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SOME INEQUALITIES AND IDENTITIES FOR φ-FRAMES

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Abstract. In this paper we introduce the concept of a φ-frame in a vector space. Some general results about φ-frame are proved. In particular, we stablish some fundamental identities and inequalities of interest.

1. Introduction and Preliminaries

The frame was first introduced by Duffin and Schaeffer [5] in the study of nonharmonic Fourier series in 1952. It is a generalization of the Riesz basis. Because frames have many nice properties, they have been widely used in signal processing, data compression, sampling theory and many mathematical fields. A frame for a complex Hilbert space $\mathcal{H}$ is a family of vectors $\{f_i\}_{i\in I}$ in $\mathcal{H}$ so that there are two positive constants A and B satisfying

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2, \ \ (f \in \mathcal{H}) . \tag{1.1}$$

The constants A and B are called the lower and upper frame bounds, repectively. A frame is said to be tight whenever $A = B$ and if we can take $A = B = 1$ it is called a Parseval frame.

If the right-hand inequality of (1.1) holds, then we say that $\{f_i\}_{i\in I}$ is a Bessel sequence for $\mathcal{H}$ with bound B.

The analytic operator associated to the frame $\{f_i\}_{i\in I}$ is defined as $T : L^2 \to \mathcal{H}$ by $T\{a_i\} = \sum\limits_{i\in I} a_i f_i$. It is easy to see that $T^* : \mathcal{H} \to L^2$ such that $T^*(f) = \{\langle f, f_i\rangle\}_{i\in I}$.

The frame operator for the frame is the positive, self adjoint invertible operator $S = TT^* : \mathcal{H} \to \mathcal{H}$ satisfying

$$Sf = \sum_{i\in I} \langle f, f_i\rangle f_i, \ \ (f \in \mathcal{H}) .$$

This provides the frame decomposition

$$f = S^{-1}Sf = \sum_{i\in I} \langle f, f_i\rangle \widetilde{f_i} = \sum_{i\in I} \left\langle f, \widetilde{f_i}\right\rangle f_i,$$

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where $\widetilde{f}_i = S^{-1} f_i$. The family $\left\{\widetilde{f}_i\right\}_{i \in I}$ is also a frame for $\mathcal{H}$, called the canonical dual frame of $\{f_i\}_{i\in I}$. If $\{f_i\}_{i\in I}$ is a Bessel sequence in $\mathcal{H}$, for every $J \subset I$ we define the operator S_J by

$$S_J f = \sum_{i \in J} \langle f, f_i \rangle f_i.$$

We refer to [3] for an introduction to the frame theory.

Later we will need the following important result from bilinear functionals. Recall that $\varphi : \mathscr{E} \times \mathscr{E} \to \mathbb{C}$ is the bilinear functional if satisfying the following two conditions:

(a) $\varphi(\alpha x_1 + \beta x_2, y) = \alpha\varphi(x_1, y) + \beta\varphi(x_2, y)$,

(b) $\varphi(x, \alpha y_1 + \beta y_2) = \overline{\alpha}\varphi(x, y_1) + \overline{\beta}\varphi(x, y_2)$.

for any scalars α and β and any $x, x_1, x_2, y, y_1, y_2 \in \mathscr{E}$.

Definition 1.1. *Let φ be a bilinear functional on vector space $\mathscr{E}$*

(a) φ is called symmetric if $\varphi(x, y) = \varphi(y, x)$.

(b) φ is called positive if $\varphi(x, y) \geq 0$ for all $x \in \mathscr{E}$.

(c) φ is called strictly if it is positive and $\varphi(x, y) > 0$ for all $x, y \in \mathscr{E}$.

(d) φ is called Cauchy Schwartz if $(\varphi(x, y))^2 \leq \varphi(x, x)\varphi(y, y)$.

Definition 1.2. *Let φ be a positive bounded bilinear functional. The corresponding quadratic form associated to φ is defined as*

$$\Phi(x) = \varphi(x, x).$$

Also, anywhere Φ is used, it means quadratic form.

Definition 1.3. *If $\mathscr{E}$ is a vector space, φ is a positive bounded bilinear functional, the following defines a semi norm on $\mathscr{E}$*

$$\|x\|_\varphi = \sqrt{\varphi(x, x)}. \tag{1.2}$$

Also, if $A \in \mathcal{B}(\mathscr{E})$ we define $\|A\|_\varphi$ to be

$$\|A\|_\varphi = \sup\left\{|\varphi(Ax, y)| : \ \|x\|_\varphi = \|y\|_\varphi = 1\right\}. \tag{1.3}$$

Lemma 1.1. *By using* (1.2)*, we have*

$$\sup\left\{|\varphi(x, y)| : \ \|x\|_\varphi = \|y\|_\varphi = 1\right\} = 1. \tag{1.4}$$

Remark 1.1. *Let $\varphi \neq \varphi'$ and φ' be a positive bilinear functional, then the semi norm of φ is defined as follows:*

$$\|\varphi\| = \sup\left\{\sqrt{\varphi'(x, x)} : \ \|x\|_{\varphi'} = 1\right\}.$$

Definition 1.4. *The operator A in $\mathcal{B}(\mathscr{E})$ is called φ-positive if for all $x \in \mathscr{E}$, $\varphi(Ax, x) \geq 0$. We note $A \geq B$ if $A - B \geq 0$.*

We would like to refer the interested reader to [4] for an extensive account on bilinear functional. Now, we generalize the notion of frame by using bilinear functional.

Definition 1.5. *A sequence $\{e_k\}_{k=1}^m$ in a vector space $\mathscr{E}$ is a basis, if the following conditions are satisfied:*

(a) $\mathscr{E} = span\,\{e_k\}_{k=1}^m$;
(b) $\{e_k\}_{k=1}^m$ is linearly independent.

Remark 1.2. *As consequence of Definition 1.5, every $f \in \mathscr{E}$ has a unique representation in terms of the elements in the basis, i.e., there exists unique scalar coefficients $\{c_k\}_{k=1}^m$ such that*

$$f = \sum_{k=1}^{m} c_k e_k.$$

Definition 1.6. *If $\{e_k\}_{k=1}^m$ is a φ-orthonormal basis, i.e., a basis for which*

$$\varphi\,(e_k, e_j) = \delta_{k,j} = \begin{cases} 1 \text{ if } k = j \\ 0 \text{ if } k \neq j \end{cases}$$

then the coefficients $\{c_k\}_{k=1}^m$ are easy to find:

$$\varphi\,(f, e_j) = \varphi\left(\sum_{k=1}^{m} c_k e_k, e_j\right) = \sum_{k=1}^{m} c_k \varphi\,(e_k, e_j) = c_j$$

so

$$f = \sum_{k=1}^{m} \varphi\,(f, e_k) e_k.$$

Definition 1.7. *A sequence $\{f_k\}_{k=1}^\infty$ in a vector space $\mathscr{E}$ is called φ-frame if there exist $A, B > 0$ such that*

$$A\Phi\,(f) \leq \sum_{k=1}^{n} \left|\varphi\,(f, f_k)\right|^2 \leq B\Phi\,(f) \tag{1.5}$$

for all $f \in \mathscr{E}$. The constants A and B are called φ-frame bounds. If $A = B$, this is a tight φ-frame and if $A = B = 1$ this is a Parseval φ-frame.

Definition 1.8. *Let $X, Y \in \mathscr{E}$ and $A : X \to Y$ be a linear operator, and let φ_1, φ_2 be bounded bilinear functionals on X and Y, respectively. In this case, A is called (φ_1, φ_2)-adjointable if there is a linear mapping $A^* : Y \to X$ that for all $x \in X$ and $y \in Y$,*

$$\varphi_2\,(Ax, y) = \varphi_1\,(x, A^* y)\,.$$

Definition 1.9. *Consider a vector space $\mathscr{E}$ equipped with a frame $\{f_k\}_{k=1}^m$ and define a linear mapping*

$$T:\mathbb{C}^m\to\mathscr{E},\ T\{c_k\}_{k=1}^m=\sum_{k=1}^m c_kf_k.$$

T is called the φ-pre-frame operator. The adjoint operator is given by

$$T^*:\mathscr{E}\to\mathbb{C}^m,\ T^*f=\{\varphi(f,f_k)\}_{k=1}^m$$

in fact by the usual inner product on $\mathbb{C}^m$ as the bilinear functional φ' we have

$$\varphi(Tx,y)=\varphi\left(\sum_{k=1}^m c_kf_k,y\right)=\sum_{k=1}^m c_k\varphi(f_k,y)$$

and

$$\varphi'(x,T^*y)=\varphi'(\{c_k\}_{k=1}^m,\{\varphi(y,f_k)\}_{k=1}^m)=\sum_{k=1}^m c_k\varphi(f_k,y).$$

T^* is called the analytic operator and by composing T with its adjoint T^*, we obtain the φ-frame operator

$$S:\mathscr{E}\to\mathscr{E},\ Sf=TT^*f=\sum_{k=1}^m\varphi(f,f_k)f_k.$$

Note that in terms of the φ-frame operator,

$$\varphi(Tf,f)=\sum_{k=1}^m|\varphi(f,f_k)|^2,\ f\in\mathscr{E}.$$

2. Discussion of the φ-frame

We start this section by pointing out the following remark:

Remark 2.1. *Let φ be a Cauchy Schwarz bounded bilinear functional, then*

$$\sum_{k=1}^m|\varphi(f,f_k)|^2\le\sum_{k=1}^m\Phi(f_k)\Phi(f)\,. \tag{2.1}$$

Proposition 2.1. *Let $\{f_k\}_{k=1}^m$ be a sequence in $\mathscr{E}$. Then $\{f_k\}_{k=1}^m$ is a φ-frame for span $\{f_k\}_{k=1}^m$.*

Proof. We can assume that none all f_k are zero. As we have seen in Remark 2.1, the upper φ-frame condition is satisfied with $B=\sum\limits_{k=1}^m\Phi(f_k)$. Now let

$$W=span\{f_k\}_{k=1}^m$$

and consider the continuous mapping

$$\psi:W\to\mathbb{R},\ \psi(f)=\sum_{k=1}^m|\varphi(f,f_k)|^2.$$

The unit ball in W is compact since, W is finite dimensional. So the function ψ takes its infimum on the unit ball W. We can find $g \in W$ with $\sqrt{\Phi(g)} = 1$ such that

$$A = \sum_{k=1}^{m} |\varphi(g, f_k)|^2 = \inf \left\{ \sum_{k=1}^{m} |\varphi(f, f_k)|^2 : \ f \in W, \ \sqrt{\Phi(f)} = 1 \right\}$$

It is clear that $A > 0$. Now given $f \in W$, $f \neq 0$, we have

$$\sum_{k=1}^{m} |\varphi(f, f_k)|^2 = \sum_{k=1}^{m} \varphi\left(\frac{f}{\sqrt{\Phi(f)}}, f_k \right)^2 |\Phi(f)| \geq A |\Phi(f)|.$$

□

Corollary 2.1. *A family of elements $\{f_k\}_{k=1}^{m}$ in $\mathscr{E}$ is a φ-frame for $\mathscr{E}$ if and only if span $\{f_k\}_{k=1}^{m} = \mathscr{E}$.*

Theorem 2.1. *Let $\{f_k\}_{k=1}^{m}$ be a φ-frame for $\mathscr{E}$ with φ-frame operator S. Then*

(a) S is invertible and self adjoint.

(b) Every $f \in \mathscr{E}$ can be represented as

$$f = \sum_{k=1}^{m} \varphi\left(f, S^{-1} f_k\right) f_k = \sum_{k=1}^{m} \varphi(f, f_k) S^{-1} f_k. \tag{2.2}$$

Proof. Since $S = TT^*$, it is clear that S is a self adjoint. We now prove that S is injective. Let $f \in \mathscr{E}$ and assume that $Sf = 0$. Then

$$0 = \varphi(Sf, f) = \sum_{k=1}^{m} |\varphi(f, f_k)|^2.$$

Implying by the φ-frame condition that $f = 0$. That S is injective actually implies that S is surjective, but let us give direct proof. The φ-frame condition implies by Corollary 2.1 that *span* $\{f_k\}_{k=1}^{m} = \mathscr{E}$, so the φ-pre frame operator T is surjective. Given $f \in \mathscr{E}$ we can therefore finde $g \in \mathscr{E}$ such that $Tg = f$; we can choose $g \in N_T^{\perp} = R_{T^*}$, so it follows that $R_S = R_{TT^*} = \mathscr{E}$. Thus S is surjective, as claimed. Each $f \in \mathscr{E}$ has the representation

$$f = SS^{-1}f = TT^*S^{-1}f = \sum_{k=1}^{m} \varphi\left(S^{-1}f, f_k\right) f_k$$

Using that S is self adjoint, we arrive at

$$f = \sum_{k=1}^{m} \varphi\left(f, S^{-1} f_k\right) f_k.$$

The second representation in (2.2) is obtained in the same way, using that $f = S^{-1}Sf$. □

Theorem 2.2. *Let $\{f_k\}_{k=1}^m$ be a φ-frame for $\mathscr{E}$ with φ-frame operator T. Then If $f \in \mathscr{E}$ also has the representation $f = \sum_{k=1}^{m} c_k f_k$ for some scalar coefficients $\{c_k\}_{k=1}^m$, then*

$$\sum_{k=1}^{m} |c_k|^2 = \sum_{k=1}^{m} \left|\varphi\left(f, T^{-1} f_k\right)\right|^2 + \sum_{k=1}^{m} \left|c_k + \varphi\left(f, T^{-1} f_k\right)\right|^2. \tag{2.3}$$

Proof. Suppose that $f = \sum_{k=1}^{m} c_k f_k$. We can write

$$\{c_k\}_{k=1}^m = \{c_k\}_{k=1}^m - \left\{\varphi\left(f, T^1 f_k\right)\right\}_{k=1}^m + \left\{\varphi\left(f, T^1 f_k\right)\right\}_{k=1}^m .$$

By the choice of $\{c_k\}_{k=1}^m$ we have

$$\sum_{k=1}^{m} \left(c_k - \varphi\left(f, T^{-1} f_k\right)\right) f_k = 0$$

i.e.,

$$\{c_k\}_{k=1}^m - \left\{\varphi\left(f, T^1 f_k\right)\right\}_{k=1}^m \in N_S = R_{S^*}^{\perp};$$

since

$$\left\{\varphi\left(f, T^{-1} f_k\right)\right\}_{k=1}^m = \left\{\varphi\left(T^{-1} f, f_k\right)\right\}_{k=1}^m \in R_{S^*}$$

we obtain (2.3). ☐

Remark 2.2. *If $\{f_k\}_{k=1}^m$ is a φ-frame but not a basis, there exist non zero sequences $\{d_k\}_{k=1}^m$ such that $\sum_{k=1}^{m} d_k f_k = 0$. Therefore $f \in \mathscr{E}$ can be written*

$$f = \sum_{k=1}^{m} \varphi\left(f, T^{-1} f_k\right) f_k + \sum_{k=1}^{m} d_k f_k$$

and

$$= \sum_{k=1}^{m} \left(\varphi\left(f, T^{-1} f_k\right) + d_k\right) f_k$$

showing that f has many representations as seperpositions of the φ-frame elements.

Proposition 2.2. *Assume that $\{f_k\}_{k=1}^m$ is a basis for $\mathscr{E}$. Then there exists a unique family $\{g_k\}_{k=1}^m$ in $\mathscr{E}$ such that*

$$f = \sum_{k=1}^{m} \varphi\left(f, g_k\right) f_k, \ \forall f \in \mathscr{E}. \tag{2.4}$$

Proof. The existence of a family $\{g_k\}_{k=1}^m$ satisfying (2.4) follows from Theorem 2.1; also uniqueness is clearly.

Remark 2.3. *Applying* (2.4) *on a fixed element f_j and using that $\{f_k\}_{k=1}^m$ is a basis, we obtain that $\varphi\left(f_j, g_k\right) = \delta_{j,k}$ for all $k = 1, 2, ..., m$.*

□

Theorem 2.3. *Let $\{f_k\}_{k=1}^m$ be a φ-frame for subspace F of the vector space $\mathscr{E}$. Then the φ-orthogonal projection of $\mathscr{E}$ onto F is given by*

$$Pf = \sum_{k=1}^{m} \varphi\left(f, T^{-1} f_k\right) f_k. \tag{2.5}$$

Proof. It is enough to prove that if we define P by (2.5), then

$$Pf = f \text{ for } f \in F \text{ and } Pf = 0 \text{ for } f \in F^{\perp}.$$

The first equation follows by Theorem 2.1, and the second by the fact that the range of T^{-1} equals F because T is a bijection on F. □

3. Some Inequality and Identity related to the φ-Frame

As an application of the Sections 1 and 2, we give the following inequalities. Also, by using the model technique of Balan et al. [1, 2] and Gavruta [6], we obtain new identity for Parseval φ-Frame.

Theorem 3.1. *Let $\{f_i\}_{i\in I}$ be a φ-frame for a vector space $\mathscr{E}$ with frame boundes A, B. Let $J \subset I$, so that $\{f_i\}_{i\in J}$ has Bessel bound $B(J) < A$. Then $\{f_i\}_{i\in J^c}$ is a φ-frame for $\mathscr{E}$.*

Proof. Since $\{f_i\}_{i\in J^c}$ has B as a Bessel bound, we only need to check its lower frame bound. For this just compute for any $f \in \mathscr{E}$

$$\begin{aligned}\sum_{i\in J^c} |\varphi(f, f_i)|^2 &= \sum_{i\in I} |\varphi(f, f_i)|^2 - \sum_{i\in J} |\varphi(f, f_i)|^2 \\ &\geq A\Phi(f) - B(J)\Phi(f) = (A - B(J))\Phi(f).\end{aligned}$$

Since $A - B(J) > 0$, we deduce the desired result. □

Corollary 3.1. *Let $\{f_i\}_{i\in I}$ be a Parseval φ-frame for $\mathscr{E}$ and $J \subset I$. In order for $\{f_i\}_{i\in J}$ to be a φ-frame for $\mathscr{E}$ is necessary and sufficient that $B(J^c) < 1$. In this case, the optimal lower frame bound for $\{f_i\}_{i\in J}$ is $1 - B(J^c)$.*

Proof. For any $f \in \mathscr{E}$ we have

$$\begin{aligned}\sum_{i\in J} |\varphi(f, f_i)|^2 &= \sum_{i\in I} |\varphi(f, f_i)|^2 - \sum_{i\in J^c} |\varphi(f, f_i)|^2 \\ &\geq \Phi(f) - B(J^c)\Phi(f) = (1 - B(J^c))\Phi(f).\end{aligned}$$

It is easy to see that the inequality above is optimal, hence the proof is complete. □

The following result can be stated as well.

Theorem 3.2. *Assume that φ is a bounded positive bilinear functional. If $U, V \in \mathcal{L}(\mathscr{E})$ are φ-self adjoint operators satisfying $U + V = 1_{\mathscr{E}}$, then for all $f \in \mathscr{E}$ we have*

$$\varphi(Uf, f) + \Phi(Vf) = \varphi(Vf, f) + \Phi(Vf) \geq \frac{3}{4}\Phi(f)$$

Proof. We have

$$\begin{aligned}
\varphi(Uf, f) + \Phi(Vf) &= \varphi(Uf, f) + \varphi(Vf, Vf) \\
&= \varphi((I_{\mathscr{E}} - V) f, f) + \varphi\left(V^2 f, f\right) \\
&= \varphi\left(\left(V^2 - V + I_{\mathscr{E}}\right) f, f\right) \\
&= \varphi(Vf, f) + \varphi(Uf, Uf) + \varphi\left((I_{\mathscr{E}} - V)^2 f, f\right) \\
&= \varphi\left(\left(V^2 f - V + I_{\mathscr{E}}\right) f, f\right) \\
&= \varphi\left(\left(\left(V - \frac{1}{2}I_{\mathscr{E}}\right)^2 + \frac{3}{4}I_{\mathscr{E}}\right) f, f\right) \\
&\geq \frac{3}{4}\Phi(f).
\end{aligned}$$

□

Remark 3.1. *We consider now $\{f_i\}_{i\in I}$, a φ-frame for $\mathscr{E}$ with S its frame operator and $\left\{\widetilde{f}_i\right\}_{i\in I}$ its canonical dual frame and $J \subset I$. We have*

$$S_J + S_{J^c} = S$$

hence

$$S^{-\frac{1}{2}} S_J S^{-\frac{1}{2}} + S^{-\frac{1}{2}} S_{J^c} S^{-\frac{1}{2}} = 1_{\mathscr{E}}$$

Proof. If in the Theorem 3.2 we take $U = S^{-\frac{1}{2}} S_J S^{-\frac{1}{2}}$, $V = S^{-\frac{1}{2}} S_{J^c} S^{-\frac{1}{2}}$ and $S^{\frac{1}{2}} f$ instead of f, we get

$$\begin{aligned}
\varphi\left(S^{-\frac{1}{2}} S_J f, S^{\frac{1}{2}} f\right) + \Phi\left(S^{-\frac{1}{2}} S_{J^c} f\right) &= \varphi\left(S^{-\frac{1}{2}} S_J f, S^{\frac{1}{2}} f\right) + \Phi\left(S^{-\frac{1}{2}} S_J f\right) \\
&\geq \frac{3}{4}\Phi\left(S^{\frac{1}{2}} f\right),
\end{aligned}$$

or

$$\begin{aligned}
\varphi(S_J f, f) + \varphi\left(S^{-\frac{1}{2}} S_{J^c} f, S^{-\frac{1}{2}} S_{J^c} f\right) &= \varphi(S_{J^c} f, f) + \varphi\left(S^{-1} S_J f, S_J f\right) \\
&\geq \frac{3}{4}\varphi(Sf, f).
\end{aligned}$$

□

The following result also holds (see [6] for the case of Hilbert space).

Theorem 3.3. *Let $\{f_i\}_{i\in I}$ be a φ-frame for $\mathscr{E}$ and $\{g_i\}_{i\in I}$ be an alternative dual of $\{f_i\}_{i\in I}$. Then for all $J\subset I$ and all $f\in\mathscr{E}$, we have*

$$\begin{aligned}
&\mathrm{Re}\sum_{i\in J}\varphi(f,g_i)\overline{\varphi(f,f_i)}+\Phi\left(\sum_{i\in J^c}\varphi(f,g_i)f_i\right)\\
&=\mathrm{Re}\sum_{i\in J}\varphi(f,g_i)\overline{\varphi(f,f_i)}+\Phi\left(\sum_{i\in J}\varphi(f,g_i)f_i\right)\\
&\geq\frac{3}{4}\Phi(f).
\end{aligned}$$

Proof. For every $J\subset I$ we define the operator L_J by

$$L_Jf=\sum_{i\in J}\varphi(f,g_i)f_i.$$

By the Cauchy-Schwartz inequality it follows that this series converges unconditionally and $L_J\in\mathcal{L}(\mathscr{E})$. Since $L_J+L_{J^c}=I_{\mathbb{E}}$,

$$\begin{aligned}
\varphi((L_J^*L_J)f,f)+\frac{1}{2}\varphi((L_{J^c}^*L_{J^c})f,f)&=\varphi((L_{J^c}^*L_{J^c})f,f)+\frac{1}{2}\varphi((L_J^*+L_J^*)f,f)\\
&\geq\frac{3}{4}\Phi(f),
\end{aligned}$$

or

$$\begin{aligned}
&\Phi\left(\sum_{i\in J}\varphi(f,g_i)f_i\right)+\frac{1}{2}\left(\overline{\varphi(L_{J^c}f,f)}+\varphi(L_{J^c}f,f)\right)\\
&=\Phi\left(\sum_{i\in J^c}\varphi(f,g_i)f_i\right)+\frac{1}{2}\left(\overline{\varphi(L_Jf,f)}+\varphi(L_Jf,f)\right)\\
&\geq\frac{3}{4}\Phi(f).
\end{aligned}$$

That is the relation states in the theorem. □

To prove Theorem 3.4, we need the following lemma.

Lemma 3.1. *If S,T are operators on $\mathbb{E}$ satisfying $S+T=I$, then $S-T=S^2-T^2$.*

$$S-T=S-(I-S)=2S-I=S^2-\left(I-2S+S^2\right)=S^2-(I-S)^2=S^2-T^2.$$

Theorem 3.4. *Let $\{f_i\}_{i\in I}$ be a φ-frame for $\mathscr{E}$ with canonical frame $\left\{\widetilde{f}_i\right\}_{i\in I}$. Then for all $J\subset I$ and for all $f\in\mathscr{E}$ we have*

$$\sum_{i\in J}|\varphi(f,f_i)|^2-\sum_{i\in I}\left|\varphi\left(S_Jf,\widetilde{f}_i\right)\right|^2=\sum_{i\in J^c}|\varphi(f,f_i)|^2-\sum_{i\in I}\left|\varphi\left(S_{J^c}f,\widetilde{f}_i\right)\right|^2.$$

Proof. Let S denote the frame operator for $\{f_i\}_{i\in I}$. Since $S = S_J + S_{J^c}$, it follows that $I = S^{-1}S_J + S^{-1}S_{J^c}$. Applying Lemma 3.1 to the two operators $S^{-1}S_J$ and $S^{-1}S_{J^c}$ yields

$$S^{-1}S_J - S^{-1}S_JS^{-1}S_J = S^{-1}S_{J^c} - S^{-1}S_{J^c}S^{-1}S_{J^c}. \tag{3.1}$$

Further, for every $f, g \in \mathscr{E}$ we obtain

$$\varphi\left(S^{-1}S_Jf, g\right) - \varphi\left(S^{-1}S_JS^{-1}S_Jf, g\right) = \varphi\left(S_Jf, S^{-1}g\right) - \varphi\left(S^{-1}S_Jf, S_JS^{-1}g\right). \tag{3.2}$$

Now, we choose g to be $g = Sf$. Then we can continuous the equality (3.2) in the following as

$$= \varphi\left(S_Jf, f\right) - \varphi\left(S^{-1}S_Jf, S_Jf\right) = \sum_{i\in J} \left|\varphi\left(f, f_i\right)\right|^2 - \sum_{i\in I} \left|\varphi\left(S_Jf, \widetilde{f}_i\right)\right|^2.$$

Setting equality (3.2) equal to the corresponding equality for J^c and using (3.1), we finally get

$$\sum_{i\in J} \left|\varphi\left(f, f_i\right)\right|^2 - \sum_{i\in I} \left|\varphi\left(S_Jf, \widetilde{f}_i\right)\right|^2 = \sum_{i\in J^c} \left|\varphi\left(f, f_i\right)\right|^2 - \sum_{i\in I} \left|\varphi\left(S_{J^c}f, \widetilde{f}_i\right)\right|^2.$$

□

Another result is as follows.

Theorem 3.5. *Let $\{f_i\}_{i\in I}$ be a Parseval φ-frame for $\mathscr{E}$. For every subset $J \subset I$ and every $f \in \mathscr{E}$, we have*

$$\sum_{i\in J} \left|\varphi\left(f, f_i\right)\right|^2 - \Phi\left(\varphi\left(f, f_i\right) f_i\right) = \sum_{i\in J^c} \left|\varphi\left(f, f_i\right)\right|^2 - \Phi\left(\sum_{i\in J^c} \varphi\left(f, f_i\right) f_i\right).$$

Proof. Let $\left\{\widetilde{f}_i\right\}_{i\in I}$ denote the dual frame of $\{f_i\}_{i\in I}$. Since $\{f_i\}_{i\in I}$ is a Parseval φ-frame, its frame operator equal identity operator and hece $\widetilde{f}_i = f_i$ for all $i \in I$. Employing Theorem 3.4 and the fact that $\{f_i\}_{i\in I}$ is a Parseval φ-frame yields

$$\begin{aligned}
\sum_{i\in J} \left|\varphi\left(f, f_i\right)\right|^2 - \Phi\left(\sum_{i\in J} \varphi\left(f, f_i\right) f_i\right) &= \sum_{i\in J} \left|\varphi\left(f, f_i\right)\right|^2 - \Phi\left(S_Jf\right) \\
&= \sum_{i\in J} \left|\varphi\left(f, f_i\right)\right|^2 - \sum_{i\in I} \left|\varphi\left(S_Jf, f_i\right)\right|^2 \\
&= \sum_{i\in J} \left|\varphi\left(f, f_i\right)\right|^2 - \sum_{i\in I} \left|\varphi\left(S_Jf, \widetilde{f}_i\right)\right|^2 \\
&= \sum_{i\in J^c} \left|\varphi\left(f, f_i\right)\right|^2 - \sum_{i\in I} \left|\varphi\left(S_{J^c}f, \widetilde{f}_i\right)\right|^2 \\
&= \sum_{i\in J^c} \left|\varphi\left(f, f_i\right)\right|^2 - \Phi\left(S_{J^c}f\right) \\
&= \sum_{i\in J^c} \left|\varphi\left(f, f_i\right)\right|^2 - \Phi\left(\sum_{i\in J^c} \varphi\left(f, f_i\right) f_i\right).
\end{aligned}$$

□

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