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SOME REVERSES OF HÖLDER VECTOR OPERATOR INEQUALITY

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ABSTRACT. In this paper we obtain some new reverses of Hölder vector inequality for positive operators on Hilbert spaces.

1. INTRODUCTION

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. We use the following notation

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},$$

the *weighted geometric mean.* When $\nu = \frac{1}{2}$ we write $A\sharp B$ for brevity.

In [6] the authors obtained the following result:

$$\begin{aligned} (1.1) \qquad \left\langle B^q\sharp_{1/p}A^p x, x\right\rangle &\le \langle A^p x, x\rangle^{1/p}\langle B^q x, x\rangle^{1/q} \\ &\le \lambda^{1/p}\left(p; \frac{m_1}{M_2^{q-1}}, \frac{M_1}{m_2^{q-1}}\right)\left\langle B^q\sharp_{1/p}A^p x, x\right\rangle \end{aligned}$$

for any $x \in H$, where $0 < m_1 I \le A \le M_1 I$, $0 < m_2 I \le B \le M_2 I$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, I is the identity operator and

$$\lambda(p; m, M) := \left[\frac{1}{p^{1/p}q^{1/q}}\frac{M^p - m^p}{(M-m)^{1/p}(mM^p - Mm^p)^{1/q}}\right]^p$$

for $0 < m < M$.

In particular, one can obtain from (1.1) the following noncommutative version of *Greub-Rheinboldt inequality*

$$(1.2) \qquad \left\langle A^2\sharp B^2 x, x\right\rangle \le \left\langle A^2 x, x\right\rangle^{1/2}\left\langle B^2 x, x\right\rangle^{1/2} \le \frac{m_1 m_2 + M_1 M_2}{2\sqrt{m_1 m_2 M_1 M_2}}\left\langle A^2\sharp B^2 x, x\right\rangle$$

for any $x \in H$.

Moreover, if A and B are replaced by $C^{1/2}$ and $C^{-1/2}$ in (1.2), then we get the *Kantorovich inequality* [20]

$$\langle Cx, x\rangle^{1/2}\left\langle C^{-1}x, x\right\rangle^{1/2} \le \frac{m+M}{2\sqrt{mM}},\ x \in H \text{ with } \|x\| = 1,$$

provided $mI \le C \le MI$ for some $0 < m < M$.

For various related inequalities, see [5]-[12] and [16]-[17].

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In this paper, by making use of some recent Young's type inequalities outlined below, we establish some reverses and a refinement of Hölder's inequality for the positive operators A, B

$$\left\langle B^q \sharp_{1/p} A^p x, x\right\rangle \leq \left\langle A^p x, x\right\rangle^{1/p} \left\langle B^q x, x\right\rangle^{1/q},\ x \in H$$

where $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

The famous *Young inequality* for scalars says that if $a,\ b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu)\, a + \nu b \tag{1.3}$$

with equality if and only if $a = b$. The inequality (1.3) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [21]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.4}$$

It is well known that $\lim_{h \to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^r\right) a^{1-\nu} b^{\nu} \leq (1-\nu)\, a + \nu b \leq S\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{1.5}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.5) is due to Tominaga [22] while the first one is due to Furuichi [7].

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.6}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^r\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu} \leq (1-\nu)\, a + \nu b \leq K^R\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu} \tag{1.7}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (1.7) was obtained by Zou et al. in [23] while the second by Liao et al. [19].

Kittaneh and Manasrah [14], [15] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a} - \sqrt{b}\right)^2 \leq (1-\nu)\, a + \nu b - a^{1-\nu} b^{\nu} \leq R\left(\sqrt{a} - \sqrt{b}\right)^2 \tag{1.8}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. The case $\nu = \frac{1}{2}$ reduces (1.8) to an identity.

In the recent paper [1] we obtained the following reverses of Young's inequality as well:

$$0 \le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le \nu(1-\nu)(a-b)(\ln a - \ln b) \tag{1.9}$$

and

$$1 \le \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \le \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{1.10}$$

where $a,\, b>0,\ \nu\in[0,1]$.

In [2] we obtained the following inequalities that improve the corresponding results of Furuichi and Minculete from [9]

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\min\{a,b\} &\le (1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\\ &\le \frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\max\{a,b\}\end{aligned} \tag{1.11}$$

and

$$\begin{aligned}\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] &\le \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\\ &\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right]\end{aligned} \tag{1.12}$$

for any $a,\, b>0$ and $\nu\in[0,1]$.

2. Some Reverse Inequalities

We have the following reverse of Hölder's inequality:

Theorem 1. *Let A and B be two positive invertible operators, $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m,\, M>0$ such that*

$$m^pB^q \le A^p \le M^pB^q. \tag{2.1}$$

Then for any $x\in H$ we have the inequality

$$\langle A^px,x\rangle^{1/p}\langle B^qx,x\rangle^{1/q} \le S\left(\left(\frac{M}{m}\right)^p\right)\left\langle B^q\sharp_{1/p}A^px,x\right\rangle. \tag{2.2}$$

Proof. Assume that $\nu\in(0,1)$. Let $a,\, b\in[t,T]\subset(0,\infty)$, then $\frac{t}{T}\le\frac{a}{b}\le\frac{T}{t}$ with $\frac{t}{T}<1<\frac{T}{t}$. If $\frac{a}{b}\in\left[\frac{t}{T},1\right)$ then $S\left(\frac{a}{b}\right)\le S\left(\frac{t}{T}\right)=S\left(\frac{T}{t}\right)$. If $\frac{a}{b}\in\left(1,\frac{T}{t}\right]$ then also $S\left(\frac{a}{b}\right)\le S\left(\frac{T}{t}\right)$. Therefore for any $a,\, b\in[t,T]$ we have by Tominaga's inequality (1.5) that

$$(1-\nu)a+\nu b \le S\left(\frac{T}{t}\right)a^{1-\nu}b^{\nu}. \tag{2.3}$$

Now, if C is an operator with $tI\le C\le TI$ then for $p>1$ we have $t^pI\le C^p\le T^pI$. Using the functional calculus we get from (2.3) for $\nu=\frac{1}{p}$ that

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}C^p \le S\left(\left(\frac{T}{t}\right)^p\right)d^{1-\frac{1}{p}}C,$$

namely, the vector inequality,

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}\langle C^py,y\rangle \le S\left(\left(\frac{T}{t}\right)^p\right)d^{1-\frac{1}{p}}\langle Cy,y\rangle, \tag{2.4}$$

for any $y \in H$, $\|y\| = 1$ and $d \in [t^p, T^p]$.

Since $d = \langle C^p y, y\rangle \in [t^p, T^p]$ for any $y \in H$, $\|y\| = 1$, hence by (2.4) we have

$$\left(1 - \frac{1}{p}\right) \langle C^p y, y\rangle + \frac{1}{p} \langle C^p y, y\rangle \leq S\left(\left(\frac{T}{t}\right)^p\right) \langle C^p y, y\rangle^{1-\frac{1}{p}} \langle Cy, y\rangle,$$

that is equivalent to

$$\langle C^p y, y\rangle \leq S\left(\left(\frac{T}{t}\right)^p\right) \langle C^p y, y\rangle^{1-\frac{1}{p}} \langle Cy, y\rangle$$

and to

$$\langle C^p y, y\rangle \leq S^p\left(\left(\frac{T}{t}\right)^p\right) \langle Cy, y\rangle^p \tag{2.5}$$

for any $y \in H$, $\|y\| = 1$.

If $z \in H$ with $z \neq 0$, then by taking $y = \frac{z}{\|z\|}$ in (2.5) we get

$$\langle C^p z, z\rangle \|z\|^{2p-2} \leq S^p\left(\left(\frac{T}{t}\right)^p\right) \langle Cz, z\rangle^p$$

for any $z \in H$, and by taking the power $\frac{1}{p}$ we get

$$\langle C^p z, z\rangle^{1/p} \langle z, z\rangle^{1/q} \leq S\left(\left(\frac{T}{t}\right)^p\right) \langle Cz, z\rangle \tag{2.6}$$

for any $z \in H$.

Now, from (2.1) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^p I \leq B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \leq M^p I$ and by taking the power $\frac{1}{p}$ we get $mI \leq \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} \leq MI$.

By writing the inequality (2.6) for $C = \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}}$, $t = m$, $T = M$ and $z = B^{\frac{q}{2}} x$, with $x \in H$, we have

$$\begin{aligned}&\left\langle B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} B^{\frac{q}{2}} x, B^{\frac{q}{2}} x\right\rangle^{1/p} \left\langle B^{\frac{q}{2}} x, B^{\frac{q}{2}} x\right\rangle^{1/q}\\ &\leq S\left(\left(\frac{M}{m}\right)^p\right) \left\langle \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} B^{\frac{q}{2}} x, B^{\frac{q}{2}} x\right\rangle,\end{aligned}$$

namely

$$\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} \leq S\left(\left(\frac{M}{m}\right)^p\right) \left\langle B^{\frac{q}{2}} \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} B^{\frac{q}{2}} x, x\right\rangle$$

for any $x \in H$, and the inequality (2.2) is proved. □

Remark 1. *We observe, for A and B two positive invertible operators, that the condition (2.2) is equivalent to following condition*

$$mI \leq \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} \leq MI. \tag{2.7}$$

If we assume that

$$rB^q \leq A^p \leq RB^q, \tag{2.8}$$

then by (2.2) we have the inequality

$$\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} \leq S\left(\frac{R}{r}\right) \left\langle B^q \sharp_{1/p} A^p x, x\right\rangle \tag{2.9}$$

for any $x \in H$.

Corollary 1. *Let A and B be two positive invertible operators and m, $M > 0$ such that*

$$mI \le \left(B^{-1}A^2B^{-1}\right)^{\frac{1}{2}} \le MI. \tag{2.10}$$

Then for any $x \in H$ we have the inequality

$$\left\langle A^2x, x\right\rangle^{1/2} \left\langle B^2x, x\right\rangle^{1/2} \le S\left(\left(\frac{M}{m}\right)^2\right)\left\langle A^2\sharp B^2x, x\right\rangle, \tag{2.11}$$

where

$$A^2\sharp B^2 = A\left(A^{-1}B^2A^{-1}\right)^{1/2}A = B^2\sharp A^2.$$

Now, by taking $A = C^{1/2}$ and $B = C^{-1/2}$, then the condition (2.10) becomes

$$mI \le C \le MI \tag{2.12}$$

and by (2.11) we get

$$\langle Cx, x\rangle^{1/2}\left\langle C^{-1}x, x\right\rangle^{1/2} \le S\left(\left(\frac{M}{m}\right)^2\right),$$

for any $x \in H$ with $\|x\| = 1$.

Corollary 2. *Assume that A and B satisfy the conditions*

$$m_1I \le A \le M_1I, \;\; m_2I \le B \le M_2I \tag{2.13}$$

for some $0 < m_1 < M_1$ and $0 < m_2 < M_2$. Then we have

$$\langle A^px, x\rangle^{1/p}\langle B^qx, B^qx\rangle^{1/q} \le S\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\left\langle B^q\sharp_{1/p}A^px, x\right\rangle, \tag{2.14}$$

for any $x \in H$.

In particular, we have

$$\left\langle A^2x, x\right\rangle^{1/2}\left\langle B^2x, x\right\rangle^{1/2} \le S\left(\left(\frac{M_1M_2}{m_1m_2}\right)^2\right)\left\langle A^2\sharp B^2x, x\right\rangle, \tag{2.15}$$

for any $x \in H$.

Proof. We have from (2.13) that

$$m_1^pI \le A^p \le M_1^pI.$$

Then

$$m_1^pM_2^{-q}I \le m_1^pB^{-q} \le B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}} \le M_1^pB^{-q} \le M_1^pm_2^{-q}I,$$

which implies that

$$m_1M_2^{-\frac{q}{p}}I \le \left(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}}\right)^{\frac{1}{p}} \le M_1m_2^{-\frac{q}{p}}I.$$

Now, on using the inequality (2.2) for $m = m_1M_2^{-\frac{q}{p}}$ and $M = M_1m_2^{-\frac{q}{p}}$, we get the desired result (2.14). □

Using Kantorovich's constant (1.6) we also have:

Theorem 2. *With the assumptions of Theorem 1 we have the inequality*

$$\langle A^px, x\rangle^{1/p}\langle B^qx, x\rangle^{1/q} \le K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(\left(\frac{M}{m}\right)^p\right)\left\langle B^q\sharp_{1/p}A^px, x\right\rangle \tag{2.16}$$

for any $x \in H$.

Proof. Assume that $\nu \in (0,1)$ and $R = \max\{1-\nu,\nu\}$. Let $a,\ b \in [t,T] \subset (0,\infty)$, then $\frac{t}{T} \le \frac{a}{b} \le \frac{T}{t}$ with $\frac{t}{T} < 1 < \frac{T}{t}$. If $\frac{a}{b} \in \left[\frac{t}{T},1\right)$ then $K^R\left(\frac{a}{b}\right) \le K^R\left(\frac{t}{T}\right) = K^R\left(\frac{T}{t}\right)$. If $\frac{a}{b} \in \left(1,\frac{T}{t}\right]$ then also $K^R\left(\frac{a}{b}\right) \le K^R\left(\frac{T}{t}\right)$. Therefore for any $a,\ b \in [t,T]$ we have by inequality (1.7) that

$$(1-\nu)a + \nu b \le K^R\left(\frac{T}{t}\right) a^{1-\nu} b^{\nu}. \tag{2.17}$$

Now, if C is an operator with $tI \le C \le TI$ then for $p>1$ we have $t^pI \le C^p \le T^pI$. Using the functional calculus we get from (2.17) for $\nu = \frac{1}{p}$ that

$$\left(1-\frac{1}{p}\right)d + \frac{1}{p}C^p \le K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(\left(\frac{T}{t}\right)^p\right) d^{1-\frac{1}{p}}C,$$

namely, the vector inequality,

$$\left(1-\frac{1}{p}\right)d + \frac{1}{p}\langle C^p y,y\rangle \le K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(\left(\frac{T}{t}\right)^p\right) d^{1-\frac{1}{p}}\langle Cy,y\rangle,$$

for any $y \in H$, $\|y\| = 1$ and $d \in [t^p, T^p]$.

Now, by employing a similar argument to the one in the proof of Theorem 1 we deduce the desired result (2.16). The details are omitted. □

Corollary 3. *With the assumptions of Corollary 1 we have*

$$\left\langle A^2x,x\right\rangle^{1/2}\left\langle B^2x,x\right\rangle^{1/2} \le \left[K\left(\left(\frac{M}{m}\right)^2\right)\right]^{1/2}\left\langle A^2\sharp B^2x,x\right\rangle, \tag{2.18}$$

for any $x \in H$.

We also have:

Corollary 4. *With the assumptions of Corollary 2 we have*

$$\begin{aligned}&\langle A^px,x\rangle^{1/p}\langle B^qx,B^qx\rangle^{1/q} \\ &\le K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\left\langle B^q\sharp_{1/p}A^px,x\right\rangle,\end{aligned} \tag{2.19}$$

for any $x \in H$.

In particular, we have

$$\left\langle A^2x,x\right\rangle^{1/2}\left\langle B^2x,x\right\rangle^{1/2} \le \left[K\left(\left(\frac{M_1M_2}{m_1m_2}\right)^2\right)\right]^{1/2}\left\langle A^2\sharp B^2x,x\right\rangle, \tag{2.20}$$

for any $x \in H$.

3. Exponential Reverses

We have:

Theorem 3. *With the assumptions of Theorem 1 we have the inequality*

$$\langle A^px,x\rangle^{1/p}\langle B^qx,x\rangle^{1/q} \le \exp\left[\frac{4}{pq}\left(K\left[\left(\frac{M}{m}\right)^p\right]-1\right)\right]\left\langle B^q\sharp_{1/p}A^px,x\right\rangle \tag{3.1}$$

for any $x \in H$.

Proof. Assume that $\nu \in (0,1)$. Let a, $b \in [t,T] \subset (0,\infty)$, then by the inequality (1.10) we have

$$(1-\nu)\,a+\nu b \le a^{1-\nu}b^{\nu}\exp\left[4\nu\,(1-\nu)\left(K\left(\frac{T}{t}\right)-1\right)\right]. \tag{3.2}$$

Now, if C is an operator with $tI \le C \le TI$ then for $p>1$ we have $t^pI \le C^p \le T^pI$. Using the functional calculus we get from (3.2) for $\nu = \frac{1}{p}$ that

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}C^p \le \exp\left[\frac{4}{pq}\left(K\left[\left(\frac{T}{t}\right)^p\right]-1\right)\right]d^{1-\frac{1}{p}}C,$$

namely, the vector inequality,

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}\left\langle C^py,y\right\rangle \le \exp\left[\frac{4}{pq}\left(K\left[\left(\frac{T}{t}\right)^p\right]-1\right)\right]d^{1-\frac{1}{p}}\left\langle Cy,y\right\rangle,$$

for any $y \in H$, $\|y\|=1$ and $d \in [t^p,T^p]$.

Now, by employing a similar argument to the one in the proof of Theorem 1 we deduce the desired result (3.1). The details are omitted. □

We have:

Corollary 5. *With the assumptions of Corollary 1 we have*

$$\left\langle A^2x,x\right\rangle^{1/2}\left\langle B^2x,x\right\rangle^{1/2} \le \exp\left(K\left[\left(\frac{M}{m}\right)^2\right]-1\right)\left\langle A^2\sharp B^2x,x\right\rangle, \tag{3.3}$$

for any $x \in H$.

Corollary 6. *With the assumptions of Corollary 2 we have*

$$\begin{aligned}&\left\langle A^px,x\right\rangle^{1/p}\left\langle B^qx,B^qx\right\rangle^{1/q}\\ &\le \exp\left[\frac{4}{pq}\left(K\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]-1\right)\right]\left\langle B^q\sharp_{1/p}A^px,x\right\rangle,\end{aligned} \tag{3.4}$$

for any $x \in H$.

In particular, we have

$$\left\langle A^2x,x\right\rangle^{1/2}\left\langle B^2x,x\right\rangle^{1/2} \le \exp\left(K\left[\left(\frac{M_1M_2}{m_1m_2}\right)^2\right]-1\right)\left\langle A^2\sharp B^2x,x\right\rangle, \tag{3.5}$$

for any $x \in H$.

Finally, we have the following reverse of Hölder's inequality as well:

Theorem 4. *With the assumptions of Theorem 1 we have the inequality*

$$\left\langle A^px,x\right\rangle^{1/p}\left\langle B^qx,x\right\rangle^{1/q} \le \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p-1\right)^2\right]\left\langle B^q\sharp_{1/p}A^px,x\right\rangle \tag{3.6}$$

for any $x \in H$.

Proof. If a, $b \in [t,T] \subset (0,\infty)$ and since

$$0 < \frac{\max\{a,b\}}{\min\{a,b\}}-1 \le \frac{T}{t}-1,$$

hence

$$\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\le\left(\frac{T}{t}-1\right)^2.$$

Therefore, by (1.12) we get

$$(1-\nu)a+\nu b\le a^{1-\nu}b^{\nu}\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{T}{t}-1\right)^2\right], \tag{3.7}$$

for any $a,\ b\in[t,T]$ and $\nu\in(0,1)$.

Now, if C is an operator with $tI\le C\le TI$ then for $p>1$ we have $t^pI\le C^p\le T^pI$. Using the functional calculus we get from (3.2) for $\nu=\frac{1}{p}$ that

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}C^p\le\exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]d^{1-\frac{1}{p}}C,$$

namely, the vector inequality,

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}\langle C^py,y\rangle\le\exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]d^{1-\frac{1}{p}}\langle Cy,y\rangle,$$

for any $y\in H$, $\|y\|=1$ and $d\in[t^p,T^p]$.

Now, by employing a similar argument to the one in the proof of Theorem 1 we deduce the desired result (3.7). The details are omitted. □

We have:

Corollary 7. *With the assumptions of Corollary 1 we have*

$$\langle A^2x,x\rangle^{1/2}\langle B^2x,x\rangle^{1/2}\le\exp\left[\frac{1}{8}\left(\left(\frac{M}{m}\right)^2-1\right)^2\right]\langle A^2\sharp B^2x,x\rangle \tag{3.8}$$

for any $x\in H$.

If $mI\le C\le MI$ for some m,M with $0<m<M$, then by (3.8) we get

$$\langle Cx,x\rangle^{1/2}\langle C^{-1}x,x\rangle^{1/2}\le\exp\left[\frac{1}{8}\left(\left(\frac{M}{m}\right)^2-1\right)^2\right]\|x\|^2, \tag{3.9}$$

for any $x\in H$.

Corollary 8. *With the assumptions of Corollary 2 we have*

$$\langle A^px,x\rangle^{1/p}\langle B^qx,x\rangle^{1/q}\le\exp\left[\frac{1}{2pq}\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q-1\right)^2\right]\langle B^q\sharp_{1/p}A^px,x\rangle,$$

for any $x\in H$.

In particular, we have

$$\langle A^2x,x\rangle^{1/2}\langle B^2x,x\rangle^{1/2}\le\exp\left[\frac{1}{8}\left(\left(\frac{M_1M_2}{m_1m_2}\right)^2-1\right)^2\right]\langle A^2\sharp B^2x,x\rangle, \tag{3.10}$$

for any $x\in H$.

References

[1] S. S. Dragomir, A note on Young's inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 126. [http://rgmia.org/papers/v18/v18a126.pdf].

[2] S. S. Dragomir, A note on new refinements and reverses of Young's inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. . [http://rgmia.org/papers/v1].

[3] S. S. Dragomir, Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces, *J. Inequal. & Appl.*, Volume **2010**, Article ID 496821, 15 pages doi:10.1155/2010/496821. Preprint RGMIA *Res. Rep. Coll.*, **11** (2008), Supliment. Art. 15. [Online http://rgmia.org/papers/v11e/RevJensenOp.pdf].

[4] S. S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type,* Springer Briefs in Mathematics, Springer, 2012.

[5] M. Fujii, S. Izumino and R. Nakamoto, Classes of operators determined by the Heinz-Kato-Furuta inequality and the Hölder-McCarthy inequality. *Nihonkai Math. J.* **5** (1994), no. 1, 61–67.

[6] M. Fuji, S. Izumino, R. Nakamoto and Y. Seo, Operator inequalities related to Cauchy-Schwarz and Hölder-McCarthy inequalities, *Nihonkai Math. J.*, **8** (1997), 117-122.

[7] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.* **20** (2012), 46–49.

[8] S. Furuichi, On refined Young inequalities and reverse inequalities, *J. Math. Inequal.* **5** (2011), 21-31.

[9] S. Furuichi and N. Minculete, Alternative reverse inequalities for Young's inequality, *J. Math Inequal.* **5** (2011), Number 4, 595–600.

[10] T. Furuta, Extensions of Hölder-McCarthy and Kantorovich inequalities and their applications. *Proc. Japan Acad. Ser. A Math. Sci.* **73** (1997), no. 3, 38–41.

[11] T. Furuta, Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities. *J. Inequal. Appl.* **2** (1998), no. 2, 137–148.

[12] T. Furuta, The Hölder-McCarthy and the Young inequalities are equivalent for Hilbert space operators. *Amer. Math. Monthly* **108** (2001), no. 1, 68–69.

[13] W. Greub and W. Rheinboldt, On a generalization of an inequaJity of L.V. Kantorovich, *Proc. Amer. Math.* **10** (1959), 407-415.

[14] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.* **361** (2010), 262-269.

[15] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra,* **59** (2011), 1031-1037.

[16] C.-S. Lin and Y. J. Cho, On Hölder-McCarthy-type inequalities with powers. *J. Korean Math. Soc.* **39** (2002), no. 3, 351–361.

[17] C.-S. Lin and Y. J. Cho, On Kantorovich inequality and Hölder-McCarthy inequalities. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **11** (2004), no. 4, 481–490.

[18] C. A. McCarthy, c_p, *Israel J. Math.* **5** (1967), 249–271.

[19] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.

[20] R. Nakamoto and M. Nakamura. Operator mean and the Kantorovich inequality, *Math. Japonica* **44** (1966), 495-498.

[21] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.*, **74** (1960), pp. 91-98.

[22] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.

[23] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.

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