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POWER AND HÖLDER TYPE TRACE INEQUALITIES FOR POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. In this paper we obtain some new power and Hölder type trace inequalities for positive operators in Hilbert spaces. As tools, we use some recent reverses and refinements of Young inequality obtained by several authors.

1. Introduction

In matrix theory, it is well known that the trace functional is *submultiplicative*, that is, for *positive semidefinite matrices* A and B in $M_n(\mathbb{C})$,

$$0 \leq \operatorname{tr}(AB) \leq \operatorname{tr}(A) \operatorname{tr}(B).$$

Therefore

$$0 \leq \operatorname{tr}(A^k) \leq [\operatorname{tr}(A)]^k,$$

where k is any positive integer.

In 2000, Yang [33] proved a matrix trace inequality

$$\operatorname{tr}\left[(AB)^k\right] \leq (\operatorname{tr} A)^k (\operatorname{tr} B)^k, \tag{1.1}$$

where A and B are positive semidefinite matrices over $\mathbb{C}$ of the same order n and k is any positive integer.

In 2001, Yang et al. [34] improved (1.1) as follows:

$$\operatorname{tr}\left[(AB)^m\right] \leq \left[\operatorname{tr}\left(A^{2m}\right) \operatorname{tr}\left(B^{2m}\right)\right]^{1/2}, \tag{1.2}$$

where A and B are positive semidefinite matrices over $\mathbb{C}$ of the same order and m is any positive integer.

In [27] the authors have proved many trace inequalities for sums and products of matrices. For instance, if A and B are positive semidefinite matrices in $M_n(\mathbb{C})$, then

$$\operatorname{tr}\left[(AB)^k\right] \leq \min\left\{\|A\|^k \operatorname{tr}\left(B^k\right), \|B\|^k \operatorname{tr}\left(A^k\right)\right\} \tag{1.3}$$

for any positive integer k. Also, if $A, B \in M_n(\mathbb{C})$ then for $r \geq 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have the following *Young type inequality*

$$\operatorname{tr}\left(|AB^*|^r\right) \leq \operatorname{tr}\left[\left(\frac{|A|^p}{p} + \frac{|B|^q}{q}\right)^r\right]. \tag{1.4}$$

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Ando [1] proved a strong form of Young's inequality - it was shown that if A and B are in $M_n(\mathbb{C})$, then there is a *unitary matrix* U such that

$$|AB^*| \le U\left(\frac{1}{p}|A|^p + \frac{1}{q}|B|^q\right)U^*,$$

where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, which immediately gives the trace inequality

$$\operatorname{tr}(|AB^*|) \le \frac{1}{p}\operatorname{tr}(|A|^p) + \frac{1}{q}\operatorname{tr}(|B|^q). \tag{1.5}$$

This inequality can also be obtained from (1.4) by taking $r = 1$.

In the general case of Hilbert spaces $(H; \langle\cdot,\cdot\rangle)$, if $\{e_i\}_{i\in I}$ is an orthonormal basis of H, we say that a bounded linear operator $A \in \mathcal{B}(H)$ is *trace class* provided

$$\|A\|_1 := \sum_{i\in I}\langle |A| e_i, e_i\rangle < \infty. \tag{1.6}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following properties are also well known:

(i) We have

$$\|A\|_1 = \|A^*\|_1 \tag{1.7}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ is an *operator ideal* in $\mathcal{B}(H)$, i.e.

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) $(\mathcal{B}_1(H), \|\cdot\|_1)$ is a *Banach space*.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I}\langle Ae_i, e_i\rangle, \tag{1.8}$$

where $\{e_i\}_{i\in I}$ is an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.8) converges absolutely and it is independent from the choice of basis.

The following results collects some properties of the trace:

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.9}$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then AT, $TA \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1\|T\|; \tag{1.10}$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\| = 1$;

(iv) $\mathcal{B}_{fin}(H)$, the space of operators of *finite rank*, is a dense subspace of $\mathcal{B}_1(H)$.

The following Hölder's type inequality has been obtained by Ruskai in [25]

$$|\operatorname{tr}(AB)| \le \operatorname{tr}(|AB|) \le [\operatorname{tr}(|A|^p)]^{1/p}[\operatorname{tr}(|B|^q)]^{1/q} \tag{1.11}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and A, $B \in \mathcal{B}(H)$ with $|A|^p$, $|B|^q \in \mathcal{B}_1(H)$.

In particular, for $p = 2$ we get the Schwarz inequality

$$|\operatorname{tr}(AB)| \le \operatorname{tr}(|AB|) \le \left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} \tag{1.12}$$

with $|A|^2$, $|B|^2 \in \mathcal{B}_1(H)$.

Assume that A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notation

$$A \sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2}, \tag{1.13}$$

for the *weighted geometric mean.* When $\nu = \frac{1}{2}$, we write $A \sharp B$ for brevity.

We have the following Hölder type trace inequality for the weighted geometric mean [12]: If A, B are positive invertible operators, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and A^p, $B^q \in \mathcal{B}_1(H)$, then $B^q \sharp_{1/p} A^p \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}\left(B^q \sharp_{1/p} A^p \right) \le \left[\operatorname{tr}(A^p) \right]^{1/p} \left[\operatorname{tr}(B^q) \right]^{1/q}. \tag{1.14}$$

In particular, if A^2, $B^2 \in \mathcal{B}_1(H)$, then $B^2 \sharp A^2 \in \mathcal{B}_1(H)$ and

$$\left[\operatorname{tr}\left(B^2 \sharp A^2 \right) \right]^2 \le \operatorname{tr}\left(A^2 \right) \operatorname{tr}\left(B^2 \right). \tag{1.15}$$

Also, if A, B are positive invertible operators, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $C \in \mathcal{B}_1(H)$, $C \ge 0$ then CA^p, CB^q, $C\left(B^q \sharp_{1/p} A^p \right) \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}\left(C \left(B^q \sharp_{1/p} A^p \right) \right) \le \left[\operatorname{tr}(CA^p) \right]^{1/p} \left[\operatorname{tr}(CB^q) \right]^{1/q}. \tag{1.16}$$

In particular, if $C \in \mathcal{B}_1(H)$, then CA^2, CB^2, $C\left(B^2 \sharp A^2 \right) \in \mathcal{B}_1(H)$ and

$$\left[\operatorname{tr}\left(C \left(B^2 \sharp A^2 \right) \right) \right]^2 \le \operatorname{tr}\left(CA^2 \right) \operatorname{tr}\left(CB^2 \right). \tag{1.17}$$

Related inequalities may be found in [12] as well.

For the theory of trace functionals and their applications the reader is referred to [29].

For some classical trace inequalities see [4], [6], [24] and [35], which are continuations of the work of Bellman [2]. For related works the reader can refer to [1], [3], [4], [14], [20], [22], [23], [26] and [32].

Motivated by the above results, we establish in this paper some new trace inequalities via recent scalar Young type inequalities.

2. Some Results Via Kittaneh-Manasrah Inequality

Kittaneh and Manasrah [18], [19] provided a refinement and a reverse for *Young's inequality* as follows:

$$r \left(\sqrt{a} - \sqrt{b} \right)^2 \le (1 - \nu) a + \nu b - a^{1-\nu} b^\nu \le R \left(\sqrt{a} - \sqrt{b} \right)^2, \tag{2.1}$$

where a, $b > 0$, $\nu \in [0, 1]$, $r = \min\{1 - \nu, \nu\}$ and $R = \max\{1 - \nu, \nu\}$. The case $\nu = \frac{1}{2}$ reduces (2.1) to an identity.

We have:

Theorem 1. *Let C be a positive operator and $P \in \mathcal{B}_1(H)$, $P > 0$. Then for any $\nu \in [0, 1]$ we have*

$$\begin{aligned} &2r \left(\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right)^{1/2} - \frac{\operatorname{tr}\left(PC^{1/2} \right)}{\operatorname{tr}(P)} \right) \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right)^{\nu - \frac{1}{2}} \\ &\le \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right)^\nu - \frac{\operatorname{tr}(PC^\nu)}{\operatorname{tr}(P)} \\ &\le 2R \left(\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right)^{1/2} - \frac{\operatorname{tr}\left(PC^{1/2} \right)}{\operatorname{tr}(P)} \right) \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right)^{\nu - \frac{1}{2}} \end{aligned} \tag{2.2}$$

and

$$r\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)^{2}\right) \tag{2.3}$$
$$\leq \nu\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}+(1-\nu)\left(\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)^{2}-\frac{\operatorname{tr}(PC^{\nu})}{\operatorname{tr}(P)}\left(\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)^{2(1-\nu)}$$
$$\leq R\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)^{2}\right)$$

where $r=\min\{1-\nu,\nu\}$ *and* $R=\max\{1-\nu,\nu\}$.

Proof. Fix $b\geq 0$, and by using the functional calculus for the operator C, we have from (2.1) that

$$r\left(\langle Cx,x\rangle-2\sqrt{b}\left\langle C^{1/2}x,x\right\rangle+b\langle x,x\rangle\right) \tag{2.4}$$
$$\leq (1-\nu)\langle Cx,x\rangle+\nu b\langle x,x\rangle-b^{\nu}\left\langle C^{1-\nu}x,x\right\rangle$$
$$\leq R\left(\langle Cx,x\rangle-2\sqrt{b}\left\langle C^{1/2}x,x\right\rangle+b\langle x,x\rangle\right)$$

for any $x\in H$.

Now, let $x=P^{1/2}e$ where $e\in H$. Then by (2.4) we get

$$r\left(\left\langle P^{1/2}CP^{1/2}e,e\right\rangle-2\sqrt{b}\left\langle P^{1/2}C^{1/2}P^{1/2}e,e\right\rangle+b\langle Pe,e\rangle\right) \tag{2.5}$$
$$\leq (1-\nu)\left\langle P^{1/2}CP^{1/2}e,e\right\rangle+\nu b\langle Pe,e\rangle-b^{\nu}\left\langle P^{1/2}C^{1-\nu}P^{1/2}e,e\right\rangle$$
$$\leq R\left(\left\langle P^{1/2}CP^{1/2}e,e\right\rangle-2\sqrt{b}\left\langle P^{1/2}C^{1/2}P^{1/2}e,e\right\rangle+b\langle Pe,e\rangle\right)$$

for any $e\in H$ and $b\geq 0$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. If we take in (2.5) $e=e_i$, $i\in I$ and summing over $i\in I$, then we get

$$r\left(\sum_{i\in I}\left\langle P^{1/2}CP^{1/2}e_i,e_i\right\rangle-2\sqrt{b}\sum_{i\in I}\left\langle P^{1/2}C^{1/2}P^{1/2}e_i,e_i\right\rangle+b\sum_{i\in I}\langle Pe_i,e_i\rangle\right)$$
$$\leq (1-\nu)\sum_{i\in I}\left\langle P^{1/2}CP^{1/2}e_i,e_i\right\rangle+\nu b\sum_{i\in I}\langle Pe_i,e_i\rangle-b^{\nu}\sum_{i\in I}\left\langle P^{1/2}C^{1-\nu}P^{1/2}e_i,e_i\right\rangle$$
$$\leq R\left(\sum_{i\in I}\left\langle P^{1/2}CP^{1/2}e_i,e_i\right\rangle-2\sqrt{b}\sum_{i\in I}\left\langle P^{1/2}C^{1/2}P^{1/2}e_i,e_i\right\rangle+b\sum_{i\in I}\langle Pe_i,e_i\rangle\right)$$

for any $b\geq 0$ and by using the properties of the trace, we obtain

$$r\left(\operatorname{tr}(PC)-2\operatorname{tr}\left(PC^{1/2}\right)\sqrt{b}+\operatorname{tr}(P)\,b\right)$$
$$\leq (1-\nu)\operatorname{tr}(PC)+\nu\operatorname{tr}(P)\,b-\operatorname{tr}\left(PC^{1-\nu}\right)b^{\nu}$$
$$\leq R\left(\operatorname{tr}(PC)-2\operatorname{tr}\left(PC^{1/2}\right)\sqrt{b}+\operatorname{tr}(P)\,b\right)$$

for any $b\geq 0$.

Dividing by $\operatorname{tr}(P)>0$ we get

$$r\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\sqrt{b}+b\right) \tag{2.6}$$
$$\leq(1-\nu)\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}+\nu b-\frac{\operatorname{tr}\left(PC^{1-\nu}\right)}{\operatorname{tr}(P)}b^{\nu}$$
$$\leq R\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\sqrt{b}+b\right)$$

for any $b\geq0$.

This inequality is of interest in itself as well.

Now, if we take in (2.6) $b=\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}$, then we get for $\nu\in[0,1]$ that

$$2r\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\sqrt{\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}}\right)$$
$$\leq\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}\left(PC^{1-\nu}\right)}{\operatorname{tr}(P)}\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{\nu}$$
$$\leq 2R\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\sqrt{\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}}\right)$$

which is equivalent to

$$2r\left(\sqrt{\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}}-\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)\sqrt{\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}} \tag{2.7}$$
$$\leq\left(\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1-\nu}-\frac{\operatorname{tr}\left(PC^{1-\nu}\right)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{\nu}$$
$$\leq 2R\left(\sqrt{\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}}-\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)\sqrt{\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}},$$

Now if we replace ν by $1-\nu$ in (2.7) we deduce (2.2).

Also, if we take in (2.6) $b=\left(\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right)^{2}$ and replace ν by $1-\nu$ then we get the inequality (2.3). □

Corollary 1. *If P, Q are positive invertible operators with $P,Q\in\mathcal{B}_1(H)$, then for any $\nu\in[0,1]$ we have*

$$2r\left(\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}\right)^{1/2}-\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}\right)^{\nu-\frac{1}{2}} \tag{2.8}$$
$$\leq\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}\right)^{\nu}-\frac{\operatorname{tr}(P\sharp_{\nu}Q)}{\operatorname{tr}(P)}$$
$$\leq 2R\left(\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}\right)^{1/2}-\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}\right)^{\nu-\frac{1}{2}},$$

$$r\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)^{2}\right) \tag{2.9}$$
$$\leq \nu\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}+(1-\nu)\left(\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)^{2}-\frac{\operatorname{tr}(P\sharp_{\nu}Q)}{\operatorname{tr}(P)}\left(\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)^{2(1-\nu)}$$
$$\leq R\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)^{2}\right)$$

where $r=\min\{1-\nu,\nu\}$ *and* $R=\max\{1-\nu,\nu\}$.

Proof. The proof follows by (2.2) on choosing $C=P^{-1/2}QP^{-1/2}$. □

Corollary 2. *Let* A *and* B *be two positive invertible operators,* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and such that* $A^{p},\ B^{q}\in\mathcal{B}_{1}(H)$. *Then*

$$2t\left(\left(\frac{\operatorname{tr}(A^{p})}{\operatorname{tr}(B^{q})}\right)^{1/2}-\frac{\operatorname{tr}(B^{q}\sharp A^{p})}{\operatorname{tr}(B^{q})}\right)\left(\frac{\operatorname{tr}(B^{q}\sharp A^{p})}{\operatorname{tr}(B^{q})}\right)^{\frac{1}{p}-\frac{1}{2}} \tag{2.10}$$
$$\leq\left(\frac{\operatorname{tr}(A^{p})}{\operatorname{tr}(B^{q})}\right)^{\frac{1}{p}}-\frac{\operatorname{tr}\left(B^{q}\sharp_{1/p}A^{p}\right)}{\operatorname{tr}(B^{q})}$$
$$\leq 2T\left(\left(\frac{\operatorname{tr}(A^{p})}{\operatorname{tr}(B^{q})}\right)^{1/2}-\frac{\operatorname{tr}(B^{q}\sharp A^{p})}{\operatorname{tr}(B^{q})}\right)\left(\frac{\operatorname{tr}(B^{q}\sharp A^{p})}{\operatorname{tr}(B^{q})}\right)^{\frac{1}{p}-\frac{1}{2}}$$

and

$$t\left(\frac{\operatorname{tr}(A^{p})}{\operatorname{tr}(B^{q})}-\left(\frac{\operatorname{tr}(B^{q}\sharp A^{p})}{\operatorname{tr}(B^{q})}\right)^{2}\right) \tag{2.11}$$
$$\leq\frac{1}{p}\frac{\operatorname{tr}(A^{p})}{\operatorname{tr}(B^{q})}+\frac{1}{q}\left(\frac{\operatorname{tr}(B^{q}\sharp A^{p})}{\operatorname{tr}(B^{q})}\right)^{2}-\frac{\operatorname{tr}\left(B^{q}\sharp_{1/p}A^{p}\right)}{\operatorname{tr}(B^{q})}\left(\frac{\operatorname{tr}(B^{q}\sharp A^{p})}{\operatorname{tr}(B^{q})}\right)^{2/q}$$
$$\leq T\left(\frac{\operatorname{tr}(A^{p})}{\operatorname{tr}(B^{q})}-\left(\frac{\operatorname{tr}(B^{q}\sharp A^{p})}{\operatorname{tr}(B^{q})}\right)^{2}\right),$$

where $t=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ *and* $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

The proof follows by Corollary 1 for $P=A^{p}$, $Q=B^{q}$ and $\nu=\frac{1}{p}$.

3. Some Results Via Tominaga Inequality

We recall that *Specht's ratio* is defined by [30]

$$S(h):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h\in(0,1)\cup(1,\infty),\\[2ex] 1 & \text{if } h=1.\end{cases} \tag{3.1}$$

It is well known that $\lim_{h\to1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a multiplicative reverse for Young's inequality

$$\left(a^{1-\nu}b^{\nu}\leq\right)(1-\nu)a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{3.2}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$. This inequality is due to Tominaga [31].

Theorem 2. *Let C be an operator with the property that*

$$mI\le C\le MI \tag{3.3}$$

for some constants m, M with $M>m>0$ and $P\in\mathcal{B}_1(H)$, $P\ge 0$ with $\operatorname{tr}(P)>0$. Then for any $p>1$ we have

$$\left(\frac{\operatorname{tr}(PC^p)}{\operatorname{tr}(P)}\right)^{1/p}\le S\left(\left(\frac{M}{m}\right)^p\right)\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}. \tag{3.4}$$

In particular, we have

$$\operatorname{tr}\left(PC^2\right)\operatorname{tr}(P)\le S^2\left(\left(\frac{M}{m}\right)^2\right)\left[\operatorname{tr}(PC)\right]^2. \tag{3.5}$$

Proof. Assume that $\nu\in(0,1)$. Let a, $b\in[m,M]\subset(0,\infty)$, then $\frac{m}{M}\le\frac{a}{b}\le\frac{M}{m}$ with $\frac{m}{M}<1<\frac{M}{m}$. If $\frac{a}{b}\in\left[\frac{m}{M},1\right)$ then $S\left(\frac{a}{b}\right)\le S\left(\frac{m}{M}\right)=S\left(\frac{M}{m}\right)$. If $\frac{a}{b}\in\left(1,\frac{M}{m}\right]$ then also $S\left(\frac{a}{b}\right)\le S\left(\frac{M}{m}\right)$. Therefore for any a, $b\in[m,M]$ we have by Tominaga's inequality (3.2) that

$$(1-\nu)a+\nu b\le S\left(\frac{M}{m}\right)a^{1-\nu}b^{\nu}. \tag{3.6}$$

Now, if C is an operator with $mI\le C\le MI$ then for $p>1$ we have $m^pI\le C^p\le M^pI$. Using the functional calculus we get from (3.6) for $\nu=\frac{1}{p}$ that

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}C^p\le S\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}C,$$

namely, the vector inequality,

$$\left(1-\frac{1}{p}\right)d\langle y,y\rangle+\frac{1}{p}\langle C^py,y\rangle\le S\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}\langle Cy,y\rangle, \tag{3.7}$$

for any $y\in H$ and $d\in[m^p,M^p]$.

Now, let $y=P^{1/2}e$ where $e\in H$. Then by (3.7) we get

$$\begin{aligned}&\left(1-\frac{1}{p}\right)d\langle Pe,e\rangle+\frac{1}{p}\left\langle P^{1/2}C^pP^{1/2}e,e\right\rangle\\&\le S\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}\left\langle P^{1/2}CP^{1/2}e,e\right\rangle,\end{aligned} \tag{3.8}$$

for any $e\in H$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. If we take in (3.8) $e=e_i$, $i\in I$ and summing over $i\in I$, then we get

$$\begin{aligned}&\left(1-\frac{1}{p}\right)d\sum_{i\in I}\langle Pe_i,e_i\rangle+\frac{1}{p}\sum_{i\in I}\left\langle P^{1/2}C^pP^{1/2}e_i,e_i\right\rangle\\&\le S\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}\sum_{i\in I}\left\langle P^{1/2}CP^{1/2}e_i,e_i\right\rangle,\end{aligned}$$

and by the properties of trace

$$\left(1-\frac{1}{p}\right)d\operatorname{tr}(P)+\frac{1}{p}\operatorname{tr}(PC^p)\le S\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}\operatorname{tr}(PC),$$

for any $d \in [m^p, M^p]$.

This inequality can be written as

$$\left(1-\frac{1}{p}\right)d+\frac{1}{p}\frac{\operatorname{tr}(PC^p)}{\operatorname{tr}(P)} \le S\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}, \tag{3.9}$$

for any $d \in [m^p, M^p]$, that is of interest in itself.

Now, if we take in (3.9) $d = \frac{\operatorname{tr}(PC^p)}{\operatorname{tr}(P)} \in [m^p, M^p]$, then we get

$$\frac{\operatorname{tr}(PC^p)}{\operatorname{tr}(P)} \le S\left(\left(\frac{M}{m}\right)^p\right)\left(\frac{\operatorname{tr}(PC^p)}{\operatorname{tr}(P)}\right)^{1-\frac{1}{p}}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)},$$

which is equivalent to (3.4). □

Corollary 3. *Let A and B be two positive invertible operators, p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and m, $M > 0$ such that $B^q \in \mathcal{B}_1(H)$ and*

$$m^p B^q \le A^p \le M^p B^q. \tag{3.10}$$

Then

$$[\operatorname{tr}(A^p)]^{1/p}[\operatorname{tr}(B^q)]^{1/q} \le S\left(\left(\frac{M}{m}\right)^p\right)\operatorname{tr}\left(B^q \sharp_{1/p} A^p\right). \tag{3.11}$$

Proof. The inequality (3.4) can be written as

$$\left[\operatorname{tr} P^{1/2}C^pP^{1/2}\right]^{1/p}[\operatorname{tr}(P)]^{1/q} \le S\left(\left(\frac{M}{m}\right)^p\right)\operatorname{tr}\left(P^{1/2}CP^{1/2}\right). \tag{3.12}$$

Now, from (3.10) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^pI \le B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}} \le M^pI$ and by taking the power $\frac{1}{p}$ we get $mI \le \left(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}}\right)^{\frac{1}{p}} \le MI$.

By writing the inequality (3.12) for $C = \left(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}}\right)^{\frac{1}{p}}$ and $P = B^q$ then we get

$$\begin{aligned}&\left[\operatorname{tr}\left(B^{q/2}\left[\left(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}}\right)^{\frac{1}{p}}\right]^pB^{q/2}\right)\right]^{1/p}[\operatorname{tr}(B^q)]^{1/q}\\&\le S\left(\left(\frac{M}{m}\right)^p\right)\operatorname{tr}\left(B^{q/2}\left(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}}\right)^{\frac{1}{p}}B^{q/2}\right),\end{aligned}$$

i.e.,

$$[\operatorname{tr}(A^p)]^{1/p}[\operatorname{tr}(B^q)]^{1/q} \le S\left(\left(\frac{M}{m}\right)^p\right)\operatorname{tr}\left(B^{q/2}\left(B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}}\right)^{\frac{1}{p}}B^{q/2}\right),$$

and the inequality (3.11) is proved. □

Corollary 4. *Let A and B be two positive invertible operators and m, $M > 0$ such that $B^2 \in \mathcal{B}_1(H)$ and*

$$m^2B^2 \le A^2 \le M^2B^2. \tag{3.13}$$

Then

$$\operatorname{tr}\left(A^2\right)\operatorname{tr}\left(B^2\right) \le S^2\left(\left(\frac{M}{m}\right)^2\right)\left[\operatorname{tr}\left(B^2\sharp A^2\right)\right]^2. \tag{3.14}$$

Remark 1. *We remark that the condition (3.13) can be written as*

$$kB^2 \leq A^2 \leq KB^2 \tag{3.15}$$

where $0 < k < K$, *then by (3.14) we have*

$$\operatorname{tr}\left(A^2\right)\operatorname{tr}\left(B^2\right) \leq S^2\left(\frac{K}{k}\right)\left[\operatorname{tr}\left(B^2\sharp A^2\right)\right]^2. \tag{3.16}$$

4. Some Results Via Liao-Wu-Zhao

We consider the *Kantorovich's constant* defined by

$$K\left(h\right) := \frac{\left(h+1\right)^2}{4h},\ h > 0. \tag{4.1}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative reverse of Young inequality in terms of Kantorovich's constant holds

$$\left(a^{1-\nu}b^{\nu} \leq\right)\left(1-\nu\right)a + \nu b \leq K^R\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{4.2}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$.

This inequality was obtained by Liao et al. in [21].

Theorem 3. *Let* C *be an operator with the property (3.3) for some constants* m, M *with* $M > m > 0$ *and* $P \in \mathcal{B}_1(H)$, $P > 0$. *Then for any* p, $q > 1$ *with* $\frac{1}{p}+\frac{1}{q} = 1$ *we have*

$$\left(\frac{\operatorname{tr}\left(PC^p\right)}{\operatorname{tr}\left(P\right)}\right)^{1/p} \leq K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(\left(\frac{M}{m}\right)^p\right)\frac{\operatorname{tr}\left(PC\right)}{\operatorname{tr}\left(P\right)}. \tag{4.3}$$

In particular, we have

$$\operatorname{tr}\left(PC^2\right)\operatorname{tr}\left(P\right) \leq K\left(\left(\frac{M}{m}\right)^2\right)\left[\operatorname{tr}\left(PC\right)\right]^2. \tag{4.4}$$

Proof. Assume that $\nu \in (0,1)$ and $R = \max\{1-\nu,\nu\}$. Let a, $b \in [m,M] \subset (0,\infty)$, then $\frac{m}{M} \leq \frac{a}{b} \leq \frac{M}{m}$ with $\frac{m}{M} < 1 < \frac{M}{m}$. If $\frac{a}{b} \in \left[\frac{m}{M},1\right)$ then $K^R\left(\frac{a}{b}\right) \leq K^R\left(\frac{m}{M}\right) = K^R\left(\frac{M}{m}\right)$. If $\frac{a}{b} \in \left(1,\frac{M}{m}\right]$ then also $K^R\left(\frac{a}{b}\right) \leq K^R\left(\frac{M}{m}\right)$. Therefore for any a, $b \in [m,M]$ we have by inequality (4.2) that

$$\left(1-\nu\right)a + \nu b \leq K^R\left(\frac{M}{m}\right)a^{1-\nu}b^{\nu}. \tag{4.5}$$

Now, if C is an operator with $mI \leq C \leq MI$ then for $p > 1$ we have $m^pI \leq C^p \leq M^pI$. Using the functional calculus we get from (4.5) for $\nu = \frac{1}{p}$ that

$$\left(1-\frac{1}{p}\right)d + \frac{1}{p}C^p \leq K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}C,$$

namely, the vector inequality,

$$\left(1-\frac{1}{p}\right)d\left\langle y,y\right\rangle + \frac{1}{p}\left\langle C^py,y\right\rangle \leq K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(\left(\frac{M}{m}\right)^p\right)d^{1-\frac{1}{p}}\left\langle Cy,y\right\rangle,$$

for any $y \in H$ and $d \in [m^p, M^p]$.

Now, by employing a similar argument to the one in the proof of Theorem 2 we deduce the desired result (4.3). The details are omitted. □

We have:

Corollary 5. *Let A and B be two positive invertible operators, $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m,\ M>0$ such that $B^q\in\mathcal{B}_1(H)$ and the condition (3.10) holds. Then*

$$[\operatorname{tr}(A^p)]^{1/p}[\operatorname{tr}(B^q)]^{1/q}\le K^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(\left(\frac{M}{m}\right)^p\right)\operatorname{tr}\left(B^q\sharp_{1/p}A^p\right). \tag{4.6}$$

If $B^2\in\mathcal{B}_1(H)$ and the condition (3.13) is valid, then

$$\operatorname{tr}\left(A^2\right)\operatorname{tr}\left(B^2\right)\le K\left(\left(\frac{M}{m}\right)^2\right)\left[\operatorname{tr}\left(B^2\sharp A^2\right)\right]^2. \tag{4.7}$$

5. Some Logarithmic Inequalities

In the recent paper [10] we obtained the following logarithmic reverse of Young's inequality:

$$0\le(1-\nu)a+\nu b-a^{1-\nu}b^\nu\le\nu(1-\nu)(a-b)(\ln a-\ln b) \tag{5.1}$$

where $a,\ b>0,\ \nu\in[0,1]$.

Theorem 4. *Let C be a positive operator and $P\in\mathcal{B}_1(H),\ P>0$. Then for any $\nu\in[0,1]$ we have*

$$\begin{aligned}0&\le\left[\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^\nu-\frac{\operatorname{tr}(PC^\nu)}{\operatorname{tr}(P)}\right]\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1-\nu}\\&\le\nu(1-\nu)\left[\frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)}\right],\end{aligned} \tag{5.2}$$

and, in particular

$$\begin{aligned}0&\le\left[\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1/2}-\frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\right]\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1/2}\\&\le\frac{1}{4}\left[\frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)}\right].\end{aligned} \tag{5.3}$$

Proof. The inequality (5.1) may be written as

$$0\le(1-\nu)a+\nu b-a^{1-\nu}b^\nu\le\nu(1-\nu)(a\ln a+b\ln b-a\ln b-b\ln a) \tag{5.4}$$

for any $a,\ b>0,\ \nu\in[0,1]$.

Fix $b>0,\ \nu\in[0,1]$. By using the functional calculus for the operator C we have

$$\begin{aligned}0&\le(1-\nu)\langle Cx,x\rangle+\nu b\langle x,x\rangle-b^\nu\left\langle C^{1-\nu}x,x\right\rangle\\&\le\nu(1-\nu)(\langle C\ln Cx,x\rangle+b\ln b\langle x,x\rangle-\ln b\langle Cx,x\rangle-b\langle\ln Cx,x\rangle)\end{aligned} \tag{5.5}$$

for any $b>0,\ \nu\in[0,1]$ and $x\in H$.

Now, let $x=P^{1/2}e$ where $e\in H$. Then by (5.5) we get

$$\begin{aligned}0&\le(1-\nu)\left\langle P^{1/2}CP^{1/2}e,e\right\rangle+\nu b\langle Pe,e\rangle-b^\nu\left\langle P^{1/2}C^{1-\nu}P^{1/2}e,e\right\rangle\\&\le\nu(1-\nu)\left[\left\langle P^{1/2}(C\ln C)P^{1/2}e,e\right\rangle+\langle Pe,e\rangle b\ln b\right.\\&\quad\left.-\left\langle P^{1/2}CP^{1/2}e,e\right\rangle\ln b-\left\langle P^{1/2}(\ln C)P^{1/2}e,e\right\rangle b\right]\end{aligned} \tag{5.6}$$

for any $b>0,\ \nu\in[0,1]$ and $e\in H$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. If we take in (5.6) $e=e_i$, $i\in I$ and summing over $i\in I$, then we get

$$\begin{aligned}(5.7)\qquad & 0\le (1-\nu)\sum_{i\in I}\left\langle P^{1/2}CP^{1/2}e_i,e_i\right\rangle+\nu b\sum_{i\in I}\langle Pe_i,e_i\rangle \\ & -b^{\nu}\sum_{i\in I}\left\langle P^{1/2}C^{1-\nu}P^{1/2}e_i,e_i\right\rangle \\ & \le \nu(1-\nu)\left[\sum_{i\in I}\left\langle P^{1/2}(C\ln C)P^{1/2}e_i,e_i\right\rangle+b\ln b\sum_{i\in I}\langle Pe_i,e_i\rangle\right. \\ & \left.-\ln b\sum_{i\in I}\left\langle P^{1/2}CP^{1/2}e_i,e_i\right\rangle-b\sum_{i\in I}\left\langle P^{1/2}(\ln C)P^{1/2}e_i,e_i\right\rangle\right].\end{aligned}$$

Using the properties of the trace, we have from (5.7) that

$$\begin{aligned}& 0\le (1-\nu)\operatorname{tr}(PC)+\nu b\operatorname{tr}(P)-b^{\nu}\operatorname{tr}\left(PC^{1-\nu}\right) \\ & \le \nu(1-\nu)\left[\operatorname{tr}(PC\ln C)+b\ln b\operatorname{tr}(P)-\ln b\operatorname{tr}(PC)-b\operatorname{tr}(P\ln C)\right],\end{aligned}$$

which by division with $\operatorname{tr}(P)>0$ produces

$$\begin{aligned}(5.8)\qquad & 0\le (1-\nu)\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}+\nu b-b^{\nu}\frac{\operatorname{tr}\left(PC^{1-\nu}\right)}{\operatorname{tr}(P)} \\ & \le \nu(1-\nu)\left[\frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)}+b\ln b-\ln b\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}-b\frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)}\right],\end{aligned}$$

for any $b>0$, $\nu\in[0,1]$.

This is an inequality of interest in itself.

Now, if we take in (5.8) $b=\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}$, then we get

$$\begin{aligned}0 & \le \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{\nu}\frac{\operatorname{tr}\left(PC^{1-\nu}\right)}{\operatorname{tr}(P)} \\ & \le \nu(1-\nu)\left[\frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)}+\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\ln\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)\right. \\ & \left.-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\ln\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)}\right] \\ & =\nu(1-\nu)\left[\frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)}\right],\end{aligned}$$

which is equivalent to

$$\begin{aligned}0 & \le \left[\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1-\nu}-\frac{\operatorname{tr}\left(PC^{1-\nu}\right)}{\operatorname{tr}(P)}\right]\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{\nu} \\ & \le \nu(1-\nu)\left[\frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)}\right],\end{aligned}$$

for any $\nu\in[0,1]$.

Now, by replacing ν with $1-\nu$ we get the desired result (5.2). □

We say that the functions $f,\ g:[a,b]\longrightarrow\mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a,b]$ if they satisfy the following condition:

$$(f(t)-f(s))(g(t)-g(s))\ge(\le)\,0 \text{ for each } t,\ s\in[a,b].$$

In recent paper [8] we obtained the following result: Let A be a selfadjoint operators on the Hilbert space H with $\operatorname{Sp}(A) \subseteq J$ and assume that the continuous functions $f, g : J \to \mathbb{R}$ are synchronous on J. If $P,\ Q \in \mathcal{B}_1(H)$ with $P,\ Q > 0$, then

$$\frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)} + \frac{\operatorname{tr}[Qf(A)g(A)]}{\operatorname{tr}(Q)} \geq \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Qg(A)]}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Qf(A)]}{\operatorname{tr}(Q)} \tag{5.9}$$

and, in particular

$$\frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)} \geq \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}. \tag{5.10}$$

Now, if we take in (5.10) $f(t) = t$, $g(t) = \ln t$, $t > 0$ and $A = C > 0$ then we get

$$0 \leq \frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)} \tag{5.11}$$

for $P \in \mathcal{B}_1(H)$, $P > 0$.

Therefore, the inequalities (5.2) and (5.3) provide refinements for (5.11).

In [9] we obtained amongst other the following Grüss type trace inequality

$$\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \leq \frac{1}{4}(M-m)(K-k) \tag{5.12}$$

provided that $k1_H \leq A \leq K1_H$, $m1_H \leq C \leq M1_H$ and $P \in \mathcal{B}_1(H)$, $P > 0$.

Therefore, if we take $A = \ln C$, then we get from (5.12) that

$$\frac{\operatorname{tr}(PC\ln C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(P\ln C)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \leq \frac{1}{4}(M-m)(\ln M - \ln m) \tag{5.13}$$

provided that $0 < m1_H \leq C \leq M1_H$ and $P \in \mathcal{B}_1(H)$, $P > 0$.

Corollary 6. *Let C be an operator such that $m1_H \leq C \leq M1_H$ for some constants $0 < m < M$ and $P \in \mathcal{B}_1(H)$, $P > 0$. Then for any $\nu \in [0,1]$ we have*

$$0 \leq \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PC^\nu)}{\operatorname{tr}(P)}\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1-\nu} \leq \frac{1}{4}\nu(1-\nu)(M-m)(\ln M - \ln m), \tag{5.14}$$

and, in particular

$$0 \leq \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)}\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1/2} \leq \frac{1}{16}(M-m)(\ln M - \ln m). \tag{5.15}$$

Remark 2. *The inequality (5.14) is equivalent to*

$$0 \leq \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^\nu - \frac{\operatorname{tr}(PC^\nu)}{\operatorname{tr}(P)} \leq \frac{1}{4}\nu(1-\nu)(M-m)(\ln M - \ln m)\left(\frac{\operatorname{tr}(P)}{\operatorname{tr}(PC)}\right)^{1-\nu}. \tag{5.16}$$

Since $\frac{\operatorname{tr}(P)}{\operatorname{tr}(PC)} \leq \frac{1}{m}$, then we get from (5.16) that

$$0 \leq \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^\nu - \frac{\operatorname{tr}(PC^\nu)}{\operatorname{tr}(P)} \leq \frac{1}{4}\nu(1-\nu)\frac{(M-m)(\ln M - \ln m)}{m^{1-\nu}}, \tag{5.17}$$

and in particular

$$0 \le \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{1/2} - \frac{\operatorname{tr}\left(PC^{1/2}\right)}{\operatorname{tr}(P)} \le \frac{1}{16}\frac{(M-m)(\ln M - \ln m)}{m^{1/2}}, \tag{5.18}$$

provided that $m1_H \le C \le M1_H$ *for some constants* $0<m<M$ *and* $P \in \mathcal{B}_1(H)$, $P>0$.

Corollary 7. *Let* A *and* B *be two positive invertible operators,* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $m,\ M>0$ *such that* $B^q \in \mathcal{B}_1(H)$ *and (3.10) is valid. Then*

$$\begin{aligned} 0 &\le [\operatorname{tr}(A^p)]^{1/p}[\operatorname{tr}(B^q)]^{1/q} - \operatorname{tr}\left(B^q \sharp_{1/p} A^p\right) \\ &\le \frac{(M^p - m^p)(\ln M - \ln m)}{4qm^{p/q}}\operatorname{tr}(B^q). \end{aligned} \tag{5.19}$$

In particular, we have

$$\begin{aligned} 0 &\le \left[\operatorname{tr}\left(A^2\right)\right]^{1/2}\left[\operatorname{tr}\left(B^2\right)\right]^{1/2} - \operatorname{tr}\left(B^2 \sharp A^2\right) \\ &\le \frac{\left(M^2-m^2\right)(\ln M - \ln m)}{8m}\operatorname{tr}\left(B^2\right) \end{aligned} \tag{5.20}$$

provided the condition (3.13) is valid.

Proof. From the inequality (5.16) we have

$$\begin{aligned} 0 &\le \left(\frac{\operatorname{tr}\left(P^{1/2}CP^{1/2}\right)}{\operatorname{tr}(P)}\right)^{\nu} - \frac{\operatorname{tr}\left(P^{1/2}C^{\nu}P^{1/2}\right)}{\operatorname{tr}(P)} \\ &\le \frac{1}{4}\nu(1-\nu)(M-m)(\ln M - \ln m)\left(\frac{\operatorname{tr}(P)}{\operatorname{tr}\left(P^{1/2}CP^{1/2}\right)}\right)^{1-\nu}, \end{aligned} \tag{5.21}$$

provided that $m1_H \le C \le M1_H$ for some constants $0<m<M$ and $P \in \mathcal{B}_1(H)$, $P>0$.

Now, from (3.10) by multiplying both sides with $B^{-\frac{q}{2}}$ we have $m^p I \le B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}} \le M^p I$.

By writing the inequality (5.21) for $C = B^{-\frac{q}{2}}A^pB^{-\frac{q}{2}}$, $P = B^q$ the bounds m^p, M^p and $\nu = \frac{1}{p}$, then we get

$$\begin{aligned} 0 &\le \left(\frac{\operatorname{tr}(A^p)}{\operatorname{tr}(B^q)}\right)^{\frac{1}{p}} - \frac{\operatorname{tr}\left(B^q\sharp_{1/p}A^p\right)}{\operatorname{tr}(B^q)} \\ &\le \frac{1}{4pq}(M^p-m^p)(\ln M^p - \ln m^p)\left(\frac{\operatorname{tr}(B^q)}{\operatorname{tr}(A^p)}\right)^{\frac{1}{q}}. \end{aligned}$$

By multiplying this with $\operatorname{tr}(B^q)>0$ we get

$$\begin{aligned} 0 &\le [\operatorname{tr}(A^p)]^{\frac{1}{p}}\operatorname{tr}(B^q)^{\frac{1}{q}} - \operatorname{tr}\left(B^q\sharp_{1/p}A^p\right) \\ &\le \frac{1}{4q}(M^p-m^p)(\ln M - \ln m)\left(\frac{\operatorname{tr}(B^q)}{\operatorname{tr}(A^p)}\right)^{\frac{1}{q}}\operatorname{tr}(B^q). \end{aligned} \tag{5.22}$$

Since $m^p\operatorname{tr}(B^q) \le \operatorname{tr}(A^p) \le M^p\operatorname{tr}(B^q)$, then $\frac{\operatorname{tr}(B^q)}{\operatorname{tr}(A^p)} \le \frac{1}{m^p}$ and by (5.22) we get the desired result (5.19). □

6. Some Exponential Inequalities

In paper [10] we also obtained the following multiplicative reverse of Young's inequality

$$1 \leq \frac{(1-\nu)\,a+\nu b}{a^{1-\nu}b^{\nu}} \leq \exp\left[4\nu\,(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{6.1}$$

where $a,\, b>0,\ \nu\in[0,1]$.

By using a similar argument to the one in the proof of Theorem 2 we can prove the following result as well:

Theorem 5. *Let C be an operator with the property (3.3) for some constants m, M with $M>m>0$ and $P\in\mathcal{B}_1(H)$, $P\geq 0$ with $\operatorname{tr}(P)>0$. Then for any $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ we have*

$$\left(\frac{\operatorname{tr}(PC^p)}{\operatorname{tr}(P)}\right)^{1/p} \leq \exp\left[\frac{4}{pq}\left(K\left(\left(\frac{M}{m}\right)^p\right)-1\right)\right]\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}. \tag{6.2}$$

In particular, we have

$$\operatorname{tr}\left(PC^2\right)\operatorname{tr}(P) \leq \exp\left[2\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right][\operatorname{tr}(PC)]^2. \tag{6.3}$$

We also have:

Corollary 8. *Let A and B be two positive invertible operators, $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m,\, M>0$ such that $B^q\in\mathcal{B}_1(H)$ and $m^pB^q\leq A^p\leq M^pB^q$. Then*

$$[\operatorname{tr}(A^p)]^{1/p}\,[\operatorname{tr}(B^q)]^{1/q} \leq \exp\left[\frac{4}{pq}\left(K\left(\left(\frac{M}{m}\right)^p\right)-1\right)\right]\operatorname{tr}\left(B^q\sharp_{1/p}A^p\right). \tag{6.4}$$

In particular, if $m^2B^2\leq A^2\leq M^2B^2$, then

$$\operatorname{tr}\left(A^2\right)\operatorname{tr}\left(B^2\right) \leq \exp\left[2\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right]\left[\operatorname{tr}\left(B^2\sharp A^2\right)\right]^2. \tag{6.5}$$

In [11] we obtained the following inequalities that improve the corresponding results of Furuichi and Minculete from [15]:

$$\begin{aligned}\exp\left[\frac{1}{2}\nu\,(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] &\leq \frac{(1-\nu)\,a+\nu b}{a^{1-\nu}b^{\nu}} \\ &\leq \exp\left[\frac{1}{2}\nu\,(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right]\end{aligned} \tag{6.6}$$

for any $a,\, b>0$ and $\nu\in[0,1]$.

Theorem 6. *Let C be an operator with the property (3.3) for some constants m, M with $M>m>0$ and $P\in\mathcal{B}_1(H)$, $P>0$. Then for any $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ we have*

$$\left(\frac{\operatorname{tr}(PC^p)}{\operatorname{tr}(P)}\right)^{1/p} \leq \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p-1\right)^2\right]\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \tag{6.7}$$

and, in particular,

$$\operatorname{tr}\left(PC^2\right)\operatorname{tr}(P) \le \exp\left[\frac{1}{pq}\left(\left(\frac{M}{m}\right)^p - 1\right)^2\right]\left[\operatorname{tr}(PC)\right]^2. \tag{6.8}$$

Proof. If a, $b \in [m, M] \subset (0, \infty)$ and since

$$0 < \frac{\max\{a,b\}}{\min\{a,b\}} - 1 \le \frac{M}{m} - 1,$$

hence

$$\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2 \le \left(\frac{M}{m} - 1\right)^2.$$

Therefore, by (6.6) we get

$$(1-\nu)\,a + \nu b \le a^{1-\nu}b^{\nu}\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M}{m} - 1\right)^2\right], \tag{6.9}$$

for any a, $b \in [m, M]$ and $\nu \in (0,1)$.

Now, if C is an operator with $mI \le C \le MI$ then for $p > 1$ we have $m^pI \le C^p \le M^pI$. Using the functional calculus we get from (6.9) for $\nu = \frac{1}{p}$ that

$$\left(1 - \frac{1}{p}\right)d + \frac{1}{p}C^p \le \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p - 1\right)^2\right]d^{1-\frac{1}{p}}C,$$

namely, the vector inequality,

$$\left(1 - \frac{1}{p}\right)d\langle y, y\rangle + \frac{1}{p}\langle C^p y, y\rangle \le \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p - 1\right)^2\right]d^{1-\frac{1}{p}}\langle Cy, y\rangle,$$

for any $y \in H$ and $d \in [m^p, M^p]$.

This is an inequality of interest in itself.

Now, by employing a similar argument to the one in the proof of Theorem 2 we deduce the desired result (6.7). The details are omitted. □

Finally, we have:

Corollary 9. *Let A and B be two positive invertible operators, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and m, $M > 0$ such that $B^q \in \mathcal{B}_1(H)$ and $m^pB^q \le A^p \le M^pB^q$. Then*

$$\left[\operatorname{tr}(A^p)\right]^{1/p}\left[\operatorname{tr}(B^q)\right]^{1/q} \le \exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^p - 1\right)^2\right]\operatorname{tr}\left(B^q\sharp_{1/p}A^p\right). \tag{6.10}$$

In particular, if $m^2B^2 \le A^2 \le M^2B^2$, then

$$\operatorname{tr}\left(A^2\right)\operatorname{tr}\left(B^2\right) \le \exp\left[\frac{1}{4}\left(\left(\frac{M}{m}\right)^2 - 1\right)^2\right]\left[\operatorname{tr}\left(B^2\sharp A^2\right)\right]^2. \tag{6.11}$$

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