literature as *Čebyšev's Inequality,* holds

$$\int_\Omega w d\nu \int_\Omega w f g d\nu \geq \int_\Omega w f d\nu \int_\Omega w g d\nu, \tag{7.2}$$

where $w(x) \geq 0$ for $\nu$ -a.e. (almost every) $x \in \Omega$.

We consider the *Čebyšev functional* $C(\cdot, w; f, g) : \mathcal{A}_\nu \to [0, \infty)$ defined by

$$C(A, w; f, g) := \int_A w d\nu \int_A w f g d\nu - \int_A w f d\nu \int_A w g d\nu. \tag{7.3}$$

The following result is known in the literature as *Korkine's identity:*

$$C(A, w; f, g) = \frac{1}{2} \int_A \int_A w(x) w(y) (f(x) - f(y)) (g(x) - g(y)) d\nu(x) d\nu(y). \tag{7.4}$$

The proof is obvious by developing the right side of (7.4) and using Fubini's theorem.

We have the following result, see [10]:

**Lemma 7.** *Let $(f, g)$ be synchronous on $\Omega$ and $f,\ g,\ fg \in L_w(\Omega, \nu)$. Then the Čebyšev functional defined by (7.3) is a supermeasure on $\mathcal{A}_\nu$.*

For $p, q \geq 1$ consider the two parameter functional $C_{p,q}(\cdot, w; \Phi, f) : \mathcal{A}_\nu \to [0, \infty)$,

$$\begin{aligned} &C_{p,q}(A, w; f, g) \\ &:= C^q(A, w; f, g) \left( \int_A w d\nu \right)^{q\left(1 - \frac{1}{p}\right)} \\ &= \left( \frac{\int_A w f g d\nu}{\int_A w d\nu} - \frac{\int_A w f d\nu}{\int_A w d\nu} \cdot \frac{\int_A w g d\nu}{\int_A w d\nu} \right)^q \left( \int_A w d\nu \right)^{q\left(3 - \frac{1}{p}\right)}. \end{aligned} \tag{7.5}$$

**Theorem 10.** *Let $(f, g)$ be synchronous on $\Omega$ and $f,\ g,\ fg \in L_w(\Omega, \nu)$ and $p, q \geq 1$. Then the functional $C_{p,q}(\cdot, w; f, g) : \mathcal{A}_\nu \to [0, \infty)$ defined by (7.5) is a supermeasure.*

The proof follows by Corollary 1 for $p, q \geq 1$.

For some Čebyšev's inequality related functionals and their properties see [3] and [17].

Let $x = (x_i)_{i \in \mathbb{N}}$ and $x = (y_i)_{i \in \mathbb{N}}$ be synchronous sequences of real numbers and $w = (w_i)_{i \in \mathbb{N}}$ a sequence of positive real numbers. Let $\Omega = \mathbb{N}$ and $\mathcal{P}_f(\mathbb{N})$ be the algebra of finite parts of natural numbers $\mathbb{N}$. By the monotonicity property of supermeasure on $\mathcal{P}_f(\mathbb{N})$ we have from the above results that the sequence

$$C_{n,p}(w; x, y) := \left( \sum_{i=0}^n w_i \sum_{i=0}^n w_i x_i y_i - \sum_{i=0}^n w_i x_i \sum_{i=0}^n w_i y_i \right) \left( \sum_{i=0}^n w_i \right)^{1 - \frac{1}{p}}$$

is monotonic nondecreasing and and we have the bound

$$C_{n,p}(w; x, y) \geq \frac{1}{2} \max_{0 \leq i \neq j \leq n} \left\{ w_i w_j (x_i - x_j)(y_i - y_j)(w_i + w_j)^{1 - \frac{1}{p}} \right\},$$

for any $p \geq 1$.

## 8. Applications for Hermite-Hadamard Inequalities

Let $I$ be an interval consisting of more than one point and $f:I\to\mathbb{R}$ a convex function. If $a,b\in I$ with $a<b$, then we have the well-known *Hermite-Hadamard inequality*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2}. \tag{8.1}$$

Suppose $f:I\to\mathbb{R}$ and for $f\in L[a,b]$ define the functionals

$$H([a,b];f):=\int_a^b f(t)\,dt-(b-a)f\left(\frac{a+b}{2}\right)$$

and

$$L([a,b];f):=\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt.$$

We have the following result concerning the properties of these mappings as functions of interval [18]:

**Lemma 8.** *Let $f:I\to\mathbb{R}$ be a convex function. Then*

*(i) For all $a,b,c\in I$ with $a\leq c\leq b$, we have*

$$0\leq H([a,c];f)+H([c,b];f)\leq H([a,b];f) \tag{8.2}$$

*and*

$$0\leq L([a,c];f)+L([c,b];f)\leq L([a,b];f), \tag{8.3}$$

*i.e. the functionals $H(\cdot;f)$ and $L(\cdot;f)$ are superadditive as functions of interval;*

*(ii) For all $[c,d]\subseteq[a,b]\subseteq I$, we have*

$$0\leq H([c,d];f)\leq H([a,b];f) \tag{8.4}$$

*and*

$$0\leq L([c,d];f)\leq L([a,b];f), \tag{8.5}$$

*i.e. the functionals $H(\cdot;f)$ and $L(\cdot;f)$ are monotonic nondecreasing as functions of interval.*

For $p,q\geq1$ consider the two parameter functionals

$$\begin{aligned}H_{p,q}([a,b];f)&:=(b-a)^{q\left(1-\frac{1}{p}\right)}H^q([a,b];f)\\&=(b-a)^{q\left(1-\frac{1}{p}\right)}\left[\int_a^b f(t)\,dt-(b-a)f\left(\frac{a+b}{2}\right)\right]^q\end{aligned} \tag{8.6}$$

and

$$\begin{aligned}L_{p,q}([a,b];f)&:=(b-a)^{q\left(1-\frac{1}{p}\right)}L^q([a,b];f)\\&=(b-a)^{q\left(1-\frac{1}{p}\right)}\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right]^q.\end{aligned} \tag{8.7}$$

**Theorem 11.** *Let $f:I\to\mathbb{R}$ be a convex function. Then for any $p,q\geq1$ we have that the mappings $H_{p,q}(\cdot;f)$ and $L_{p,q}(\cdot;f)$ are superadditive and monotonic nondecreasing as functions of interval.*

The proof follows by Corollary 1 for $p,q\geq 1$.

For an arbitrary function $f:I\to\mathbb{R}$ we introduce the mapping

$$S([a,b];f):=(b-a)\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right],$$

where $a,b\in I$ with $a<b$.

We have [18]:

**Lemma 9.** *Let $f:I\to\mathbb{R}$ a convex function. Then the mapping $S(\cdot;f)$ is a superadditive and monotonic nondecreasing function of interval.*

For $p,q\geq 1$ consider the two parameter functional

$$S_{p,q}([a,b];f):=(b-a)^{q\left(2-\frac{1}{p}\right)}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]^{q}.$$

Using Corollary 1 for $p,q\geq 1$, we then have:

**Theorem 12.** *Let $f:I\to\mathbb{R}$ be a convex function. Then for any $p,q\geq 1$ we have that the mapping $S_{p,q}(\cdot;f)$ is superadditive and monotonic nondecreasing as function of interval.*

If we use the superadditivity of the functional $H_{p,q}([a,b];f)$ with $p,q\geq 1$ on the intervals $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$ then we get

$$\begin{aligned}&(b-a)^{q\left(1-\frac{1}{p}\right)}\left[\int_a^b f(t)\,dt-(b-a)f\left(\frac{a+b}{2}\right)\right]^q\\&\geq\left(\frac{b-a}{2}\right)^{q\left(1-\frac{1}{p}\right)}\left[\int_a^{\frac{a+b}{2}} f(t)\,dt-\frac{b-a}{2}f\left(\frac{3a+b}{4}\right)\right]^q\\&+\left(\frac{b-a}{2}\right)^{q\left(1-\frac{1}{p}\right)}\left[\int_{\frac{a+b}{2}}^b f(t)\,dt-\frac{b-a}{2}f\left(\frac{a+3b}{4}\right)\right]^q,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\left[\int_a^b f(t)\,dt-(b-a)f\left(\frac{a+b}{2}\right)\right]^q\\&\geq\frac{1}{2^{q\left(1-\frac{1}{p}\right)}}\left\{\left[\int_a^{\frac{a+b}{2}} f(t)\,dt-\frac{b-a}{2}f\left(\frac{3a+b}{4}\right)\right]^q\right.\\&\qquad\left.+\left[\int_{\frac{a+b}{2}}^b f(t)\,dt-\frac{b-a}{2}f\left(\frac{a+3b}{4}\right)\right]^q\right\}\end{aligned}\tag{8.8}$$

for $p,q\geq 1$.

Similarly, we have

$$\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right]^q \geq \frac{1}{2^{q\left(1-\frac{1}{p}\right)}}\left\{\left[\frac{f(a)+f\left(\frac{a+b}{2}\right)}{4}(b-a)-\int_a^{\frac{a+b}{2}} f(t)\,dt\right]^q + \left[\frac{f\left(\frac{a+b}{2}\right)+f(b)}{4}(b-a)-\int_{\frac{a+b}{2}}^b f(t)\,dt\right]^q\right\} \tag{8.9}$$

and

$$\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]^q \geq \frac{1}{2^{q\left(2-\frac{1}{p}\right)}}\left[\frac{f(a)+f\left(\frac{a+b}{2}\right)}{2}-f\left(\frac{3a+b}{4}\right)\right]^q + \left[\frac{f\left(\frac{a+b}{2}\right)+f(b)}{2}-f\left(\frac{a+3b}{4}\right)\right]^q \tag{8.10}$$

for $p,q\geq 1$.

By the monotonicity property we also have the inequalities:

$$(b-a)^{1-\frac{1}{p}}\left[\int_a^b f(t)\,dt-(b-a)\,f\left(\frac{a+b}{2}\right)\right] \geq (d-c)^{1-\frac{1}{p}}\left[\int_c^d f(t)\,dt-(d-c)\,f\left(\frac{c+d}{2}\right)\right] \tag{8.11}$$

$$(b-a)^{1-\frac{1}{p}}\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right] \geq (d-c)^{1-\frac{1}{p}}\left[\frac{f(c)+f(d)}{2}(d-c)-\int_c^d f(t)\,dt\right] \tag{8.12}$$

and

$$(b-a)^{2-\frac{1}{p}}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right] \geq (d-c)^{2-\frac{1}{p}}\left[\frac{f(c)+f(d)}{2}-f\left(\frac{c+d}{2}\right)\right] \tag{8.13}$$

for any $[c,d]\subset[a,b]$.

## 9. Applications for Convex Functions

Consider a convex function $f:I\subset\mathbb{R}\to\mathbb{R}$ defined on the interval $I$ of the real line $\mathbb{R}$ and two distinct elements $a,b\in I$ with $a<b$. We denote by $[a,b]$ the closed segment defined by $\{(1-t)\,a+tb,\ t\in[0,1]\}$. We also define the functional of interval

$$C_t([a,b];f):=(1-t)\,f(a)+tf(b)-f((1-t)\,a+tb)\geq 0 \tag{9.1}$$

where $a, b \in I$ with $a < b$ and $t \in [0,1]$ is fixed.

We have [10] and [9]:

**Lemma 10.** *Let $f : I \to \mathbb{R}$ be a convex function and $t \in (0,1)$. Then the mapping $C_t(\cdot; f)$ is a superadditive and monotonic nondecreasing function of interval.*

For $t = \frac{1}{2}$ we consider the functional

$$C([a,b];f) := C_{\frac{1}{2}}([a,b];f) = \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right).$$

**Corollary 4.** *Let $f : I \to \mathbb{R}$ be a convex function. Then the mapping $C(\cdot; f)$ is a superadditive and monotonic nondecreasing function of interval.*

We can also define the symmetric functional

$$\begin{aligned} D_t([a,b];f) &:= \frac{1}{2}\left[C_t([a,b];f) + C_{1-t}([a,b];f)\right] \\ &= \frac{f(a)+f(b)}{2} - \frac{1}{2}\left[f((1-t)a+tb) + f((1-t)b+ta)\right], \end{aligned}$$

where $t \in [0,1]$ is fixed.

**Corollary 5.** *Let $f : I \to \mathbb{R}$ be a convex function and $t \in (0,1)$. Then the mapping $D_t(\cdot; f)$ is a superadditive and monotonic nondecreasing function of interval.*

Perhaps the most interesting functional we can consider from the above is the following one:

$$\begin{aligned} \Theta([a,b];f) &:= \frac{f(a)+f(b)}{2} - \int_0^1 f((1-t)a+tb)\,dt \\ &= \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(s)\,ds \geq 0, \end{aligned} \tag{9.2}$$

which is related to the second Hermite-Hadamard inequality.

We observe that

$$\Theta([a,b];f) = \int_0^1 C_t([a,b];f)\,dt. \tag{9.3}$$

**Corollary 6.** *Let $f : I \to \mathbb{R}$ be a convex function. Then the mapping $\Theta(\cdot; f)$ is a superadditive and monotonic nondecreasing function of interval.*

We can define the associated two parameter $p, q \geq 1$ functionals

$$\begin{aligned} C_{p,q,t}([a,b];f) &:= (b-a)^{q\left(1-\frac{1}{p}\right)}\left[C_t([a,b];f)\right]^q \\ &= (b-a)^{q\left(1-\frac{1}{p}\right)}\left[(1-t)f(a) + tf(b) - f((1-t)a+tb)\right]^q, \end{aligned}$$

$$\begin{aligned} C_{p,q}([a,b];f) &:= (b-a)^{q\left(1-\frac{1}{p}\right)}\left[C([a,b];f)\right]^q \\ &= (b-a)^{q\left(1-\frac{1}{p}\right)}\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right]^q, \end{aligned}$$

$$
\begin{aligned}
D_t([a,b];f) &:= \frac{1}{2^q}(b-a)^{q\left(1-\frac{1}{p}\right)}\left[C_t([a,b];f)+C_{1-t}([a,b];f)\right]^q \\
&= (b-a)^{q\left(1-\frac{1}{p}\right)} \\
&\quad \times \left[\frac{f(a)+f(b)}{2}-\frac{1}{2}\left[f((1-t)a+tb)+f((1-t)b+ta)\right]\right]^q
\end{aligned}
$$

and

$$
\begin{aligned}
\Theta_{p,q}([a,b];f) &:= (b-a)^{q\left(1-\frac{1}{p}\right)}\left[\Theta([a,b];f)\right]^q \\
&= (b-a)^{q\left(1-\frac{1}{p}\right)}\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(s)\,ds\right]^q .
\end{aligned}
$$

Using Corollary 1 for $p,q\geq 1$, we then have:

**Theorem 13.** *Let $f:I\to\mathbb{R}$ be a convex function and $t\in(0,1)$. Then for any $p,q\geq 1$ we have that the mappings $C_{p,q,t}(\cdot;f)$, $C_{p,q}(\cdot;f)$, $D_t(\cdot;f)$ and $\Theta_{p,q}(\cdot;f)$ are superadditive and monotonic nondecreasing as functions of interval.*

Various inequalities similar with the ones from the previous section can be stated, however we do not present them here.

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