Received 09/10/15

NEW GENERAL INTEGRAL INEQUALITIES FOR LIPSCHITZIAN FUNCTIONS VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS AND APPLICATIONS

MEHMET KUNT AND İMDAT İŞCAN

Abstract. In this paper, new estimates on generalization of Hermite-Hadamard, Ostorowski and Simpson type inequalities for Lipschitzian functions via Riemann-Liouville fractional integrals are obtained. Also, some applications to special means of two positive real numbers are given.

1. Introduction

In recent years, many authors have studied errors estimations for Hermite-Hadamard, Ostorowski and Simpson inequalities, see [1, 2, 3, 4, 7, 14].

But this inequalities rarely studied for M-Lipschitzian functions. In our observation, some works for this subject as follows:

In [5], Dragomir et. al. gave some inequalities of Hadamard's type for M-Lipschitzian functions and gave some applications which are connected some special means of two positive numbers. In [21], Yang and Tseng established several inequalities of Hadamard's type for Lipschitzian mappings. In [17, 18], Wang studied several inequalities of Hadamard's type for Lipschitzian mappings and gave some applications. In [16], Tseng et. al. established some Hermite-type and Bullen-type inequalities for Lipschitzian functions and gave several applications for special means. In [6], Hwang et. al. established some Hadamard-type inequalities for Lipschitzian functions in one and two wariables and gave several applications for special means. In [11], İşcan studied Hadamard, Ostorowski and Simpson type inequalities for Lipschitzian functions via Hadamard fractional integrals and gave some applications to special means of positive real numbers.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [8, 9, 15, 19, 20].

In this work, we study Hadamard, Ostorowski and Simpson type inequalities for Lipschitzian functions via Riemann-Liouville fractional integrals and gave some applications to special means of positive real numbers.

2. Preliminaries and General Conditions

Let a real function f be defined on some nonempty interval I of real line $\mathbb{R}$. The function f is said to be convex on I if inequality

$$f\left(tx+(1-t)\,y\right)\leq tf\left(x\right)+(1-t)\,f\left(y\right)$$

holds for all $x,y\in I$ and $t\in[0,1]$.

2000 *Mathematics Subject Classification.* 26A51, 26A33, 26D10.

Key words and phrases. Hermite-Hadamard inequality, Ostorowski inequality, Simpson inequality, Riemann-Liouville fractional integral.

The following inequalities are well known in the literature as Hermite-Hadamard, Ostorowski and Simpson inequalities respectively.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2.1}$$

Theorem 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping differentiable in I°, the interior of I, and let $a, b \in I^{\circ}$ with $a < b$. If $|f'(x)| \leq M$, $x \in [a, b]$; then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{M}{b-a}\left[\frac{(x-a)^2 + (b-x)^2}{2}\right] \tag{2.2}$$

for all $x \in [a, b]$.

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be four times continuously differentiable mapping on (a, b) and $\left\|f^{(4)}\right\|_{\infty} = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f((b))}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4. \tag{2.3}$$

The following definition of M-Lipschitzian function is well known in the literature.

Definition 1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ called an M-Lipschitzian function on the interval I of real numbers with $M \geq 0$ if*

$$|f(x) - f(y)| \leq M\,|x - y| \tag{2.4}$$

for all $x, y \in I$.

The following definitions and mathematical preliminaries of fractional calculus theory are used further in this paper.

Definition 2. [13]. *Let $f \in L[a, b]$. The Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt,\ x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt,\ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-t} t^{\alpha-1} dt$ and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

In [10], İşcan gave definition of harmonically convex functions and established following Hermite-Hadamard type inequality for harmonically convex functions as follows:

Definition 3. *Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y)+(1-t)f(x) \tag{2.5}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (2.5) is reversed, then f is said to be harmonically concave.

Theorem 4. [10]. *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt \leq \frac{f(a)+f(b)}{2}. \tag{2.6}$$

In [12], Kunt and İşcan established Hermite-Hadamard's inequalities for harmonically convex functions in Riemann-Liouville fractional integral forms as follows:

Theorem 5. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c} J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(1/a) \\ +J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ g)(1/b)\end{array}\right\} \\ &\leq \frac{f(a)+f(b)}{2} \end{aligned} \tag{2.7}$$

with $\alpha > 0$ and $g(x) = 1/x$, $x \in \left[\frac{1}{b}, \frac{1}{a}\right]$.

3. General Results

Let $I \subseteq (0,\infty)$ be a real interval and $f : I \to \mathbb{R}$ be a M-Lipschitzian function on I; throughout this section, we will take

$$\begin{aligned} I_f(x,\lambda,\alpha,a,b) &= (1-\lambda)\left[\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha}+\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha}\right]f(x) \\ &\quad +\lambda\left[f(a)\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha}+f(b)\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha}\right] \\ &\quad -\Gamma(\alpha+1)\left[J^{\alpha}_{\frac{1}{x}+}(f\circ g)(1/a)+J^{\alpha}_{\frac{1}{x}-}(f\circ g)(1/b)\right] \end{aligned} \tag{3.1}$$

$$\begin{aligned} S_f(x,y,\alpha,a,b) &= f(x)+f(y)-\left(\frac{2ab}{b-a}\right)^{\alpha}\Gamma(\alpha+1) \\ &\quad \times\left[J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ g)(1/b)\right] \end{aligned} \tag{3.2}$$

where $a, b \in I$ with $a < b$, $x, y \in [a,b]$, $g(t) = 1/t$, $\lambda \in [0,1]$, $\alpha > 0$ and Γ is Euler Gamma function.

Theorem 6. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a M-Lipschitzian function on I and $a,b \in I$ with $a<b$. Then for all $x\in[a,b]$, $\lambda\in[0,1]$ and $\alpha>0$ we have the following inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned} \left|I_f(x,\lambda,\alpha,a,b)\right| \leq\ & M\left\{[(1-\lambda)x-\lambda a]\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha}\right. \\ & +\alpha(2\lambda-1)\int_{\frac{1}{x}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}\frac{1}{t}dt+[\lambda b-(1-\lambda)x] \\ & \left.\times\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha}+\alpha(1-2\lambda)\int_{\frac{1}{b}}^{\frac{1}{x}}\left(t-\frac{1}{b}\right)^{\alpha-1}\frac{1}{t}dt\right\} \qquad (3.3) \end{aligned}$$

Proof. Since f is M-Lipschitzian function, we have the following inequality:

$$\begin{aligned} \left|I_f(x,\lambda,\alpha,a,b)\right| =\ & \left|(1-\lambda)\left[\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha}+\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha}\right]f(x)\right. \\ & +\lambda\left[f(a)\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha}+f(b)\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha}\right] \\ & \left.-\Gamma(\alpha+1)\left[J^{\alpha}_{\frac{1}{x}+}(f\circ g)(1/a)+J^{\alpha}_{\frac{1}{x}-}(f\circ g)(1/b)\right]\right| \\ \leq\ & (1-\lambda)\left|\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha}f(x)-\alpha\int_{\frac{1}{x}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}f\left(\frac{1}{t}\right)dt\right. \\ & \left.+\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha}f(x)-\alpha\int_{\frac{1}{b}}^{\frac{1}{x}}\left(t-\frac{1}{b}\right)^{\alpha-1}f\left(\frac{1}{t}\right)dt\right| \\ & +\lambda\left|\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha}f(a)-\alpha\int_{\frac{1}{x}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}f\left(\frac{1}{t}\right)dt\right. \\ & \left.+\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha}f(b)-\alpha\int_{\frac{1}{b}}^{\frac{1}{x}}\left(t-\frac{1}{b}\right)^{\alpha-1}f\left(\frac{1}{t}\right)dt\right| \\ \leq\ & \alpha(1-\lambda)\left\{\int_{\frac{1}{x}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}\left|f(x)-f\left(\frac{1}{t}\right)\right|dt\right. \\ & \left.+\int_{\frac{1}{b}}^{\frac{1}{x}}\left(t-\frac{1}{b}\right)^{\alpha-1}\left|f(x)-f\left(\frac{1}{t}\right)\right|dt\right\} \\ & +\alpha\lambda\left\{\int_{\frac{1}{x}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}\left|f(a)-f\left(\frac{1}{t}\right)\right|dt\right. \\ & \left.+\int_{\frac{1}{b}}^{\frac{1}{x}}\left(t-\frac{1}{b}\right)^{\alpha-1}\left|f(b)-f\left(\frac{1}{t}\right)\right|dt\right\} \\ \leq\ & \alpha(1-\lambda)M\left\{\int_{\frac{1}{x}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}\left(x-\frac{1}{t}\right)dt\right. \\ & \left.+\int_{\frac{1}{b}}^{\frac{1}{x}}\left(t-\frac{1}{b}\right)^{\alpha-1}\left(\frac{1}{t}-x\right)dt\right\} \\ & +\alpha\lambda M\left\{\int_{\frac{1}{x}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}\left(\frac{1}{t}-a\right)dt\right. \\ & \left.+\int_{\frac{1}{b}}^{\frac{1}{x}}\left(t-\frac{1}{b}\right)^{\alpha-1}\left(b-\frac{1}{t}\right)dt\right\} \end{aligned}$$

By a simple computation from this inequality we have the inequality (3.3). This completes the proof. □

Corollary 1. *In theorem 6, if we take $\lambda = 0$, we have*

$$\begin{aligned}
&\left| \frac{\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha}+\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha}}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^{\alpha} f(x) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^{\alpha} \right. \\
&\left. \times \left[J^{\alpha}_{\frac{1}{x}+} (f\circ g)(1/a) + J^{\alpha}_{\frac{1}{x}-} (f\circ g)(1/b) \right] \right| \\
\leq\ & \frac{M}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^{\alpha} \left\{ x\left[\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha} - \left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha} \right] \right. \\
&\left. +\alpha \left[\int_{\frac{1}{b}}^{\frac{1}{x}} \left(t-\frac{1}{b}\right)^{\alpha-1} \frac{1}{t} dt - \int_{\frac{1}{x}}^{\frac{1}{a}} \left(\frac{1}{a}-t\right)^{\alpha-1} \frac{1}{t} dt \right] \right\}. \qquad (3.4)
\end{aligned}$$

In (3.4),
(i) if we take $\alpha = 1$, then

$$\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt \right| \leq M \frac{ab}{b-a} \left\{ \frac{x(a+b)}{ab} - 2 + \ln \frac{ab}{x^2} \right\}, \qquad (3.5)$$

(ii) if we take $x = \frac{2ab}{a+b}$, then

$$\begin{aligned}
&\left| f\left(\frac{2ab}{a+b}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^{\alpha} \left[J^{\alpha}_{\frac{a+b}{2ab}+} (f\circ g)(1/a) + J^{\alpha}_{\frac{a+b}{2ab}-} (f\circ g)(1/b) \right] \right| \\
\leq\ & \frac{M}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^{\alpha} \alpha \left\{ \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left(t-\frac{1}{b}\right)^{\alpha-1} \frac{1}{t} dt - \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left(\frac{1}{a}-t\right)^{\alpha-1} \frac{1}{t} dt \right\}, \qquad (3.6)
\end{aligned}$$

(iii) if we take $\alpha = 1$ and $x = \frac{2ab}{a+b}$, then

$$\left| f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt \right| \leq M \frac{ab}{b-a} \ln \frac{(a+b)^2}{4ab} \qquad (3.7)$$

Corollary 2. *In theorem 6, if we take $\lambda = 1$, we have*

$$\begin{aligned}
&\left| \frac{f(a)\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha}+f(b)\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha}}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^{\alpha} - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^{\alpha} \right. \\
&\left. \times \left[J^{\alpha}_{\frac{1}{x}+} (f\circ g)(1/a) + J^{\alpha}_{\frac{1}{x}-} (f\circ g)(1/b) \right] \right| \\
\leq\ & \frac{M}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^{\alpha} \left\{ \left[b\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha} - a\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha} \right] \right. \\
&\left. +\alpha \left[\int_{\frac{1}{x}}^{\frac{1}{a}} \left(\frac{1}{a}-t\right)^{\alpha-1} \frac{1}{t} dt - \int_{\frac{1}{b}}^{\frac{1}{x}} \left(t-\frac{1}{b}\right)^{\alpha-1} \frac{1}{t} dt \right] \right\}. \qquad (3.8)
\end{aligned}$$

In (3.8),

(i) if we take $x=\frac{2ab}{a+b}$*, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\right.\\&\left.\times\left[J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ g)(1/b)\right]\right|\\\leq\ &\frac{M}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{(b-a)\left(\frac{b-a}{2ab}\right)^{\alpha}\right.\\&\left.+\alpha\left[\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}\frac{1}{t}dt-\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(t-\frac{1}{b}\right)^{\alpha-1}\frac{1}{t}dt\right]\right\},\qquad(3.9)\end{aligned}$$

(ii) if we take $\alpha=1$ *and* $x=\frac{2ab}{a+b}$*, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt\right|\\\leq\ &M\frac{ab}{b-a}\left\{\frac{(b-a)^2}{2ab}+\ln\frac{4ab}{(a+b)^2}\right\}\qquad(3.10)\end{aligned}$$

Remark 1. *In* (3.6) *and* (3.7) *we get new inequalities about the left hand side of Hermite-Hadamard's inequalities of* (2.7) *and* (2.6)*, in* (3.9) *and* (3.10) *we get new inequalities about the right hand side of Hermite-Hadamard's inequalities of* (2.7) *and* (2.6) *respectively for* M*-Lipschitzian functions.*

Corollary 3. *In theorem* 6*,*
(1) if we take $\lambda=\frac{1}{3}$*,* $x=\frac{2ab}{a+b}$*, then*

$$\begin{aligned}&\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{2ab}{a+b}\right)\right]-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\right.\\&\left.\times\left[J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ g)(1/b)\right]\right|\\\leq\ &\frac{M}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\frac{b-a}{3}\left(\frac{b-a}{2ab}\right)^{\alpha}\right.\\&\left.+\frac{\alpha}{3}\left(\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(t-\frac{1}{b}\right)^{\alpha-1}\frac{1}{t}dt-\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}\frac{1}{t}dt\right)\right\},\qquad(3.11)\end{aligned}$$

Specially if we take $\alpha=1$ *in* (3.11) *then we have*

$$\begin{aligned}&\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{2ab}{a+b}\right)\right]-\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt\right|\\\leq\ &M\frac{ab}{b-a}\left\{\frac{(b-a)^2}{6ab}+\frac{1}{3}\ln\frac{(a+b)^2}{4ab}\right\}\subset.\qquad(3.12)\end{aligned}$$

(2)if we take $\lambda=\frac{1}{2}$*,* $x=\frac{2ab}{a+b}$*, then*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{2ab}{a+b}\right)\right]-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\right.\\&\left.\times\left[J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ g)(1/b)\right]\right|\\\leq\ &M\frac{b-a}{4}\qquad(3.13)\end{aligned}$$

Specially if we take $\alpha = 1$ in (3.13) then we have

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{2ab}{a+b}\right)\right]-\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt\right|\leq M\frac{b-a}{4} \tag{3.14}$$

Corollary 4. *In theorem 6, if we take $\alpha = 1$, then*

$$\begin{aligned}&\left|(1-\lambda)f(x)+\lambda\left[\frac{f(a)\left(\frac{1}{a}-\frac{1}{x}\right)+f(b)\left(\frac{1}{x}-\frac{1}{b}\right)}{\frac{b-a}{ab}}\right]-\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt\right|\\ \leq\ & M\frac{ab}{b-a}\left\{[(1-\lambda)x-\lambda a]\left(\frac{1}{a}-\frac{1}{x}\right)+[\lambda b-(1-\lambda)x]\left(\frac{1}{x}-\frac{1}{b}\right)\right.\\ &\left.+(2\lambda-1)\ln\frac{x^2}{ab}\right\}.\end{aligned} \tag{3.15}$$

Specially, if we take $x = \frac{2ab}{a+b}$ in (3.15) then we have

$$\begin{aligned}&\left|(1-\lambda)f\left(\frac{2ab}{a+b}\right)+\lambda\left[\frac{f(a)+f(b)}{2}\right]-\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt\right|\\ \leq\ & M\frac{ab}{b-a}\left\{\lambda\frac{(b-a)^2}{2ab}+(2\lambda-1)\ln\frac{4ab}{(a+b)^2}\right\}.\end{aligned} \tag{3.16}$$

Remark 2. *If we take $\lambda = 0$, $\lambda = 1$, $\lambda = \frac{1}{3}$ and $\lambda = \frac{1}{2}$ in inequality (3.16) we obtain inequalities (3.7), (3.10), (3.12) and (3.14), respectively.*

Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a M-Lipschitzian function on I and $a, b \in I$ with $a < b$. In the next theorem $a \leq x \leq y \leq b$ and define $U_\alpha(x,y)$, $\alpha > 0$ as follows:
(1) If $a \leq \frac{2ab}{a+b} \leq x \leq y \leq b$, then

$$\begin{aligned}U_\alpha(x,y) &= \frac{x}{\alpha}\left(\frac{b-a}{2ab}\right)^\alpha-\frac{y}{\alpha}\left[2\left(\frac{b-y}{by}\right)^\alpha-\left(\frac{b-a}{2ab}\right)^\alpha\right]\\ &-\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}\frac{dt}{t}+\int_{\frac{1}{b}}^{\frac{1}{y}}\left(t-\frac{1}{b}\right)^{\alpha-1}\frac{dt}{t}\\ &-\int_{\frac{1}{y}}^{\frac{a+b}{2ab}}\left(t-\frac{1}{b}\right)^{\alpha-1}\frac{dt}{t}\end{aligned} \tag{3.17}$$

(2) If $a \leq x \leq \frac{2ab}{a+b} \leq y \leq b$, then

$$\begin{aligned}U_\alpha(x,y) &= \frac{x}{\alpha}\left[2\left(\frac{x-a}{ax}\right)^\alpha-\left(\frac{b-a}{2ab}\right)^\alpha\right]-\frac{y}{\alpha}\left[2\left(\frac{b-y}{by}\right)^\alpha-\left(\frac{b-a}{2ab}\right)^\alpha\right]\\ &+\int_{\frac{a+b}{2ab}}^{\frac{1}{x}}\left(\frac{1}{a}-t\right)^{\alpha-1}\frac{dt}{t}-\int_{\frac{1}{x}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}\frac{dt}{t}\\ &+\int_{\frac{1}{b}}^{\frac{1}{y}}\left(t-\frac{1}{b}\right)^{\alpha-1}\frac{dt}{t}-\int_{\frac{1}{y}}^{\frac{a+b}{2ab}}\left(t-\frac{1}{b}\right)^{\alpha-1}\frac{dt}{t}\end{aligned} \tag{3.18}$$

(2) If $a \leq x \leq y \leq \frac{2ab}{a+b} \leq b$, then

$$\begin{aligned} U_{\alpha}(x,y) &= \frac{x}{\alpha}\left[2\left(\frac{x-a}{ax}\right)^{\alpha}-\left(\frac{b-a}{2ab}\right)^{\alpha}\right]-\frac{y}{\alpha}\left(\frac{b-a}{2ab}\right)^{\alpha} \\ &+\int_{\frac{a+b}{2ab}}^{\frac{1}{x}}\left(\frac{1}{a}-t\right)^{\alpha-1}\frac{dt}{t}-\int_{\frac{1}{x}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}\frac{dt}{t} \\ &+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(t-\frac{1}{b}\right)^{\alpha-1}\frac{dt}{t} \end{aligned} \tag{3.19}$$

Theorem 7. *Let x,y,α, $U_{\alpha}(x,y)$ and function f be defined as above. Then we have following inequality for Riemann-Louville fractional integrals:*

$$|S_f(x,y,\alpha,a,b)|\leq M\alpha\left(\frac{2ab}{b-a}\right)^{\alpha}U_{\alpha}(x,y) \tag{3.20}$$

Proof. Since f is M-Lipschitzian function, we have the following inequality:

$$\begin{aligned} |S_f(x,y,\alpha,a,b)| &= \alpha\left(\frac{2ab}{b-a}\right)^{\alpha}\left|\frac{\left(\frac{b-a}{2ab}\right)^{\alpha}}{\alpha}f(x)+\frac{\left(\frac{b-a}{2ab}\right)^{\alpha}}{\alpha}f(y)\right. \\ &\left.-\Gamma(\alpha)\left[J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(1/a)+J^{\alpha\alpha}_{\frac{a+b}{2ab}-\frac{1}{x}-}(f\circ g)(1/b)\right]\right| \\ &= \alpha\left(\frac{2ab}{b-a}\right)^{\alpha}\left|\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}f(x)\,dt\right. \\ &+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(t-\frac{1}{b}\right)^{\alpha-1}f(y)\,dt-\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}f\left(\frac{1}{t}\right)dt \\ &\left.-\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(t-\frac{1}{b}\right)^{\alpha-1}f\left(\frac{1}{t}\right)dt\right| \\ &\leq \alpha\left(\frac{2ab}{b-a}\right)^{\alpha}\left\{\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}\left|f(x)-f\left(\frac{1}{t}\right)\right|dt\right. \\ &\left.+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(t-\frac{1}{b}\right)^{\alpha-1}\left|f(y)-f\left(\frac{1}{t}\right)\right|dt\right\} \\ &\leq M\alpha\left(\frac{2ab}{b-a}\right)^{\alpha}\left\{\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}\left|x-\frac{1}{t}\right|dt\right. \\ &\left.+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(t-\frac{1}{b}\right)^{\alpha-1}\left|y-\frac{1}{t}\right|dt\right\} \end{aligned} \tag{3.21}$$

Using (3.17), (3.18) and (3.19), by a simple calculations, we have

$$\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\alpha-1}\left|x-\frac{1}{t}\right|dt+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(t-\frac{1}{b}\right)^{\alpha-1}\left|y-\frac{1}{t}\right|dt=U_{\alpha}(x,y) \tag{3.22}$$

Now using (3.21) and (3.22) we obtain (3.20). This completes the proof. □

With using assumptions of Theorem 7 we have the following corollary and remarks.

Corollary 5. *In Theorem 7, if we take $\alpha = 1$, then the inequality (3.20) reduces the following inequality:*

$$\left| f(x) + f(y) - \frac{2ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt \right| \leq M \frac{2ab}{b-a} U_1(x, y) \tag{3.23}$$

Remark 3. *In Theorem 7, if we take $x = y = \frac{2ab}{a+b}$, then the inequality (3.20) reduces the inequality (3.6).*

Remark 4. *In Theorem 7, if we take $x = a$ and $y = b$ then the inequality (3.20) reduces the inequality (3.9).*

4. Applications to Special Means

Let us recall the following special means of positive numbers a, b with $a < b$.

(1) The arithmetic mean:

$$A = A(a, b) := \frac{a+b}{2}. \tag{4.1}$$

(2) The geometric mean:

$$G = G(a, b) := \sqrt{ab}. \tag{4.2}$$

(3) The harmonic mean:

$$H = H(a, b) := \frac{2ab}{a+b}. \tag{4.3}$$

(4) The logarithmic mean:

$$L = L(a, b) := \frac{b-a}{\ln b - \ln a}. \tag{4.4}$$

(5) The identric mean:

$$I = I(a, b) := \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}}. \tag{4.5}$$

To prove the results of this section, we need the following lemma.

Lemma 1. *(see [16]). Let $f : [a, b] \to \mathbb{R}$ be differentiable with $\|f'\|_\infty < \infty$. Then f is a M-Lipschitzian function on $[a, b]$ where $M = \|f'\|_\infty$.*

Proposition 1. *For $b > a > 0$, $\lambda \in [0, 1]$ and $n \geq 1$, we have*

$$\begin{aligned} & \left| (1-\lambda) H^n(a, b) + \lambda A(a^n, b^n) - ab \frac{L(a^{n-1}, b^{n-1})}{L(a, b)} \right| \\ \leq \; & nb^{n-1} \left\{ \lambda \frac{b-a}{2} + (2\lambda - 1) \frac{ab}{b-a} \ln \frac{H(a, b)}{A(a, b)} \right\}. \end{aligned} \tag{4.6}$$

Proof. The proof follows by inequality (3.16) applied for the Lipschitzian function $f(x) = x^n$ on $[a, b]$. □

Remark 5. *Let $\lambda = 0$ and $\lambda = 1$ in inequality (4.6). Then, using inequality (2.6), we have the following inequalities respectively,*

$$0 \leq ab \frac{L(a^{n-1}, b^{n-1})}{L(a, b)} - H^n(a, b) \leq -nb^{n-1} \frac{ab}{b-a} \ln \frac{H(a, b)}{A(a, b)}, \tag{4.7}$$

$$\begin{aligned} 0 &\leq A\left(a^{n},b^{n}\right)-ab\frac{L\left(a^{n-1},b^{n-1}\right)}{L\left(a,b\right)} \\ &\leq nb^{n-1}\left\{\frac{b-a}{2}+\frac{ab}{b-a}\ln\frac{H\left(a,b\right)}{A\left(a,b\right)}\right\}. \end{aligned} \tag{4.8}$$

Proposition 2. *For $b>a>0$ and $\lambda\in[0,1]$, we have*

$$\begin{aligned} &\left|(1-\lambda)H^{2}\left(a,b\right)e^{H(a,b)}+\lambda A\left(a^{2}e^{a},b^{2}e^{b}\right)-abL\left(e^{a},e^{b}\right)\right| \\ \leq\ &\left(2b+b^{2}\right)e^{b}\left\{\lambda\frac{b-a}{2}+(2\lambda-1)\frac{ab}{b-a}\ln\frac{H\left(a,b\right)}{A\left(a,b\right)}\right\}. \end{aligned} \tag{4.9}$$

Proof. The proof follows by inequality (3.16) applied for the Lipschitzian function $f(x)=x^{2}e^{x}$ on $[a,b]$. □

Remark 6. *Let $\lambda=0$ and $\lambda=1$ in inequality (4.9). Then, using inequality (2.6), we have the following inequalities respectively,*

$$0\leq abL\left(e^{a},e^{b}\right)-H^{2}\left(a,b\right)e^{H(a,b)}\leq-\left(2b+b^{2}\right)e^{b}\frac{ab}{b-a}\ln\frac{H\left(a,b\right)}{A\left(a,b\right)}, \tag{4.10}$$

$$\begin{aligned} 0 &\leq A\left(a^{2}e^{a},b^{2}e^{b}\right)-abL\left(e^{a},e^{b}\right) \\ &\leq \left(2b+b^{2}\right)e^{b}\left\{\frac{b-a}{2}+\frac{ab}{b-a}\ln\frac{H\left(a,b\right)}{A\left(a,b\right)}\right\}. \end{aligned} \tag{4.11}$$

Proposition 3. *For $b>a>0$ and $\lambda\in[0,1]$, we have*

$$\begin{aligned} &\left|(1-\lambda)H\left(a,b\right)+\lambda A\left(a,b\right)-abL^{-1}\left(a,b\right)\right| \\ \leq\ &\lambda\frac{b-a}{2}+(2\lambda-1)\frac{ab}{b-a}\ln\frac{H\left(a,b\right)}{A\left(a,b\right)}. \end{aligned} \tag{4.12}$$

Proof. The proof follows by inequality (3.16) applied for the Lipschitzian function $f(x)=x$ on $[a,b]$. □

Remark 7. *Let $\lambda=0$ and $\lambda=1$ in inequality (4.12). Then, using inequality (2.6), we have the following inequalities respectively,*

$$0\leq abL^{-1}\left(a,b\right)-H\left(a,b\right)\leq-\frac{ab}{b-a}\ln\frac{H\left(a,b\right)}{A\left(a,b\right)}, \tag{4.13}$$

$$0\leq A\left(a,b\right)-abL^{-1}\left(a,b\right)\leq\frac{b-a}{2}+\frac{ab}{b-a}\ln\frac{H\left(a,b\right)}{A\left(a,b\right)}. \tag{4.14}$$

Proposition 4. *For $b>a>0$ and $\lambda\in[0,1]$, we have*

$$\begin{aligned} &\left|(1-\lambda)H\left(a,b\right)\ln H\left(a,b\right)+\lambda A\left(a\ln a,b\ln b\right)-\frac{ab\ln ab}{2}L^{-1}\left(a,b\right)\right| \\ \leq\ &\ln eb\frac{ab}{b-a}\left\{\lambda\frac{(b-a)^{2}}{2ab}+(2\lambda-1)\ln\frac{H\left(a,b\right)}{A\left(a,b\right)}\right\}. \end{aligned} \tag{4.15}$$

Proof. The proof follows by inequality (3.16) applied for the Lipschitzian function $f(x)=x\ln x$ on $[a,b]$. □

Remark 8. *Let* $\lambda = 0$ *and* $\lambda = 1$ *in inequality* (4.15)*. Then, using inequality* (2.6)*, we have the following inequalities respectively,*

$$0 \leq \frac{ab\ln ab}{2}L^{-1}(a,b) - H(a,b)\ln H(a,b) \leq -\ln eb\frac{ab}{b-a}\ln\frac{H(a,b)}{A(a,b)}, \quad (4.16)$$

$$\begin{aligned} 0 &\leq A(a\ln a, b\ln b) - \frac{ab\ln ab}{2}L^{-1}(a,b) \\ &\leq \ln eb\left\{\frac{b-a}{2} + \frac{ab}{b-a}\ln\frac{H(a,b)}{A(a,b)}\right\}. \end{aligned} \quad (4.17)$$

Proposition 5. *For* $b > a > 0$ *and* $\lambda \in [0,1]$*, we have*

$$\begin{aligned} &\left|(1-\lambda)H^2(a,b)\ln H(a,b) + \lambda A\left(a^2\ln a, b^2\ln b\right) - ab\ln I(a,b)\right| \\ \leq\ & (b + 2b\ln b)\frac{ab}{b-a}\left\{\lambda\frac{(b-a)^2}{2ab} + (2\lambda-1)\ln\frac{H(a,b)}{A(a,b)}\right\}. \end{aligned} \quad (4.18)$$

Proof. The proof follows by inequality (3.16) applied for the Lipschitzian function $f(x) = x^2\ln x$ on $[a,b]$. □

Remark 9. *Let* $\lambda = 0$ *and* $\lambda = 1$ *in inequality* (4.18)*. Then, using inequality* (2.6)*, we have the following inequalities respectively,*

$$\begin{aligned} 0 &\leq \frac{ab\ln ab}{2}L^{-1}(a,b) - H^2(a,b)\ln H(a,b) \\ &\leq -(b + 2b\ln b)\frac{ab}{b-a}\ln\frac{H(a,b)}{A(a,b)}, \end{aligned} \quad (4.19)$$

$$\begin{aligned} 0 &\leq A\left(a^2\ln a, b^2\ln b\right) - ab\ln I(a,b) \\ &\leq (b + 2b\ln b)\left\{\frac{b-a}{2} + \frac{ab}{b-a}\ln\frac{H(a,b)}{A(a,b)}\right\}. \end{aligned} \quad (4.20)$$

References

[1] M. Alomari, M. Darus, S. S. Dragomir, and P. Cerone, Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, Applied Mathematics Letters, vol. 23, no. 9, pp. 1071–1076, 2010.

[2] M. Avci, H. Kavurmaci, and M. E. ¨ Ozdemir, New inequalities of Hermite-Hadamard type via s-convex functions in the second sense with applications, Applied Mathematics and Computation, vol. 217, no. 12, pp. 5171–5176, 2011.

[3] Y. Chu, G. Wang, and X. Zhang, Schur convexity and hadamard's inequality, Mathematical Inequalities and Applications, vol. 13, no. 4, pp. 725–731, 2010.

[4] Y.-M. Chu, X.-M. Zhang, and X.-H. Zhang, "The hermite hadamard type inequality of GA-convex functions and its application," Journal of Inequalities and Applications, vol. 2010, Article ID 507560, 11 pages, 2010.

[5] S. S. Dragomir, Y. J. Cho, and S. S. Kim, Inequalities of Hadamard's type for Lipschitzian mappings and their applications, Journal of Mathematical Analysis and Applications, vol.245, no. 2, pp. 489–501, 2000.

[6] S.-R. Hwang, K.-C. Hsu, and K.-L. Tseng, Hadamard-type inequalities for Lipschitzian functions in one and two variables with applications, Journal of Mathematical Analysis and Applications, vol. 405, no. 2, pp. 546–554, 2013.

[7] İ. İşcan, New estimates on generalization of some integral inequalities for s-convex functions and their applications, Int. J. Pure Appl. Math., 86(4) (2013), 727-746.

[8] İ. İşcan, Generalization of different type integral inequalities for s-convex functions via fractional integrals, Applicable Analysis, 2013. doi: 10.1080/00036811.2013.851785.

[9] İ. İşcan, On generalization of different type integral inequalities for s-convex functions via fractional integrals, Mathematical Sciences and Applications E-Notes, 2(1) (2014), 55-67.

[10] İ. İşcan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacet. J. Math. Stat., 43 (6) (2014), 935-942.

[11] İ. İşcan, New general integral inequalities for Lipschitzian functions via Hadamard fractional integrals, International Jurnal of Analysis, vol. 2014, Article ID 353924, 2014.
[12] M. Kunt, İ. İşcan, On new inequalities of Hermite-Hadamard-Fejer type for harmonically convex functions via fractional integrals, RGMIA Research Report Collection, 18(2015), Article 109, 17 pp.
[13] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations. Elsevier, Amsterdam (2006).
[14] E. Set,M. E. Ozdemir, and M. Z. Sarkıaya, On new inequalities of Simpson's type for quasi convex functions with applications, Tamkang Journal of Mathematics, vol. 43, no. 3, pp. 357–364, 2012.
[15] M.Z. Sarıkaya, E. Set, H. Yaldız and N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling, 57(9) (2013), 2403-2407.
[16] K.-L. Tseng, S.-R. Hwang, and K.-C. Hsu, Hadamard-type and Bullen-type inequalities for Lipschitzian functions and their applications, Computers and Mathematics with Applications, vol. 64, no. 4, pp. 651–660, 2012.
[17] L.-C. Wang, New inequalities of Hadamard's type for Lipschitzian mappings, Journal of Inequalities in Pure and Applied Mathematics, vol. 6, Iss. 2(37) , 2005.
[18] L.-C. Wang, On new inequalities of Hadamard-type for Lipschitzian mappings and their applications, vol. 8, Iss. 1(30) , 2007.
[19] J. Wang, X. Li, M. Fečkan and Y. Zhou, Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity, Appl. Anal., 92(11) (2012), 2241-2253. doi:10.1080/00036811.2012.727986
[20] J. Wang, C. Zhu and Y. Zhou, New generalized Hermite-Hadamard type inequalities and applications to special means, J. Inequal. Appl., 2013(325) (2013), 15 pages.
[21] G.-S. Yang and K.-L. Tseng, Inequalities of hadamard's type for Lipschitzian mappings, Journal of Mathematical Analysis and Applications, vol. 260, no. 1, pp. 230–238, 2001.

Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, Trabzon, Turkey
E-mail address: mkunt@ktu.edu.t

Department of Mathematics, Faculty of Sciences and Arts, Giresun University, Giresun, Turkey
E-mail address: imdat.iscan@giresun.edu.tr;imdati@yahoo.com