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SOME ESTIMATES ON THE HERMITE-HADAMARD INEQUALITY THROUGH GEOMETRICALLY QUASI-CONVEX FUNCTIONS

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Abstract. In present paper, by proving a new integral identity, using Hölder's inequality and mathematical analysis, we prove some Hermite-Hadamard type integral inequalities for geometrically quasi-convex functions which give better estimates to those already proven for the right-side of a Hermite-Hadamard type inequality established for geometrically convex functions in earlier works. A numerical example is also provided to support our claim. Applications of the results to special means of positive numbers are given.

1. Introduction

A function $g:\varphi\neq J\subseteq\mathbb{R}\to\mathbb{R}$ is said to be convex on J if

$$g\left(\sigma l+\left(1-\sigma\right)m\right)\leq\sigma g\left(l\right)+\left(1-\sigma\right)g\left(m\right)$$

holds for every $l,m\in J$ and $\sigma\in[0,1]$. There are several accomplishments of the role of convexity towards the field of inequalities and in the other branches of pure and applied mathematics but one of them is the celebrated Hermite Hadamard inequality, which is expressed as follows

$$g\left(\frac{r+w}{2}\right)\leq\frac{1}{w-r}\int_{r}^{w}g\left(l\right)dl\leq\frac{g\left(r\right)+g\left(w\right)}{2},\tag{1.1}$$

where $g:\varphi\neq J\subseteq\mathbb{R}\to\mathbb{R}$ is a convex function and $r,\ w\in J$ with $r<w$. Undeniably, the inequality (1.1) specifies a necessary and sufficient conditions for a function g to be convex on the interval $[r,w]$.

The approach of quasi-convex functions speculates the concept of convex functions, that is a function $g:\varphi\neq J\subseteq\mathbb{R}\to\mathbb{R}$ is quasi-convex on J if the following relaxation of Jensen's inequality holds

$$g\left(\sigma l+\left(1-\sigma\right)m\right)\leq\max\left\{g\left(l\right),g\left(m\right)\right\}$$

for $0\leq\sigma\leq1$.

A large number of papers have been written to provide refinements and generalizations of the inequalities (1.1) during the past few years by using convexity and quasi-convexity see for instance [1, 2], [4]-[8], [12], [15] and the references therein.

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Many researchers have tried to generalize the notions of usual convexity and quasi-convexity in a number of approaches, one of the generalizations of the usual convexity is the geometric convexity which is avowed in the definition below.

Definition 1. [11] *A function $g : J \subseteq (0,\infty) \to (0,\infty)$ is said to be geometrically convex on J if*

$$g\left(l^{\sigma}m^{1-\sigma}\right) \leq [g(l)]^{\sigma}[g(m)]^{1-\sigma}$$

holds for x, $y \in J$ and $\sigma \in [0,1]$.

The definition was further generalized by Qi and Xi in [10] as follows.

Definition 2. [10] *A function $g : J \subseteq [0,\infty) \to [0,\infty)$ is called geometrically quasi-convex function on J if*

$$g\left(l^{\sigma}m^{1-\sigma}\right) \leq \sup\left\{g(l), g(m)\right\}$$

holds for $l, m \in J$ and $\sigma \in [0,1]$.

Remark 1. [10]*If $g : J \subseteq [0,\infty) \to [0,\infty)$ is decreasing and geometrically quasi-convex on J, then it is quasi-convex on J. If $g : J \subseteq [0,\infty) \to [0,\infty)$ is increasing and quasi-convex on J, then it is geometrically quasi-convex on J.*

Some of the main results from [10] for geometrically quasi-convex functions are given in following theorems.

Theorem 1. [10]*Let $g : J \subseteq \mathbb{R}_{+} \to \mathbb{R}$ be a differentiable on J° and $g^{'} \in L([r,w])$ for r, $w \in J^{\circ}$ with $r < w$. If $\left|g^{'}\right|^{q}$ is a convex function on $[r,w]$, then*

$$\left|\frac{(\ln w)g(w)-(\ln r)g(r)}{\ln w-\ln r}-\frac{1}{\ln w-\ln r}\int_{r}^{w}\frac{g(l)}{l}dl\right| \leq N(r,w)\sup\left\{\left|g^{'}(r)\right|,\left|g^{'}(w)\right|\right\}, \tag{1.2}$$

where

$$N(r,w)=\int_{0}^{1}r^{1-\sigma}w^{\sigma}\left|\ln\left(r^{1-\sigma}w^{\sigma}\right)\right|d\sigma = \begin{cases} \frac{w\ln w-r\ln r-(w-r)}{2(\ln w-\ln r)}, & r\geq 1,\\ \frac{w\ln w+r\ln r+2-w-r}{\ln w-\ln r}, & r<1<w,\\ \frac{w-r-(w\ln w-r\ln r)}{2(\ln w-\ln r)}, & w\leq 1. \end{cases}$$

Theorem 2. [10]*Let $g : J \subseteq \mathbb{R}_{+} \to \mathbb{R}$ be a differentiable mapping on J° and $g^{'} \in L[r,w]$ for $r, w \in J^{\circ}$ with $r < w$. If $\left|g^{'}\right|^{q}$ is a geometrically quasi-convex on $[r,w]$ for $q > 1$, then*

$$\left|\frac{(\ln w)g(w)-(\ln r)g(r)}{\ln w-\ln r}-\frac{1}{\ln w-\ln r}\int_{r}^{w}\frac{g(l)}{l}dl\right| \leq \left\{\frac{q-1}{q}N\left(r^{\frac{q}{q-1}},w^{\frac{q}{q-1}}\right)\right\}^{1-\frac{1}{q}}[M(r,w)]^{\frac{1}{q}}\sup\left\{\left|g^{'}(r)\right|,\left|g^{'}(w)\right|\right\}, \tag{1.3}$$

where $N(r,w)$ *is defined in Theorem 1 and*

$$M(r,w)=\int_0^1\left|\ln\left(r^{1-\sigma}w^{\sigma}\right)\right|d\sigma$$
$$=\begin{cases}\frac{w\ln w+r\ln r}{2}, & r\geq 1,\\ \frac{(\ln w)^2+(\ln r)^2}{\ln w-\ln r}, & r<1<w,\\ -\frac{w\ln w+r\ln r}{2}, & w\leq 1.\end{cases}$$

Theorem 3. [10]*Let* $g:J\subseteq\mathbb{R}_+\to\mathbb{R}$ *be a differentiable on* J° *and* $g^{'}\in L[r,w]$ *for* $r,w\in J^\circ$ *with* $r<w$. *If* $\left|g^{'}\right|^q$ *is a geometrically convex on* $[r,w]$ *for* $q>1$ *and* ℓ *is a real number such that* $q>\ell>0$. *Then*

$$\left|\frac{(\ln w)\,g(w)-(\ln r)\,g(r)}{\ln w-\ln r}-\frac{1}{\ln w-\ln r}\int_r^w\frac{g(l)}{l}dl\right|\leq\left(\frac{q-1}{q-\ell}\right)^{1-\frac{1}{q}}\left(\frac{1}{\ell}\right)^{\frac{1}{q}}$$
$$\times\left[N\left(r^{\frac{q-\ell}{q-1}},w^{\frac{q-\ell}{q-1}}\right)\right]^{1-\frac{1}{q}}\left[N\left(r^{\ell},w^{\ell}\right)\right]^{\frac{1}{q}}\sup\left\{\left|g^{'}(r)\right|,\left|g^{'}(w)\right|\right\},\tag{1.4}$$

where $M(r,w)$ *and* $N(r,w)$ *are defined in Theorem 1 and Theorem 2.*

The main objective of the present paper is to provide new Hermite-Hadamard type inequalities for geometrically quasi-convex functions which provide refinements of the results given in [10]. A numerical example is provided to justify our claim. Applications of our results to special means are presented as well in Section 3.

2. Inequalities for Geometrically Quasi-Convex Functions

The following two Lemmas are requisite to authorize our major determinations in this section:

Lemma 1. *Suppose* $J\subseteq\mathbb{R}\to\mathbb{R}$ *be a differentiable on* J° *and* $g^{'}\in L([r,w])$, *for* r, $w\in J^\circ$ *and* $r<w$, *then*

$$\frac{(\ln w)g(w)-(\ln r)g(r)}{\ln w-\ln r}-\frac{1}{\ln w-\ln r}\int_r^w\frac{g(l)}{l}dl$$
$$=\frac{1}{2}\int_0^1 r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)g^{'}\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)d\sigma$$
$$+\frac{1}{2}\int_0^1 r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)g^{'}\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)d\sigma\tag{2.1}$$

Proof. Let

$$J_1=\frac{1}{2}\int_0^1 r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)g^{'}\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)d\sigma$$

and

$$J_2=\frac{1}{2}\int_0^1 r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)g^{'}\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)d\sigma.$$

Let $l = r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}$ in J_1 and integrating by parts, we get

$$
\begin{aligned}
(2.2)\quad J_1 &= \frac{1}{\ln w - \ln r} \int_{\sqrt{rw}}^{w} (\ln l)\, g^{'}(l)\, dl \\
&= \frac{1}{\ln w - \ln r} \left[(\ln w)\, g(w) - \ln\left(\sqrt{rw}\right) g\left(\sqrt{rw}\right) - \int_{\sqrt{rw}}^{w} \frac{g(l)}{l}\, dl \right]
\end{aligned}
$$

Now let $l = r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}$ in J_2 and integrating by parts, we have

$$
\begin{aligned}
(2.3)\quad J_2 &= \frac{1}{\ln w - \ln r} \int_{r}^{\sqrt{rw}} (\ln l)\, g^{'}(l)\, dl \\
&= \frac{1}{\ln w - \ln r} \left[\ln\left(\sqrt{rw}\right) g\left(\sqrt{rw}\right) - (\ln r)\, g(r) - \int_{r}^{\sqrt{rw}} \frac{g(l)}{l}\, dl \right]
\end{aligned}
$$

By subtracting (2.3) from (2.2), we get required result. □

Lemma 2. *For $w > r > 0$, we have*

$$
\begin{aligned}
\xi(r, w) &:= \int_0^1 \left| \ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}} \right) \right| d\sigma \\
&= \begin{cases}
-\frac{\ln r + 3\ln w}{4}, & r < w < 1, \\
\frac{\ln r + 3\ln w}{4}, & 1 < r < w, \\
\frac{5(\ln w)^2 + (\ln r)^2 + 2(\ln r)(\ln w)}{4(\ln w - \ln r)}, & r < 1 < w, rw < 1, \\
\frac{(\ln w)^2}{\ln w - \ln r}, & r < 1 < w, rw = 1
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
\eta(r, w) &:= \int_0^1 r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}} \left| \ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}} \right) \right| d\sigma \\
&= \begin{cases}
\frac{2\left[w - \sqrt{rw} + \sqrt{rw}\ln\left(\sqrt{rw}\right) - w\ln w\right]}{\ln w - \ln r}, & r < w < 1, \\
\frac{2\left[-w + \sqrt{rw} - \sqrt{rw}\ln\left(\sqrt{rw}\right) + w\ln w\right]}{\ln w - \ln r}, & 1 < r < w, \\
\frac{2\left[2 + \sqrt{rw}\ln\left(\sqrt{rw}\right) - \sqrt{rw} + w\ln w - w\right]}{\ln w - \ln r}, & r < 1 < w, rw < 1, \\
\frac{2[w\ln w - w + 1]}{\ln w - \ln r}, & r < 1 < w, rw = 1.
\end{cases}
\end{aligned}
$$

Proof. The proof can be done by simple computations. □

Theorem 4. *Let $g : J \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on J° and $g^{'} \in L[r, w]$ for $r, w \in J^\circ$ with $r < w$. If $\left|g^{'}\right|$ is a geometrically quasi-convex on $[r, w]$,* *then*

$$\begin{aligned}(2.4)\quad &\left|\frac{(\ln w)g\left(w\right)-(\ln r)g\left(r\right)}{\ln w-\ln r}-\frac{1}{\ln w-\ln r}\int_{r}^{w}\frac{g\left(l\right)}{l}dl\right|\\ &\leq\frac{\eta(r,w)}{2}\left(\sup\left\{\left|g^{'}\left(\sqrt{rw}\right)\right|,\left|g^{'}(w)\right|\right\}\right)+\frac{\eta\left(w,r\right)}{2}\left(\sup\left\{\left|g^{'}\left(r\right)\right|,\left|g^{'}\left(\sqrt{rw}\right)\right|\right\}\right),\end{aligned}$$

where $\eta\left(r,w\right)$ *is defined in Lemma 2.*

Proof. From Lemma 1 and using Hölder inequality, It is easy to observe that the following inequality holds

$$\begin{aligned}(2.5)\quad &\left|\frac{(\ln w)g\left(w\right)-(\ln r)g\left(r\right)}{\ln w-\ln r}-\frac{1}{\ln w-\ln r}\int_{r}^{w}\frac{g\left(l\right)}{l}dl\right|\\ &\leq\frac{1}{2}\int_{0}^{1}r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|\left|g^{'}\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|d\sigma\\ &\quad+\frac{1}{2}\int_{0}^{1}r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|\left|g^{'}\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|d\sigma.\end{aligned}$$

By using the geometric quasi-convexity of $\left|g^{'}\right|$ on $[r,w]$, we have

$$\left|g^{'}\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|\leq\sup\left\{|g^{'}\left(\sqrt{rw}\right)|,|g^{'}(w)|\right\}$$

and

$$\left|g^{'}\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|\leq\sup\left\{\left|g^{'}\left(r\right)\right|,\left|g^{'}\left(\sqrt{rw}\right)\right|\right\}.$$

By using Lemma 2 and the above results, we obtain

$$\begin{aligned}(2.6)\quad &\int_{0}^{1}r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|\left|g^{'}\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|d\sigma\\ &\leq\sup\left\{\left|g^{'}\left(\sqrt{rw}\right)\right|,\left|g^{'}(w)\right|\right\}\int_{0}^{1}r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|d\sigma\\ &=\eta\left(r,w\right)\left(\sup\left\{\left|g^{'}\left(\sqrt{rw}\right)\right|,\left|g^{'}(w)\right|\right\}\right)\end{aligned}$$

and

$$\begin{aligned}(2.7)\quad &\int_{0}^{1}r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|\left|g^{'}\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|d\sigma\\ &\leq\sup\left\{\left|g^{'}\left(r\right)\right|,\left|g^{'}\left(\sqrt{rw}\right)\right|\right\}\int_{0}^{1}r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|d\sigma\\ &=\eta\left(w,r\right)\left(\sup\left\{\left|g^{'}\left(r\right)\right|,\left|g^{'}\left(\sqrt{rw}\right)\right|\right\}\right).\end{aligned}$$

The inequality (2.4) can be obtained by using (2.6) and (2.7) in (2.5). □

Theorem 5. *Let* $g:J\subseteq\mathbb{R}_{+}\rightarrow\mathbb{R}$ *be a differentiable mapping on* J° *and* $g^{'}\in L\left[r,w\right]$ *for* $r,w\in J^{\circ}$ *with* $r<w$. *If* $\left|g^{'}\right|^{q}$ *is a geometrically quasi-convex on* $[r,w]$

for $q>1$*, then*

$$\begin{aligned}(2.8)\quad &\left|\frac{(\ln w)\,g(w)-(\ln r)\,g(r)}{\ln w-\ln r}-\frac{1}{\ln w-\ln r}\int_r^w \frac{g(l)}{l}dl\right|\\ &\le \frac{1}{2}\left\{\frac{q-1}{q}\eta\left(r^{\frac{q}{q-1}},w^{\frac{q}{q-1}}\right)\right\}^{1-\frac{1}{q}}\left[\xi(r,w)\left(\sup\left\{\left|g^{'}\left(\sqrt{rw}\right)\right|^q,\left|g^{'}(w)\right|^q\right\}\right)\right]^{\frac{1}{q}}\\ &+\frac{1}{2}\left\{\frac{q-1}{q}\eta\left(w^{\frac{q}{q-1}},r^{\frac{q}{q-1}}\right)\right\}^{1-\frac{1}{q}}\left[\xi(w,r)\left(\sup\left\{\left|g^{'}(w)\right|^q,\left|g^{'}\left(\sqrt{rw}\right)\right|^q\right\}\right)\right]^{\frac{1}{q}},\end{aligned}$$

where $\xi(u,v)$ *and* $\eta(u,v)$ *are defined in Lemma 2.*

Proof. From Lemma 1 and using Hölder inequality, we have

$$\begin{aligned}(2.9)\quad &\left|\frac{(\ln w)\,g(w)-(\ln r)\,g(r)}{\ln w-\ln r}-\frac{1}{\ln w-\ln r}\int_r^w \frac{g(l)}{l}dl\right|\\ &\le \frac{1}{2}\int_0^1 r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|\left|g^{'}\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|d\sigma\\ &+\frac{1}{2}\int_0^1 r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|\left|g^{'}\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|d\sigma\\ &\le \frac{1}{2}\left(\int_0^1 r^{\frac{q(1-\sigma)}{2(q-1)}}w^{\frac{q(1+\sigma)}{2(q-1)}}\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|d\sigma\right)^{1-\frac{1}{q}}\\ &\times\left(\int_0^1\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|\left|g^{'}\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|^q d\sigma\right)^{\frac{1}{q}}\\ &+\frac{1}{2}\left(\int_0^1 r^{\frac{q(1+\sigma)}{2(q-1)}}w^{\frac{q(1-\sigma)}{2(q-1)}}\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|d\sigma\right)^{1-\frac{1}{q}}\\ &\times\left(\int_0^1\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|\left|g^{'}\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|^q d\sigma\right)^{\frac{1}{q}}.\end{aligned}$$

By using the geometric quasi-convexity of $\left|g^{'}\right|^q$ on $[r,w]$, we obtain

$$\begin{aligned}(2.10)\quad &\int_0^1\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|\left|g^{'}\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|^q d\sigma\\ &\le\left(\sup\left\{\left|g^{'}\left(\sqrt{rw}\right)\right|^q,\left|g^{'}(w)\right|^q\right\}\right)\int_0^1\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|d\sigma\\ &=\xi(r,w)\left(\sup\left\{\left|g^{'}\left(\sqrt{rw}\right)\right|^q,\left|g^{'}(w)\right|^q\right\}\right)\end{aligned}$$

and

$$\begin{aligned}(2.11)\quad &\int_0^1\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|\left|g^{'}\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|^q d\sigma\\ &\le\left(\sup\left\{\left|g^{'}(r)\right|^q,\left|g^{'}\left(\sqrt{rw}\right)\right|^q\right\}\right)\int_0^1\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|d\sigma\\ &=\xi(w,r)\left(\sup\left\{\left|g^{'}(r)\right|^q,\left|g^{'}\left(\sqrt{rw}\right)\right|^q\right\}\right).\end{aligned}$$

Now an application of Lemma 2, gives us the following results

$$\int_0^1 r^{\frac{q(1-\sigma)}{2(q-1)}} w^{\frac{q(1+\sigma)}{2(q-1)}} \left|\ln\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| d\sigma = \frac{q-1}{q}\eta\left(r^{\frac{q}{q-1}}, w^{\frac{q}{q-1}}\right) \tag{2.12}$$

$$\int_0^1 r^{\frac{q(1+\sigma)}{2(q-1)}} w^{\frac{q(1-\sigma)}{2(q-1)}} \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| d\sigma = \frac{q-1}{q}\eta\left(w^{\frac{q}{q-1}}, r^{\frac{q}{q-1}}\right), \tag{2.13}$$

$$\int_0^1 \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| d\sigma = \xi\left(r,w\right) \text{ and } \int_0^1 \left|\ln\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| d\sigma = \xi\left(w,r\right). \tag{2.14}$$

A combination of the results (2.9)-(2.14) gives us the required inequality (2.8). Hence the proof of the theorem is complete. ☐

Theorem 6. *Let* $g : J \subseteq \mathbb{R}_+ \to \mathbb{R}$ *be a differentiable on* J° *and* $g' \in L\left[r,w\right]$ *for* $r, w \in J^\circ$ *with* $r < w$. *If* $\left|g'\right|^q$ *is a geometrically convex on* $\left[r,w\right]$ *for* $q > 1$ *and* ℓ *is a real number such that* $q > \ell > 0$. *Then*

$$\begin{aligned} &\left|\frac{\left(\ln w\right) g\left(w\right) - \left(\ln r\right) g\left(r\right)}{\ln w - \ln r} - \frac{1}{\ln w - \ln r}\int_r^w \frac{g\left(l\right)}{l}dl\right| \le \left(\frac{q-1}{q-\ell}\right)^{1-\frac{1}{q}} \left(\frac{1}{\ell}\right)^{\frac{1}{q}} \\ &\times \frac{1}{2}\left\{\left[\eta\left(r^{\frac{q-\ell}{q-1}}, w^{\frac{q-\ell}{q-1}}\right)\right]^{1-\frac{1}{q}} \left[\eta\left(r^\ell, w^\ell\right)\left(\sup\left\{\left|g'\left(\sqrt{rw}\right)\right|^q, \left|g'\left(w\right)\right|^q\right\}\right)\right]^{\frac{1}{q}}\right. \\ &\left. + \left[\eta\left(w^{\frac{q-\ell}{q-1}}, r^{\frac{q-\ell}{q-1}}\right)\right]^{1-\frac{1}{q}} \left[\eta\left(w^\ell, r^\ell\right)\left(\sup\left\{\left|g'\left(r\right)\right|^q, \left|g'\left(\sqrt{rw}\right)\right|^q\right\}\right)\right]^{\frac{1}{q}}\right\}, \end{aligned} \tag{2.15}$$

where $\eta\left(u,v\right)$ *is defined in Lemma 2.*

Proof.

$$\begin{aligned} &\left|\frac{\left(\ln w\right) g\left(w\right) - \left(\ln r\right) g\left(r\right)}{\ln w - \ln r} - \frac{1}{\ln w - \ln r}\int_r^w \frac{g\left(l\right)}{l}dl\right| \\ &\le \frac{1}{2}\int_0^1 r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}} \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| \left|g'\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| d\sigma \\ &+ \frac{1}{2}\int_0^1 r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}} \left|\ln\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| \left|g'\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| d\sigma \\ &\le \frac{1}{2}\left(\int_0^1 r^{\frac{(q-\ell)(1-\sigma)}{2(q-1)}} w^{\frac{(q-\ell)(1+\sigma)}{2(q-1)}} \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| d\sigma\right)^{1-\frac{1}{q}} \\ &\times \left(\int_0^1 r^{\frac{\ell(1-\sigma)}{2}} w^{\frac{\ell(1+\sigma)}{2}} \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| \left|g'\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right|^q d\sigma\right)^{\frac{1}{q}} \\ &+ \frac{1}{2}\left(\int_0^1 r^{\frac{(q-\ell)(1+\sigma)}{2(q-1)}} w^{\frac{(q-\ell)(1-\sigma)}{2(q-1)}} \left|\ln\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| d\sigma\right)^{1-\frac{1}{q}} \\ &\times \left(\int_0^1 r^{\frac{\ell(1+\sigma)}{2}} w^{\frac{\ell(1-\sigma)}{2}} \left|\ln\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| \left|g'\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right|^q d\sigma\right)^{\frac{1}{q}}. \end{aligned} \tag{2.16}$$

Now by using the quasi-geometric convexity of $\left|g^{'}\right|^q$ on $[r,w]$ and Lemma 2, we have

$$\begin{aligned}(2.17)\quad &\int_0^1 r^{\frac{\ell(1-\sigma)}{2}} w^{\frac{\ell(1+\sigma)}{2}} \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| \left|g^{'}\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right|^q d\sigma \\ &\qquad \le \frac{1}{\ell}\eta\left(r^\ell, w^\ell\right)\left(\sup\left\{\left|g^{'}\left(\sqrt{rw}\right)\right|^q, \left|g^{'}\left(w\right)\right|^q\right\}\right),\end{aligned}$$

$$\begin{aligned}(2.18)\quad &\int_0^1 r^{\frac{\ell(1+\sigma)}{2}} w^{\frac{\ell(1-\sigma)}{2}} \left|\ln\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| \left|g^{'}\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right|^q d\sigma \\ &\qquad \le \frac{1}{\ell}\eta\left(w^\ell, r^\ell\right)\left(\sup\left\{\left|g^{'}\left(r\right)\right|^q, \left|g^{'}\left(\sqrt{rw}\right)\right|^q\right\}\right),\end{aligned}$$

$$(2.19)\quad \int_0^1 r^{\frac{(q-\ell)(1-\sigma)}{2(q-1)}} w^{\frac{(q-\ell)(1+\sigma)}{2(q-1)}} \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| d\sigma = \left(\frac{q-1}{q-\ell}\right)\eta\left(r^{\frac{q-\ell}{q-1}}, w^{\frac{q-\ell}{q-1}}\right)$$

and

$$(2.20)\quad \int_0^1 r^{\frac{(q-\ell)(1+\sigma)}{2(q-1)}} w^{\frac{(q-\ell)(1-\sigma)}{2(q-1)}} \left|\ln\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| d\sigma = \left(\frac{q-1}{q-\ell}\right)\eta\left(w^{\frac{q-\ell}{q-1}}, r^{\frac{q-\ell}{q-1}}\right).$$

Applying (2.17)-(2.20) in (2.16), we get (2.15). This completes the proof of the theorem. □

Theorem 7. *Let $g:\mathbb{R}_+\to\mathbb{R}_0$ be a geometrically quasi-convex function on $[r,w]$ and $g\in L([r,w])$. Then*

$$\begin{aligned}(2.21)\quad g\left((rw)^{\frac{1}{2}}\right) &\le \frac{1}{\ln w-\ln r}\int_r^w \frac{g\left(l\right)}{l}\,dl \\ &\le \frac{1}{2}\left[\sup\left\{g\left(r\right), g\left(\sqrt{rw}\right)\right\} + \sup\left\{g(\sqrt{rw}), g(w)\right\}\right].\end{aligned}$$

Proof. It is easy to observe that

$$(rw)^{\frac{1}{2}} = r^{\frac{1}{2}\left(\frac{1-\sigma}{2}\right)} w^{\frac{1}{2}\left(\frac{1+\sigma}{2}\right)} r^{\frac{1}{2}\left(\frac{1+\sigma}{2}\right)} w^{\frac{1}{2}\left(\frac{1-\sigma}{2}\right)}$$

for $0\le\sigma\le1$. Since $g(l)$ is geometrically quasi- convex on $[r,w]$, we have

$$(2.22)\quad g\left((rw)^{\frac{1}{2}}\right) \le \sup\left\{g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right), g\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right\}.$$

Now we consider the two cases:
Case I
If

$$\sup\left\{g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right), g\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right\} = g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)$$

then

$$(2.23)\quad g\left((rw)^{\frac{1}{2}}\right) \le g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right).$$

Integrating (2.23) with respect to σ over $[0,1]$, we obtain

$$\begin{aligned}(2.24)\quad g\left((rw)^{\frac{1}{2}}\right) &\le \frac{2}{\ln w-\ln r}\int_{\sqrt{rw}}^w \frac{g\left(l\right)}{l}\,dl = \int_0^1 g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right) d\sigma \\ &\le \sup\left\{g\left(\sqrt{rw}\right), g\left(w\right)\right\}.\end{aligned}$$

$$\int_0^1 g(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}})\, d\sigma = \int_0^1 g(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}})\, d\sigma = \frac{1}{\ln w - \ln r}\int_r^w \frac{g(l)}{l}\, dl \tag{2.25}$$

Case II
If

$$\sup\left\{g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right), g\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right\} = g\left(r^{\frac{1+\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)$$

then

$$g\left((rw)^{\frac{1}{2}}\right) \le g\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right). \tag{2.26}$$

Integrating (2.26) with respect to σ over $[0,1]$, we get

$$\begin{aligned} g\left((rw)^{\frac{1}{2}}\right) \le \frac{2}{\ln w - \ln r}\int_r^{\sqrt{rw}} \frac{g\left(l\right)}{l}\, dl = \int_0^1 g\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right) d\sigma \\ \le \sup\left\{g\left(r\right), g\left(\sqrt{rw}\right)\right\}. \end{aligned} \tag{2.27}$$

Adding (2.24) and (2.27) and dividing the resulting inequality by 2, we get the inequality (2.21). Thus the proof of the theorem is completed. □

Theorem 8. *Let $g, h : [r, w] \subseteq \mathbb{R} \to \mathbb{R}$ be a geometrically convex function on $[r, w]$ and $gh \in L([r, w])$, then the following inequality holds*

$$\begin{aligned} &\frac{1}{\ln w - \ln r}\int_r^w \frac{g\left(l\right)}{l} h(l)\, dl \\ &\le \frac{1}{2}\left[\sup\left\{g(\sqrt{rw})h(\sqrt{rw}), g(\sqrt{rw})h(w), g(w)h(\sqrt{rw}), g(w)h(w)\right\}\right. \\ &\quad \left. + \sup\left\{g\left(\sqrt{rw}\right)h\left(\sqrt{rw}\right), g\left(r\right)h\left(\sqrt{rw}\right), g\left(\sqrt{rw}\right)h\left(r\right), g\left(r\right)h\left(r\right)\right\}\right]. \end{aligned} \tag{2.28}$$

Proof. Let $l = r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}$, $0 \le \sigma \le 1$ and using the geometric quasi-convex of g and h on $[r, w]$, we have

$$\begin{aligned} \frac{2}{\ln w - \ln r}\int_{\sqrt{rw}}^w \frac{g\left(l\right)h\left(l\right)}{l} dl = \int_0^1 g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right) h\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right) d\sigma \\ \le \sup\left\{g\left(\sqrt{rw}\right)h\left(\sqrt{rw}\right), g\left(w\right)h\left(\sqrt{rw}\right), g\left(\sqrt{rw}\right)h\left(w\right), g\left(w\right)h\left(w\right)\right\}. \end{aligned} \tag{2.29}$$

Similarly, the substitution $l = r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}$, $0 \le \sigma \le 1$, gives us the following result

$$\begin{aligned} \frac{2}{\ln w - \ln r}\int_r^{\sqrt{rw}} \frac{g\left(l\right)h\left(l\right)}{l} dl = \int_0^1 g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right) h\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right) d\sigma \\ \le \sup\left\{g\left(\sqrt{rw}\right)h\left(\sqrt{rw}\right), g\left(r\right)h\left(\sqrt{rw}\right), g\left(\sqrt{rw}\right)h\left(r\right), g\left(r\right)h\left(r\right)\right\}. \end{aligned} \tag{2.30}$$

Adding (2.29) and (2.30) and dividing the resulting inequality by 2, we get (2.28). □

3. Applications to Special Means

Let $w > r > 0$ and $s \in \mathbb{R}$. Consider the following means

$$G\left(r, w\right) = \sqrt{rw},$$

$$I\left(r, w\right) = \begin{cases} \frac{1}{e}\left(\frac{w^w}{r^r}\right)^{\frac{1}{w-r}}, & r \ne w, \\ a, & r = w, \end{cases}$$

$$L(a,b)=\begin{cases}\frac{w-r}{\ln w-\ln r}, & r\neq w,\\ \\ r, & r=w,\end{cases}$$

and

$$L_s(r,w)=\begin{cases}\left[\frac{w^{s+1}+r^{s+1}}{(s+1)(w-r)}\right]^{\frac{1}{s}}, & s\neq -1,0,\\ \\ L(r,w), & s=-1,\\ \\ I(r,w), & s=0,\end{cases}$$

The above means are known as respectively the geometric, exponential, logarithmic and generalized logarithmic means of two positive real numbers a and b.

We now give applications of our results to the above means.

Theorem 9. *Let $w>r>0$ and $s\in\mathbb{R}$.*

(1) *If $r<w<1$ and $s>0$, then*

$$\ln I\left(r^{s+1},w^{s+1}\right)\leq\frac{|s+1|\ln I(G(a,b),w)}{(\ln w-\ln r)\left[L_s\left(r^{s+1},w^{s+1}\right)\right]^s}\times\left\{w^s(G(a,b)-w)+[G(a,b)]^s(G(a,b)-r)\right\}. \tag{3.1}$$

(2) *If $1<r<w$ and $s<-1$, then*

$$\ln I\left(r^{s+1},w^{s+1}\right)\leq-\frac{|s+1|\ln I(G(a,b),w)}{(\ln w-\ln r)\left[L_s\left(r^{s+1},w^{s+1}\right)\right]^s}\times\left\{[G(a,b)]^s(G(a,b)-w)+r^s(G(a,b)-r)\right\}. \tag{3.2}$$

Proof. Let $f(l)=l^{s+1}$, $l\in\mathbb{R}_+$ and $s\in\mathbb{R}$, $s\neq-1$. If $m>l>0$, then

$$\left|f^{'}\left(l^{\frac{\sigma-1}{2}}m^{\frac{\sigma+1}{2}}\right)\right|^s\leq\begin{cases}|s+1|\,m^s, & ,s\geq 0\\ \\ |s+1|\,(lm)^{\frac{s}{2}}, & s<0.\end{cases}$$

Also

$$\left|f^{'}\left(l^{\frac{\sigma+1}{2}}m^{\frac{\sigma-1}{2}}\right)\right|^s\leq\begin{cases}|s+1|\,(lm)^{\frac{s}{2}}, & ,s\geq 0\\ \\ |s+1|\,l^s, & s<0.\end{cases}$$

This shows that the function $\left|f^{'}(l)\right|=|s+1|\,l^s$ for $s\in\mathbb{R}$, $s\neq-1$, is geometrically quasi-convex on $\mathbb{R}_+$.

Now

$$\frac{(\ln w)g(w)-(\ln r)g(r)}{\ln w-\ln r}-\frac{1}{\ln w-\ln r}\int_r^w\frac{g(l)}{l}dl=\left[L_s\left(r^{s+1},w^{s+1}\right)\right]^s\ln I\left(r^{s+1},w^{s+1}\right), \tag{3.3}$$

$$\begin{aligned}\frac{1}{2}\eta(r,w)\left(\sup\left\{\left|g^{'}\left(\sqrt{rw}\right)\right|,\left|g^{'}(w)\right|\right\}\right)&=\frac{|s+1|\,w^s\left[w-\sqrt{rw}+\sqrt{rw}\ln\left(\sqrt{rw}\right)-w\ln w\right]}{\ln w-\ln r}\\&=\frac{|s+1|\,w^s\left(\sqrt{rw}-w\right)\ln I\left(\sqrt{rw},w\right)}{\ln w-\ln r}\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\frac{1}{2}\eta(w,r)\left(\sup\left\{\left|g^{'}(r)\right|,\left|g^{'}\left(\sqrt{rw}\right)\right|\right\}\right) & \\
&= \frac{|s+1|\,(rw)^{\frac{s}{2}}\left[r-\sqrt{rw}+\sqrt{rw}\ln\left(\sqrt{rw}\right)-r\ln r\right]}{\ln w-\ln r} \\
&= \frac{|s+1|\,(rw)^{\frac{s}{2}}\left(\sqrt{rw}-r\right)\ln I\left(\sqrt{rw},r\right)}{\ln w-\ln r}.
\end{aligned} \tag{3.5}$$

Substituting (3.3), (3.4) and (3.5) in Theorem 4, we get (3.1). The inequality (3.2) can be obtained in similar way. □

The following numerical example illustrates that our results provide refinements of the results given in [10].

For the geometrically quasi-convex function $g\left(x\right)=x^3$, $x\in\mathbb{R}_+$, the following table is prepared using mathematica.

Function	Error Bound of Theorem 4	Error Bound of Theorem 1
$g\left(x\right)=x^3, a=\frac{1}{2}, b=\frac{3}{2}$	0.363473	0.418996
$g\left(x\right)=x^3, a=\frac{3}{2}, b=2$	10.4408	11.6002
$g\left(x\right)=x^3, a=\frac{1}{2}, b=\frac{3}{2}$	0.389078	1.60745

Similarly we can make tables for comparison of the results given in Theorem 2 and Theorem 3 with those given in Theorem 5 and Theorem 6 respectively.

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