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ON SOME HÖLDER TYPE TRACE INEQUALITIES FOR OPERATOR WEIGHTED GEOMETRIC MEAN

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ABSTRACT. In this paper we obtain some Hölder type trace inequalities for operator weighted geometric mean. Some vector inequalities are also given.

1. Introduction

If $\{e_i\}_{i\in I}$ is an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is *trace class* provided

$$\|A\|_1:=\sum_{i\in I}\langle|A|e_i,e_i\rangle<\infty. \tag{1.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following properties are also well known:

(i) We have

$$\|A\|_1=\|A^*\|_1 \tag{1.2}$$

for any $A\in\mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ is an *operator ideal* in $\mathcal{B}(H)$, i.e.

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H)\subseteq\mathcal{B}_1(H);$$

(iii) $(\mathcal{B}_1(H),\|\cdot\|_1)$ is a *Banach space.*

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{1.3}$$

where $\{e_i\}_{i\in I}$ is an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.3) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

(i) If $A\in\mathcal{B}_1(H)$ then $A^*\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.4}$$

(ii) If $A\in\mathcal{B}_1(H)$ and $T\in\mathcal{B}(H)$, then AT, $TA\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\text{ and }|\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.5}$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\|=1$;

(iv) $\mathcal{B}_{fin}(H)$, the space of operators of *finite rank*, is a dense subspace of $\mathcal{B}_1(H)$.

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Now, for the finite dimensional case, it is well known that the trace functional is *submultiplicative*, that is, for *positive semidefinite matrices* A and B in $M_n(\mathbb{C})$,

$$0 \leq \operatorname{tr}(AB) \leq \operatorname{tr}(A)\operatorname{tr}(B).$$

Therefore

$$0 \leq \operatorname{tr}(A^k) \leq [\operatorname{tr}(A)]^k,$$

where k is any positive integer.

In 2000, Yang [22] proved a matrix trace inequality

$$\operatorname{tr}\left[(AB)^k\right] \leq (\operatorname{tr} A)^k (\operatorname{tr} B)^k, \tag{1.6}$$

where A and B are positive semidefinite matrices over $\mathbb{C}$ of the same order n and k is any positive integer.

If $(H, \langle\cdot,\cdot\rangle)$ is a separable infinite-dimensional Hilbert space then the inequality (1.6) is also valid for any positive operators $A, B \in \mathcal{B}_1(H)$. This result was obtained by L. Liu in 2007, see [13].

In 2001, Yang et al. [23] improved (1.6) as follows:

$$\operatorname{tr}\left[(AB)^m\right] \leq \left[\operatorname{tr}\left(A^{2m}\right)\operatorname{tr}\left(B^{2m}\right)\right]^{1/2}, \tag{1.7}$$

where A and B are positive semidefinite matrices over $\mathbb{C}$ of the same order and m is any positive integer.

In [18] the authors have proved many trace inequalities for sums and products of matrices. For instance, if A and B are positive semidefinite matrices in $M_n(\mathbb{C})$ then

$$\operatorname{tr}\left[(AB)^k\right] \leq \min\left\{\|A\|^k \operatorname{tr}\left(B^k\right), \|B\|^k \operatorname{tr}\left(A^k\right)\right\} \tag{1.8}$$

for any positive integer k. Also, if $A, B \in M_n(\mathbb{C})$ then for $r \geq 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have the following *Young type inequality*

$$\operatorname{tr}\left(|AB^*|^r\right) \leq \operatorname{tr}\left[\left(\frac{|A|^p}{p} + \frac{|B|^q}{q}\right)^r\right]. \tag{1.9}$$

Ando [1] proved a strong form of Young's inequality - it was shown that if A and B are in $M_n(\mathbb{C})$, then there is a *unitary matrix* U such that

$$|AB^*| \leq U\left(\frac{1}{p}|A|^p + \frac{1}{q}|B|^q\right)U^*,$$

where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, which immediately gives the trace inequality

$$\operatorname{tr}\left(|AB^*|\right) \leq \frac{1}{p}\operatorname{tr}\left(|A|^p\right) + \frac{1}{q}\operatorname{tr}\left(|B|^q\right). \tag{1.10}$$

This inequality can also be obtained from (1.9) by taking $r = 1$.

The following Hölder's type inequality has been obtained by Ruskai in [16]

$$|\operatorname{tr}(AB)| \leq \operatorname{tr}(|AB|) \leq \left[\operatorname{tr}\left(|A|^p\right)\right]^{1/p}\left[\operatorname{tr}\left(|B|^q\right)\right]^{1/q} \tag{1.11}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and A, $B \in \mathcal{B}(H)$ with $|A|^p$, $|B|^q \in \mathcal{B}_1(H)$.

In particular, for $p = 2$ we get the Schwarz inequality

$$|\operatorname{tr}(AB)| \leq \operatorname{tr}(|AB|) \leq \left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} \tag{1.12}$$

with $|A|^2$, $|B|^2 \in \mathcal{B}_1(H)$.

For the theory of trace functionals and their applications the reader is referred to [20].

For some classical trace inequalities see [4], [6], [15] and [24], which are continuations of the work of Bellman [2]. For related works the reader can refer to [1], [3], [4], [9], [12], [13], [14], [17] and [21].

2. Some Hölder Type Trace Inequalities

Assume that A, B are positive invertible operators on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. We use the following notation

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},$$

for the *weighted geometric mean.* When $\nu = \frac{1}{2}$, we write $A\sharp B$ for brevity.

We have the following Hölder type trace inequality:

Theorem 1. *If A, B are positive invertible operators, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $A^p, B^q \in \mathcal{B}_1(H)$, then $B^q\sharp_{1/p}A^p \in \mathcal{B}_1(H)$ and*

$$\operatorname{tr}\left(B^q\sharp_{1/p}A^p\right) \le \left[\operatorname{tr}(A^p)\right]^{1/p}\left[\operatorname{tr}(B^q)\right]^{1/q}. \tag{2.1}$$

In particular, if $A^2, B^2 \in \mathcal{B}_1(H)$, then $B^2\sharp A^2 \in \mathcal{B}_1(H)$ and

$$\left[\operatorname{tr}\left(B^2\sharp A^2\right)\right]^2 \le \operatorname{tr}\left(A^2\right)\operatorname{tr}\left(B^2\right). \tag{2.2}$$

Proof. In [8], the authors obtained the following Hölder's type inequality for the weighted geometric mean:

$$\left\langle B^q\sharp_{1/p}A^p x, x\right\rangle \le \langle A^p x, x\rangle^{1/p}\langle B^q x, x\rangle^{1/q} \tag{2.3}$$

for any $x \in H$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. Then by (2.3) and Hölder's inequality we have

$$\begin{aligned}
\operatorname{tr}\left(B^q\sharp_{1/p}A^p\right) &= \sum_{i\in I}\left\langle B^q\sharp_{1/p}A^p e_i, e_i\right\rangle \\
&\le \sum_{i\in I}\langle A^p e_i, e_i\rangle^{1/p}\langle B^q e_i, e_i\rangle^{1/q} \\
&\le \left(\sum_{i\in I}\left[\langle A^p e_i, e_i\rangle^{1/p}\right]^p\right)^{1/p}\left(\sum_{i\in I}\left[\langle B^q e_i, e_i\rangle^{1/q}\right]^q\right)^{1/q} \\
&= \left(\sum_{i\in I}\langle A^p e_i, e_i\rangle\right)^{1/p}\left(\sum_{i\in I}\langle B^q e_i, e_i\rangle\right)^{1/q} = \left[\operatorname{tr}(A^p)\right]^{1/p}\left[\operatorname{tr}(B^q)\right]^{1/q},
\end{aligned}$$

which proves the desired inequality (2.1). □

Corollary 1. *If A_k, B_k are positive invertible operators, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and A_k^p, $B_k^q \in \mathcal{B}_1(H)$ for $k\in\{1,...,n\}$, then $B_k^q\sharp_{1/p}A_k^p \in \mathcal{B}_1(H)$ for $k \in \{1,...,n\}$ and for any $p_k \ge 0$, $k\in\{1,...,n\}$ we have*

$$\operatorname{tr}\left(\sum_{k=1}^n p_k B_k^q\sharp_{1/p}A_k^p\right) \le \left(\operatorname{tr}\left(\sum_{k=1}^n p_k A_k^p\right)\right)^{1/p}\left(\operatorname{tr}\left(\sum_{k=1}^n p_k B_k^q\right)\right)^{1/q}. \tag{2.4}$$

In particular, if $A_k^2,\ B_k^2 \in \mathcal{B}_1(H)$ *for* $k \in \{1,...,n\}$ *then* $B_k^2 \sharp A_k^2 \in \mathcal{B}_1(H)$ *for* $k \in \{1,...,n\}$ *and for any* $p_k \geq 0,\ k \in \{1,...,n\}$ *we have*

$$\left[\operatorname{tr}\left(\sum_{k=1}^{n} p_k B_k^2 \sharp A_k^2\right)\right]^2 \leq \operatorname{tr}\left(\sum_{k=1}^{n} p_k A_k^2\right) \operatorname{tr}\left(\sum_{k=1}^{n} p_k B_k^2\right). \tag{2.5}$$

Proof. Using Hölder's weighted discrete inequality we have

$$\begin{aligned}
\operatorname{tr}\left(\sum_{k=1}^{n} p_k B_k^q \sharp_{1/p} A_k^p\right) &= \sum_{k=1}^{n} p_k \operatorname{tr}\left(B_k^q \sharp_{1/p} A_k^p\right) \leq \sum_{k=1}^{n} p_k \left[\operatorname{tr}\left(A_k^p\right)\right]^{1/p} \left[\operatorname{tr}\left(B_k^q\right)\right]^{1/q} \\
&\leq \left(\sum_{k=1}^{n} p_k \left(\left[\operatorname{tr}\left(A_k^p\right)\right]^{1/p}\right)^p\right)^{1/p} \left(\sum_{k=1}^{n} p_k \left(\left[\operatorname{tr}\left(B_k^q\right)\right]^{1/q}\right)^q\right)^{1/q} \\
&= \left(\sum_{k=1}^{n} p_k \operatorname{tr}\left(A_k^p\right)\right)^{1/p} \left(\sum_{k=1}^{n} p_k \operatorname{tr}\left(B_k^q\right)\right)^{1/q} \\
&= \left(\operatorname{tr}\left(\sum_{k=1}^{n} p_k A_k^p\right)\right)^{1/p} \left(\operatorname{tr}\left(\sum_{k=1}^{n} p_k B_k^q\right)\right)^{1/q}
\end{aligned}$$

and the inequality (2.4) is proved. □

Theorem 2. *If* $A,\ B$ *are positive invertible operators,* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $C \in \mathcal{B}_1(H),\ C \geq 0$ *then* $CA^p,\ CB^q,\ C\left(B^q \sharp_{1/p} A^p\right) \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}\left(C\left(B^q \sharp_{1/p} A^p\right)\right) \leq \left[\operatorname{tr}\left(CA^p\right)\right]^{1/p} \left[\operatorname{tr}\left(CB^q\right)\right]^{1/q}. \tag{2.6}$$

In particular, if $C \in \mathcal{B}_1(H)$, *then* $CA^2,\ CB^2,\ C\left(B^2 \sharp A^2\right) \in \mathcal{B}_1(H)$ *and*

$$\left[\operatorname{tr}\left(C\left(B^2 \sharp A^2\right)\right)\right]^2 \leq \operatorname{tr}\left(CA^2\right) \operatorname{tr}\left(CB^2\right). \tag{2.7}$$

Proof. From the inequality (2.3) we have

$$\left\langle B^q \sharp_{1/p} A^p C^{1/2} x, C^{1/2} x\right\rangle \leq \left\langle A^p C^{1/2} x, C^{1/2} x\right\rangle^{1/p} \left\langle B^q C^{1/2} x, C^{1/2} x\right\rangle^{1/q}$$

for any $x \in H$, which is equivalent to

$$\left\langle C^{1/2} B^q \sharp_{1/p} A^p C^{1/2} x, x\right\rangle \leq \left\langle C^{1/2} A^p C^{1/2} x, x\right\rangle^{1/p} \left\langle C^{1/2} B^q C^{1/2} x, x\right\rangle^{1/q} \tag{2.8}$$

for any $x \in H$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. Then by (2.8) and Hölder's inequality we have

$$\begin{aligned}
&\operatorname{tr}\left(C\left(B^q\sharp_{1/p}A^p\right)\right)\\
&=\operatorname{tr}\left(C^{1/2}\left(B^q\sharp_{1/p}A^p\right)C^{1/2}\right)=\sum_{i\in I}\left\langle C^{1/2}\left(B^q\sharp_{1/p}A^p\right)C^{1/2}e_i,e_i\right\rangle\\
&\leq\sum_{i\in I}\left\langle C^{1/2}A^pC^{1/2}e_i,e_i\right\rangle^{1/p}\left\langle C^{1/2}B^qC^{1/2}e_i,e_i\right\rangle^{1/q}\\
&\leq\left(\sum_{i\in I}\left[\left\langle C^{1/2}A^pC^{1/2}e_i,e_i\right\rangle^{1/p}\right]^p\right)^{1/p}\left(\sum_{i\in I}\left[\left\langle C^{1/2}B^qC^{1/2}e_i,e_i\right\rangle^{1/q}\right]^q\right)^{1/q}\\
&=\left(\sum_{i\in I}\left\langle C^{1/2}A^pC^{1/2}e_i,e_i\right\rangle\right)^{1/p}\left(\sum_{i\in I}\left\langle C^{1/2}B^qC^{1/2}e_i,e_i\right\rangle\right)^{1/q}\\
&=\left[\operatorname{tr}\left(C^{1/2}A^pC^{1/2}\right)\right]^{1/p}\left[\operatorname{tr}\left(C^{1/2}B^qC^{1/2}\right)\right]^{1/q}=\left[\operatorname{tr}\left(CA^p\right)\right]^{1/p}\left[\operatorname{tr}\left(CB^q\right)\right]^{1/q},
\end{aligned}$$

which proves the desired result (2.6). □

Corollary 2. *If A_k, B_k are positive invertible operators, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $C_k\in\mathcal{B}_1(H)$, $C_k\geq 0$ for $k\in\{1,...,n\}$ then $C_kA_k^p$, $C_kB_k^q$, $C_k\left(B_k^q\sharp_{1/p}A_k^p\right)\in\mathcal{B}_1(H)$ for $k\in\{1,...,n\}$ and we have*

$$\operatorname{tr}\left(\sum_{k=1}^n C_k\left(B_k^q\sharp_{1/p}A_k^p\right)\right)\leq\left(\operatorname{tr}\left(\sum_{k=1}^n C_kA_k^p\right)\right)^{1/p}\left(\operatorname{tr}\left(\sum_{k=1}^n C_kB_k^q\right)\right)^{1/q}. \tag{2.9}$$

In particular, $C_kA_k^2$, $C_kB_k^2$, $C_k\left(B_k^2\sharp A_k^2\right)\in\mathcal{B}_1(H)$ for $k\in\{1,...,n\}$ and we have

$$\left[\operatorname{tr}\left(\sum_{k=1}^n C_k\left(B_k^2\sharp A_k^2\right)\right)\right]^2\leq\operatorname{tr}\left(\sum_{k=1}^n C_kA_k^2\right)\operatorname{tr}\left(\sum_{k=1}^n C_kB_k^2\right). \tag{2.10}$$

The proof follows by (2.6) on making use of a similar argument to the one in the proof of Corollary 1.

3. Some Reverse Vector Inequalities

We have the following reverse of Hölder's vector inequality for operators:

Theorem 3. *Let A and B be two positive invertible operators, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and m, $M>0$ such that*

$$m^pB^q\leq A^p\leq M^pB^q. \tag{3.1}$$

Then

$$\left\langle A^px,x\right\rangle^{1/p}\left\langle B^qx,x\right\rangle^{1/q}\leq\exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p-1\right)^2\right]\left\langle B^q\sharp_{1/p}A^px,x\right\rangle \tag{3.2}$$

for any $x\in H$.

Proof. In [7] we proved the following double inequality that provides a refinement and a reverse of the *arithmetic mean - geometric mean* inequality:

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right]\le\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\le\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right] \tag{3.3}$$

for any $a,\ b>0$ and $\nu\in[0,1]$.

If $a,\ b\in[t,T]\subset(0,\infty)$ and since

$$0<\frac{\max\{a,b\}}{\min\{a,b\}}-1\le\frac{T}{t}-1,$$

hence

$$\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\le\left(\frac{T}{t}-1\right)^2.$$

Therefore, by (3.3) we get

$$(1-\nu)a+\nu b\le a^{1-\nu}b^{\nu}\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{T}{t}-1\right)^2\right], \tag{3.4}$$

for any $a,\ b\in[t,T]$ and $\nu\in(0,1)$.

Now, if C is an operator with $tI\le C\le TI$ then for $p>1$ we have $t^pI\le C^p\le T^pI$. Using the functional calculus we get from (3.4) for $\nu=\frac{1}{p}$ that

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}C^p\le\exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]d^{1-\frac{1}{p}}C,$$

namely, the vector inequality,

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}\langle C^py,y\rangle\le\exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]d^{1-\frac{1}{p}}\langle Cy,y\rangle, \tag{3.5}$$

for any $y\in H$, $\|y\|=1$ and $d\in[t^p,T^p]$.

Since $d=\langle C^py,y\rangle\in[t^p,T^p]$ for any $y\in H$, $\|y\|=1$, hence by (3.5) we have

$$\left(1-\frac{1}{p}\right)\langle C^py,y\rangle+\frac{1}{p}\langle C^py,y\rangle\le\exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]\langle C^py,y\rangle^{1-\frac{1}{p}}\langle Cy,y\rangle,$$

that is equivalent to

$$\langle C^py,y\rangle\le\exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p-1\right)^2\right]\langle C^py,y\rangle^{1-\frac{1}{p}}\langle Cy,y\rangle,$$

and by division with $\langle C^p y, y\rangle^{1-\frac{1}{p}} > 0$, $y \in H$, $\|y\| = 1$, to

$$\langle C^p y, y\rangle^{1/p} \le \exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p - 1\right)^2\right]\langle Cy, y\rangle . \tag{3.6}$$

If $z \in H$ with $z \neq 0$, then by taking $y = \frac{z}{\|z\|}$ in (3.6) we get

$$\langle C^p z, z\rangle^{1/p}\langle z, z\rangle^{1/q} \le \exp\left[\frac{1}{2pq}\left(\left(\frac{T}{t}\right)^p - 1\right)^2\right]\langle Cz, z\rangle , \tag{3.7}$$

for any $z \in H$.

Now, from (3.1) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^p I \le B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} \le M^p I$ and by taking the power $\frac{1}{p}$ we get $mI \le \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} \le MI$.

By writing the inequality (3.7) for $C = \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}}$, $t = m$, $T = M$ and $z = B^{\frac{q}{2}}x$, with $x \in H$, we have

$$\begin{aligned}&\left\langle B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}} B^{\frac{q}{2}}x, B^{\frac{q}{2}}x\right\rangle^{1/p}\left\langle B^{\frac{q}{2}}x, B^{\frac{q}{2}}x\right\rangle^{1/q}\\ &\le \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p - 1\right)^2\right]\left\langle\left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} B^{\frac{q}{2}}x, B^{\frac{q}{2}}x\right\rangle ,\end{aligned}$$

namely

$$\begin{aligned}&\langle A^p x, x\rangle^{1/p}\langle B^q x, x\rangle^{1/q}\\ &\le \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p - 1\right)^2\right]\left\langle B^{\frac{q}{2}}\left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} B^{\frac{q}{2}}x, x\right\rangle ,\end{aligned}$$

for any $x \in H$, and the inequality (3.2) is proved. □

Remark 1. *We observe, for A and B two positive invertible operators, that the condition (3.1) is equivalent to following condition*

$$mI \le \left(B^{-\frac{q}{2}} A^p B^{-\frac{q}{2}}\right)^{\frac{1}{p}} \le MI. \tag{3.8}$$

If we assume that $rB^q \le A^p \le RB^q$, then by (3.2) we have the inequality

$$\langle A^p x, x\rangle^{1/p}\langle B^q x, x\rangle^{1/q} \le \exp\left[\frac{1}{2pq}\left(\frac{R}{r} - 1\right)^2\right]\left\langle B^q \sharp_{1/p} A^p x, x\right\rangle \tag{3.9}$$

for any $x \in H$.

The following particular case is related to Schwarz's trace inequality:

Corollary 3. *Let A and B be two positive invertible operators and m, $M > 0$ such that*

$$mI \le \left(B^{-1}A^2B^{-1}\right)^{\frac{1}{2}} \le MI, \tag{3.10}$$

then we have

$$\left\langle A^2 x, x\right\rangle^{1/2}\left\langle B^2 x, x\right\rangle^{1/2} \le \exp\left[\frac{1}{8}\left(\left(\frac{M}{m}\right)^2 - 1\right)^2\right]\left\langle A^2 \sharp B^2 x, x\right\rangle \tag{3.11}$$

for any $x \in H$.

Under more suitable conditions for the operators involved, we have:

Corollary 4. *Assume that A and B satisfy the conditions*

$$m_1 I \leq A \leq M_1 I,\ \ m_2 I \leq B \leq M_2 I \tag{3.12}$$

for some $0 < m_1 < M_1$ and $0 < m_2 < M_2$. Then we have

$$\begin{aligned} &\langle A^p x, x\rangle^{1/p} \langle B^q x, x\rangle^{1/q} \\ &\leq \exp\left[\frac{1}{2pq}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q - 1\right)^2\right] \left\langle B^q \sharp_{1/p} A^p x, x\right\rangle, \end{aligned} \tag{3.13}$$

for any $x \in H$.

In particular, we have

$$\left\langle A^2 x, x\right\rangle^{1/2} \left\langle B^2 x, x\right\rangle^{1/2} \leq \exp\left[\frac{1}{8}\left(\left(\frac{M_1 M_2}{m_1 m_2}\right)^2 - 1\right)^2\right] \left\langle A^2 \sharp B^2 x, x\right\rangle, \tag{3.14}$$

for any $x \in H$.

References

[1] T. Ando, Matrix Young inequalities, *Oper. Theory Adv. Appl.* **75** (1995), 33–38.

[2] R. Bellman, Some inequalities for positive definite matrices, in: E.F. Beckenbach (Ed.), *General Inequalities 2*, Proceedings of the 2nd International Conference on General Inequalities, Birkhäuser, Basel, 1980, pp. 89–90.

[3] E. V. Belmega, M. Jungers and S. Lasaulce, A generalization of a trace inequality for positive definite matrices. *Aust. J. Math. Anal. Appl.* **7** (2010), no. 2, Art. 26, 5 pp.

[4] D. Chang, A matrix trace inequality for products of Hermitian matrices, *J. Math. Anal. Appl.* **237** (1999) 721–725.

[5] L. Chen and C. Wong, Inequalities for singular values and traces, *Linear Algebra Appl.* **171** (1992), 109–120.

[6] I. D. Coop, On matrix trace inequalities and related topics for products of Hermitian matrix, *J. Math. Anal. Appl.* **188** (1994) 999–1001.

[7] S. S. Dragomir, A note on new refinements and reverses of Young's inequality, Preprint *RGMIA Res. Rep. Coll.* , 18 (2015), Art. 131. `Online http://rgmia.org/papers/v18/v18a131.pdf.`

[8] M. Fuji, S. Izumino, R. Nakamoto and Y. Seo, Operator inequalities related to Cauchy-Schwarz and Hölder-McCarthy inequalities, *Nihonkai Math. J.*, **8** (1997), 117-122.

[9] S. Furuichi and M. Lin, Refinements of the trace inequality of Belmega, Lasaulce and Debbah. *Aust. J. Math. Anal. Appl.* **7** (2010), no. 2, Art. 23, 4 pp.

[10] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[11] W. Greub and W. Rheinboldt, On a generalisation of an inequality of L.V. Kantorovich, *Proc. Amer. Math. Soc.,* **10** (1959), 407-415.

[12] H. D. Lee, On some matrix inequalities, *Korean J. Math.* **16** (2008), No. 4, pp. 565-571.

[13] L. Liu, A trace class operator inequality, *J. Math. Anal. Appl.* 328 (2007) 1484–1486.

[14] S. Manjegani, Hölder and Young inequalities for the trace of operators, *Positivity* **11** (2007), 239–250.

[15] H. Neudecker, A matrix trace inequality, *J. Math. Anal. Appl.* **166** (1992) 302–303.

[16] M. B. Ruskai, Inequalities for traces on von Neumann algebras, *Commun. Math. Phys.* **26**(1972), 280—289.

[17] K. Shebrawi and H. Albadawi, Operator norm inequalities of Minkowski type, *J. Inequal. Pure Appl. Math.* **9**(1) (2008), 1–10, article 26.

[18] K. Shebrawi and H. Albadawi, Trace inequalities for matrices, *Bull. Aust. Math. Soc.* **87** (2013), 139–148.

[19] O. Shisha and B. Mond, Bounds on differences of means, *Inequalities I,* New York-London, 1967, 293-308.

[20] B. Simon, *Trace Ideals and Their Applications,* Cambridge University Press, Cambridge, 1979.

[21] Z. Ulukök and R. Türkmen, On some matrix trace inequalities. *J. Inequal. Appl.* **2010**, Art. ID 201486, 8 pp.

[22] X. Yang, A matrix trace inequality, *J. Math. Anal. Appl.* **250** (2000) 372–374.

[23] X. M. Yang, X. Q. Yang and K. L. Teo, A matrix trace inequality, *J. Math. Anal. Appl.* **263** (2001), 327–331.

[24] Y. Yang, A matrix trace inequality, *J. Math. Anal. Appl.* **133** (1988) 573–574.

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