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TRACE INEQUALITIES FOR POSITIVE OPERATORS VIA RECENT REFINEMENTS AND REVERSES OF YOUNG'S INEQUALITY

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Abstract. In this paper we obtain some trace inequalities for positive operators via recent refinements and reverses of Young's inequality due to Kittaneh-Manasrah, Liao-Wu-Zhao, Zuo-Shi-Fujii, Tominaga and Furuichi.

1. Introduction

If $\{e_i\}_{i\in I}$ is an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is *trace class* provided

$$\|A\|_1 := \sum_{i\in I}\langle |A| e_i, e_i\rangle < \infty. \tag{1.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following properties are also well known:

(i) We have

$$\|A\|_1 = \|A^*\|_1 \tag{1.2}$$

for any $A\in\mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ is an *operator ideal* in $\mathcal{B}(H)$, i.e.

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H)\subseteq\mathcal{B}_1(H);$$

(iii) $(\mathcal{B}_1(H), \|\cdot\|_1)$ is a *Banach space.*

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I}\langle Ae_i, e_i\rangle, \tag{1.3}$$

where $\{e_i\}_{i\in I}$ is an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.3) converges absolutely and it is independent from the choice of basis.

The following results collects some properties of the trace:

(i) If $A\in\mathcal{B}_1(H)$ then $A^*\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.4}$$

(ii) If $A\in\mathcal{B}_1(H)$ and $T\in\mathcal{B}(H)$, then $AT, TA\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1\|T\|; \tag{1.5}$$

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(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\| = 1$;

(iv) $\mathcal{B}_{fin}(H)$, the space of operators of *finite rank*, is a dense subspace of $\mathcal{B}_1(H)$.

Now, for the finite dimensional case, it is well known that the trace functional is *submultiplicative*, that is, for *positive semidefinite matrices* A and B in $M_n(\mathbb{C})$,

$$0 \leq \operatorname{tr}(AB) \leq \operatorname{tr}(A) \operatorname{tr}(B).$$

Therefore

$$0 \leq \operatorname{tr}(A^k) \leq [\operatorname{tr}(A)]^k,$$

where k is any positive integer.

In 2000, Yang [30] proved a matrix trace inequality

$$\operatorname{tr}\left[(AB)^k\right] \leq (\operatorname{tr} A)^k (\operatorname{tr} B)^k, \tag{1.6}$$

where A and B are positive semidefinite matrices over $\mathbb{C}$ of the same order n and k is any positive integer.

If $(H, \langle \cdot, \cdot \rangle)$ is a separable infinite-dimensional Hilbert space then the inequality (1.6) is also valid for any positive operators $A, B \in \mathcal{B}_1(H)$. This result was obtained by L. Liu in 2007, see [19].

In 2001, Yang et al. [31] improved (1.6) as follows:

$$\operatorname{tr}[(AB)^m] \leq \left[\operatorname{tr}\left(A^{2m}\right) \operatorname{tr}\left(B^{2m}\right)\right]^{1/2}, \tag{1.7}$$

where A and B are positive semidefinite matrices over $\mathbb{C}$ of the same order and m is any positive integer.

In [24] the authors have proved many trace inequalities for sums and products of matrices. For instance, if A and B are positive semidefinite matrices in $M_n(\mathbb{C})$, then

$$\operatorname{tr}\left[(AB)^k\right] \leq \min\left\{\|A\|^k \operatorname{tr}\left(B^k\right), \|B\|^k \operatorname{tr}\left(A^k\right)\right\} \tag{1.8}$$

for any positive integer k. Also, if $A, B \in M_n(\mathbb{C})$ then for $r \geq 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have the following *Young type inequality*

$$\operatorname{tr}(|AB^*|^r) \leq \operatorname{tr}\left[\left(\frac{|A|^p}{p} + \frac{|B|^q}{q}\right)^r\right]. \tag{1.9}$$

Ando [1] proved a strong form of Young's inequality - it was shown that if A and B are in $M_n(\mathbb{C})$, then there is a *unitary matrix* U such that

$$|AB^*| \leq U\left(\frac{1}{p}|A|^p + \frac{1}{q}|B|^q\right)U^*,$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, which immediately gives the trace inequality

$$\operatorname{tr}(|AB^*|) \leq \frac{1}{p}\operatorname{tr}(|A|^p) + \frac{1}{q}\operatorname{tr}(|B|^q). \tag{1.10}$$

This inequality can also be obtained from (1.9) by taking $r = 1$.

The following Hölder's type inequality has been obtained by Ruskai in [22]

$$|\operatorname{tr}(AB)| \leq \operatorname{tr}(|AB|) \leq [\operatorname{tr}(|A|^p)]^{1/p} [\operatorname{tr}(|B|^q)]^{1/q} \tag{1.11}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $A, B \in \mathcal{B}(H)$ with $|A|^p, |B|^q \in \mathcal{B}_1(H)$.

In particular, for $p = 2$ we get the Schwarz inequality

$$|\operatorname{tr}(AB)| \leq \operatorname{tr}(|AB|) \leq \left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2} \left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} \tag{1.12}$$

with $|A|^2, |B|^2 \in \mathcal{B}_1(H)$.

Assume that A, B are positive invertible operators on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. We use the following notation

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2}, \tag{1.13}$$

for the *weighted geometric mean.* When $\nu = \frac{1}{2}$, we write $A\sharp B$ for brevity.

We have the following Hölder type trace inequality for the weighted geometric mean [9]: If A, B are positive invertible operators, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and A^p, $B^q \in \mathcal{B}_1(H)$, then $B^q\sharp_{1/p}A^p \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}\left(B^q\sharp_{1/p}A^p\right) \le \left[\operatorname{tr}(A^p)\right]^{1/p}\left[\operatorname{tr}(B^q)\right]^{1/q}. \tag{1.14}$$

In particular, if A^2, $B^2 \in \mathcal{B}_1(H)$, then $B^2\sharp A^2 \in \mathcal{B}_1(H)$ and

$$\left[\operatorname{tr}\left(B^2\sharp A^2\right)\right]^2 \le \operatorname{tr}\left(A^2\right)\operatorname{tr}\left(B^2\right). \tag{1.15}$$

Also, if A, B are positive invertible operators, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $C \in \mathcal{B}_1(H)$, $C \ge 0$ then CA^p, CB^q, $C\left(B^q\sharp_{1/p}A^p\right) \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}\left(C\left(B^q\sharp_{1/p}A^p\right)\right) \le \left[\operatorname{tr}(CA^p)\right]^{1/p}\left[\operatorname{tr}(CB^q)\right]^{1/q}. \tag{1.16}$$

In particular, if $C \in \mathcal{B}_1(H)$, then CA^2, CB^2, $C\left(B^2\sharp A^2\right) \in \mathcal{B}_1(H)$ and

$$\left[\operatorname{tr}\left(C\left(B^2\sharp A^2\right)\right)\right]^2 \le \operatorname{tr}\left(CA^2\right)\operatorname{tr}\left(CB^2\right). \tag{1.17}$$

Related inequalities may be found in [9] as well.

For the theory of trace functionals and their applications the reader is referred to [26].

For some classical trace inequalities see [4], [6], [21] and [32], which are continuations of the work of Bellman [2]. For related works the reader can refer to [1], [3], [4], [12], [17], [19], [20], [23] and [29].

Motivated by the above results, we establish in this paper some new trace inequalities via recent scalar Young type inequalities.

2. Trace Inequalities Via Kittaneh-Manasrah Results

Kittaneh and Manasrah [15], [16] provided a refinement and a reverse for *Young's inequality* as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \le (1-\nu)a + \nu b - a^{1-\nu}b^\nu \le R\left(\sqrt{a}-\sqrt{b}\right)^2, \tag{2.1}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$. The case $\nu = \frac{1}{2}$ reduces (2.1) to an identity.

We can give a simple direct proof for (2.1) as follows. Recall the following result obtained by the author in 2006 [7] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$
\begin{aligned}
(2.2)\qquad & n \min_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right]\\
&\le \frac{1}{P_n}\sum_{j=1}^{n}p_j\Phi(x_j)-\Phi\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\right)\\
&\le n \max_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right],
\end{aligned}
$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$. For $n = 2$, we deduce from (2.2) that

$$
\begin{aligned}
(2.3)\qquad & 2\min\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]\\
&\le \nu\Phi(x)+(1-\nu)\Phi(y)-\Phi[\nu x+(1-\nu)y]\\
&\le 2\max\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]
\end{aligned}
$$

for any x, $y \in \mathbb{R}$ and $\nu \in [0,1]$. If we take $\Phi(x) = \exp(x)$, then we get from (2.3)

$$
\begin{aligned}
(2.4)\qquad & 2\min\{\nu,1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2}-\exp\left(\frac{x+y}{2}\right)\right]\\
&\le \nu\exp(x)+(1-\nu)\exp(y)-\exp[\nu x+(1-\nu)y]\\
&\le 2\max\{\nu,1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2}-\exp\left(\frac{x+y}{2}\right)\right]
\end{aligned}
$$

for any x, $y \in \mathbb{R}$ and $\nu \in [0,1]$. Further, denote $\exp(x) = a$, $\exp(y) = b$ with a, $b > 0$, then from (2.4) we obtain the inequality (2.1).

We have:

Theorem 1. *Let A, B be two positive operators and P, $Q \in \mathcal{B}_1(H)$ with P, $Q > 0$. Then for any $\nu \in [0,1]$ we have*

$$
\begin{aligned}
(2.5)\qquad & r\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}\right)\\
&\le (1-\nu)\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}+\nu\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}-\frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)}\\
&\le R\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}\right),
\end{aligned}
$$

where $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$.

Proof. Fix $b>0$, and by using the functional calculus for the operator A, we have from (2.1) that

$$\begin{aligned} &r\left(\langle Ax,x\rangle-2\sqrt{b}\left\langle A^{1/2}x,x\right\rangle+b\langle x,x\rangle\right)\\ &\leq(1-\nu)\langle Ax,x\rangle+\nu b\langle x,x\rangle-b^{\nu}\left\langle A^{1-\nu}x,x\right\rangle\\ &\leq R\left(\langle Ax,x\rangle-2\sqrt{b}\left\langle A^{1/2}x,x\right\rangle+b\langle x,x\rangle\right) \end{aligned} \tag{2.6}$$

for any $x\in H$.

Now, fix $x\in H\setminus\{0\}$. Then by using the functional calculus for the operator B, we have by (2.6) that

$$\begin{aligned} &r\left(\langle Ax,x\rangle\|y\|^2-2\left\langle A^{1/2}x,x\right\rangle\left\langle B^{1/2}y,y\right\rangle+\|x\|^2\langle By,y\rangle\right)\\ &\leq(1-\nu)\langle Ax,x\rangle\|y\|^2+\nu\|x\|^2\langle By,y\rangle-\langle B^{\nu}y,y\rangle\left\langle A^{1-\nu}x,x\right\rangle\\ &\leq R\left(\langle Ax,x\rangle\|y\|^2-2\left\langle A^{1/2}x,x\right\rangle\left\langle B^{1/2}y,y\right\rangle+\|x\|^2\langle By,y\rangle\right) \end{aligned} \tag{2.7}$$

for any $x,\ y\in H$ and $\nu\in[0,1]$.

This inequality is of interest in itself as well.

Now, let $x=P^{1/2}e$, $y=Q^{1/2}f$ where $e,\ f\in H$. Then by (2.7) we get

$$\begin{aligned} &r\Big(\left\langle P^{1/2}AP^{1/2}e,e\right\rangle\langle Qf,f\rangle\\ &-2\left\langle P^{1/2}A^{1/2}P^{1/2}e,e\right\rangle\left\langle Q^{1/2}B^{1/2}Q^{1/2}f,f\right\rangle\\ &+\langle Pe,e\rangle\left\langle Q^{1/2}BQ^{1/2}f,f\right\rangle\Big)\\ &\leq(1-\nu)\left\langle P^{1/2}AP^{1/2}e,e\right\rangle\langle Qf,f\rangle+\nu\langle Pe,e\rangle\left\langle Q^{1/2}BQ^{1/2}f,f\right\rangle\\ &-\left\langle P^{1/2}A^{1-\nu}P^{1/2}e,e\right\rangle\left\langle Q^{1/2}B^{\nu}Q^{1/2}f,f\right\rangle\\ &\leq R\Big(\left\langle P^{1/2}AP^{1/2}e,e\right\rangle\langle Qf,f\rangle\\ &-2\left\langle P^{1/2}A^{1/2}P^{1/2}e,e\right\rangle\left\langle Q^{1/2}B^{1/2}Q^{1/2}f,f\right\rangle\\ &+\langle Pe,e\rangle\left\langle Q^{1/2}BQ^{1/2}f,f\right\rangle\Big) \end{aligned} \tag{2.8}$$

for any $e,\ f\in H$.

Let $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ be two orthonormal bases of H. If we take in (2.8) $e=e_i$, $i\in I$ and $f=f_j$, $j\in J$ and summing over $i\in I$ and $j\in J$, then we get

$$\begin{aligned}
(2.9)\qquad & r\left(\sum_{i\in I}\left\langle P^{1/2}AP^{1/2}e_i,e_i\right\rangle\sum_{j\in J}\langle Qf_j,f_j\rangle\right.\\
&-2\sum_{i\in I}\left\langle P^{1/2}A^{1/2}P^{1/2}e_i,e_i\right\rangle\sum_{j\in J}\left\langle Q^{1/2}B^{1/2}Q^{1/2}f_j,f_j\right\rangle\\
&\left.+\sum_{i\in I}\langle Pe_i,e_i\rangle\sum_{j\in J}\left\langle Q^{1/2}BQ^{1/2}f_j,f_j\right\rangle\right)\\
&\le(1-\nu)\sum_{i\in I}\left\langle P^{1/2}AP^{1/2}e_i,e_i\right\rangle\sum_{j\in J}\langle Qf_j,f_j\rangle\\
&+\nu\sum_{i\in I}\langle Pe_i,e_i\rangle\sum_{j\in J}\left\langle Q^{1/2}BQ^{1/2}f_j,f_j\right\rangle\\
&-\sum_{i\in I}\left\langle P^{1/2}A^{1-\nu}P^{1/2}e_i,e_i\right\rangle\sum_{j\in J}\left\langle Q^{1/2}B^{\nu}Q^{1/2}f_j,f_j\right\rangle\\
&\le R\left(\sum_{i\in I}\left\langle P^{1/2}AP^{1/2}e_i,e_i\right\rangle\sum_{j\in J}\langle Qf_j,f_j\rangle\right.\\
&-2\sum_{i\in I}\left\langle P^{1/2}A^{1/2}P^{1/2}e_i,e_i\right\rangle\sum_{j\in J}\left\langle Q^{1/2}B^{1/2}Q^{1/2}f_j,f_j\right\rangle\\
&\left.+\sum_{i\in I}\langle Pe_i,e_i\rangle\sum_{j\in J}\left\langle Q^{1/2}BQ^{1/2}f_j,f_j\right\rangle\right).
\end{aligned}$$

Using the properties of the trace we get

$$\begin{aligned}
& r\left(\operatorname{tr}(PA)\operatorname{tr}(Q)-2\operatorname{tr}\left(PA^{1/2}\right)\operatorname{tr}\left(QB^{1/2}\right)+\operatorname{tr}(P)\operatorname{tr}(QB)\right)\\
&\le(1-\nu)\operatorname{tr}(PA)\operatorname{tr}(Q)+\nu\operatorname{tr}(P)\operatorname{tr}(QB)-\operatorname{tr}\left(PA^{1-\nu}\right)\operatorname{tr}(QB^{\nu})\\
&\le R\left(\operatorname{tr}(PA)\operatorname{tr}(Q)-2\operatorname{tr}\left(PA^{1/2}\right)\operatorname{tr}\left(QB^{1/2}\right)+\operatorname{tr}(P)\operatorname{tr}(QB)\right)
\end{aligned}$$

and the inequality (2.5) is proved. □

Corollary 1. *Let A be a positive operator and $P\in\mathcal{B}_1(H)$ with $P>0$. Then for any $\nu\in[0,1]$ we have*

$$\begin{aligned}
(2.10)\qquad 2r\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\right)^2\right)&\le\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PA^{\nu})}{\operatorname{tr}(P)}\\
&\le 2R\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\right)^2\right),
\end{aligned}$$

where $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

Remark 1. *If P, Q are positive invertible operators with $P,\, Q\in\mathcal{B}_1(H)$, then by (2.10) for $A=P^{-1/2}QP^{-1/2}$ we get*

$$2r\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)^2\right)\le\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(P\sharp_{1-\nu}Q)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\sharp_{\nu}Q)}{\operatorname{tr}(P)}\le 2R\left(\frac{\operatorname{tr}(Q)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(P\sharp Q)}{\operatorname{tr}(P)}\right)^2\right),\tag{2.11}$$

where the operator weighted geometric mean is defined in (1.13).

Corollary 2. *Let A, B two positive operators and $P,\, Q\in\mathcal{B}_1(H)$ with $P,\, Q>0$. If $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\begin{aligned}&t\left(\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}\left(PA^{p/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{q/2}\right)}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}(QB^q)}{\operatorname{tr}(Q)}\right)\\&\le\frac{1}{p}\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}+\frac{1}{q}\frac{\operatorname{tr}(QB^q)}{\operatorname{tr}(Q)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}\\&\le T\left(\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}\left(PA^{p/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{q/2}\right)}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}(QB^q)}{\operatorname{tr}(Q)}\right),\end{aligned}\tag{2.12}$$

where $t=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ and $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

The proof follows by (2.5) on replacing A with A^p, B with B^q and $\nu=\frac{1}{q}$.

Remark 2. *If P, Q, S, V are positive invertible operators with $P,\, Q,\, S,\, V\in\mathcal{B}_1(H)$, then by (2.12) we get for $A=P^{-1/2}SP^{-1/2}$ and $B=Q^{-1/2}VQ^{-1/2}$ that*

$$\begin{aligned}&t\left(\frac{\operatorname{tr}(P\sharp_p S)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}\left(P\sharp_{p/2}S\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\sharp_{q/2}V\right)}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}(Q\sharp_q V)}{\operatorname{tr}(Q)}\right)\\&\le\frac{1}{p}\frac{\operatorname{tr}(P\sharp_p S)}{\operatorname{tr}(P)}+\frac{1}{q}\frac{\operatorname{tr}(Q\sharp_q V)}{\operatorname{tr}(Q)}-\frac{\operatorname{tr}(S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(V)}{\operatorname{tr}(Q)}\\&\le T\left(\frac{\operatorname{tr}(P\sharp_p S)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}\left(P\sharp_{p/2}S\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\sharp_{q/2}V\right)}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}(Q\sharp_q V)}{\operatorname{tr}(Q)}\right),\end{aligned}\tag{2.13}$$

where $t=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ and $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

In particular, if we take in (2.13) $S=Q$ and $V=P$, then we get

$$\begin{aligned}&t\left(\frac{\operatorname{tr}(P\sharp_p Q)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}\left(P\sharp_{p/2}Q\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\sharp_{q/2}P\right)}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}(Q\sharp_q P)}{\operatorname{tr}(Q)}\right)\\&\le\frac{1}{p}\frac{\operatorname{tr}(P\sharp_p Q)}{\operatorname{tr}(P)}+\frac{1}{q}\frac{\operatorname{tr}(Q\sharp_q P)}{\operatorname{tr}(Q)}-1\\&\le T\left(\frac{\operatorname{tr}(P\sharp_p Q)}{\operatorname{tr}(P)}-2\frac{\operatorname{tr}\left(P\sharp_{p/2}Q\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\sharp_{q/2}P\right)}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}(Q\sharp_q P)}{\operatorname{tr}(Q)}\right),\end{aligned}\tag{2.14}$$

where $t=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ and $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

3. Trace Inequalities Via Liao-Wu-Zhao and Zuo-Shi-Fujii Results

We consider the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h>0. \tag{3.1}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h>0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^r\left(\frac{a}{b}\right)a^{1-\nu}b^\nu \leq (1-\nu)a+\nu b \leq K^R\left(\frac{a}{b}\right)a^{1-\nu}b^\nu, \tag{3.2}$$

where $a,\ b>0$, $\nu \in [0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (3.2) was obtained by Zou et al. in [33] while the second by Liao et al. [18].

We can give a simple direct proof for (3.2) as follows.

Indeed, if we write the inequality (2.3) for the convex function $\Phi(x) = -\ln x$, and for the positive numbers a and b we get

$$\begin{aligned}
&2\min\{\nu,1-\nu\}\left[\ln\left(\frac{a+b}{2}\right)-\frac{\ln a+\ln b}{2}\right]\\
&\leq \ln[\nu b+(1-\nu)a]-(1-\nu)\ln a-\nu\ln b\\
&\leq 2\max\{\nu,1-\nu\}\left[\ln\left(\frac{a+b}{2}\right)-\frac{\ln a+\ln b}{2}\right]
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
\min\{\nu,1-\nu\}\ln\left(\frac{a+b}{2\sqrt{ab}}\right)^2 &\leq \ln\left[\frac{\nu b+(1-\nu)a}{a^{1-\nu}b^\nu}\right]\\
&\leq \max\{\nu,1-\nu\}\ln\left(\frac{a+b}{2\sqrt{ab}}\right)^2
\end{aligned}$$

and to (3.2), as stated.

If $a\in[m_1,M_1]$ and $b\in[m_2,M_2]$ with $0<m_1<M_1$, $0<m_2<M_2$ then

$$\frac{m_1}{M_2} \leq \frac{a}{b} \leq \frac{M_1}{m_2}.$$

Denote

$$m =: \min_{(a,b)\in[m_1,M_1]\times[m_2,M_2]} K\left(\frac{a}{b}\right) \text{ and } M =: \max_{(a,b)\in[m_1,M_1]\times[m_2,M_2]} K\left(\frac{a}{b}\right).$$

Taking into account the properties of Kantorovich's ratio we have

$$m := \begin{cases} K\left(\frac{M_1}{m_2}\right) > 1 \text{ if } \frac{M_1}{m_2} < 1,\\[2mm] 1 \text{ if } \frac{m_1}{M_2} \leq 1 \leq \frac{M_1}{m_2},\\[2mm] K\left(\frac{m_1}{M_2}\right) > 1 \text{ if } 1 < \frac{m_1}{M_2},\end{cases} = \begin{cases} K\left(\frac{M_1}{m_2}\right) > 1 \text{ if } \frac{M_1}{m_2} < 1,\\[2mm] 1 \text{ if } \frac{m_1}{M_2} \leq 1 \leq \frac{M_1}{m_2},\\[2mm] K\left(\frac{M_2}{m_1}\right) > 1 \text{ if } 1 < \frac{m_1}{M_2}\end{cases} \tag{3.3}$$

and

$$M:=\begin{cases} K\left(\frac{m_1}{M_2}\right)>1 \text{ if } \frac{M_1}{m_2}<1, \\ \\ \max\left\{K\left(\frac{m_1}{M_2}\right),K\left(\frac{M_1}{m_2}\right)\right\}>1 \text{ if } \frac{m_1}{M_2}\le 1\le \frac{M_1}{m_2}, \\ \\ K\left(\frac{M_1}{m_2}\right)>1 \text{ if } 1<\frac{m_1}{M_2}, \end{cases} \tag{3.4}$$

$$=\begin{cases} K\left(\frac{M_2}{m_1}\right)>1 \text{ if } \frac{M_1}{m_2}<1, \\ \\ \max\left\{K\left(\frac{M_2}{m_1}\right),K\left(\frac{M_1}{m_2}\right)\right\}>1 \text{ if } \frac{m_1}{M_2}\le 1\le \frac{M_1}{m_2}, \\ \\ K\left(\frac{M_1}{m_2}\right)>1 \text{ if } 1<\frac{m_1}{M_2}. \end{cases}$$

We have the following result:

Theorem 2. *Let A, B be two operators such that*

$$0<m_1I\le A<M_1I,\ \ 0<m_2I\le B\le M_2I \tag{3.5}$$

and P, $Q\in\mathcal{B}_1(H)$ with P, $Q>0$. Then for any $\nu\in[0,1]$, we have for m, M as defined by (3.3) and (3.4) that

$$m^r\frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)}\le(1-\nu)\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}+\nu\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} \le M^R\frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)}, \tag{3.6}$$

where $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

In particular, we have

$$m^{1/2}\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)}\le\frac{1}{2}\left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}+\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}\right] \le M^{1/2}\frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)}. \tag{3.7}$$

Proof. From (3.2) we have

$$m^ra^{1-\nu}b^{\nu}\le(1-\nu)a+\nu b\le M^Ra^{1-\nu}b^{\nu}, \tag{3.8}$$

where $a\in[m_1,M_1]$, $b\in[m_2,M_2]$ and $\nu\in[0,1]$.

Using the functional calculus for the operator A, we have

$$m^rb^{\nu}\left\langle A^{1-\nu}x,x\right\rangle\le(1-\nu)\langle Ax,x\rangle+\nu b\|x\|^2\le M^Rb^{\nu}\left\langle A^{1-\nu}x,x\right\rangle, \tag{3.9}$$

for any $x\in H$, $b\in[m_2,M_2]$ and $\nu\in[0,1]$.

Using the functional calculus for B we get from (3.9) that

$$m^r\left\langle A^{1-\nu}x,x\right\rangle\langle B^{\nu}y,y\rangle\le(1-\nu)\langle Ax,x\rangle\|y\|^2+\nu\|x\|^2\langle By,y\rangle \le M^R\left\langle A^{1-\nu}x,x\right\rangle\langle B^{\nu}y,y\rangle, \tag{3.10}$$

for any $x,y\in H$ and $\nu\in[0,1]$.

This is an inequality of interest in itself as well.

Further, let $x = P^{1/2}e$, $y = Q^{1/2}f$ where e, $f \in H$. Then by (3.10) we have

$$m^r \left\langle P^{1/2}A^{1-\nu}P^{1/2}e, e\right\rangle \left\langle Q^{1/2}B^{\nu}Q^{1/2}f, f\right\rangle \leq (1-\nu)\left\langle P^{1/2}AP^{1/2}e, e\right\rangle \langle Qf, f\rangle + \nu \langle Pe, e\rangle \left\langle Q^{1/2}BQ^{1/2}f, f\right\rangle \leq M^R \left\langle P^{1/2}A^{1-\nu}P^{1/2}e, e\right\rangle \left\langle Q^{1/2}B^{\nu}Q^{1/2}f, f\right\rangle, \tag{3.11}$$

for any e, $f \in H$ and $\nu \in [0, 1]$.

Now, on making use of a similar argument as in the proof of Theorem 1, we get the desired result (3.6). □

Remark 3. *Let A, B be two operators such that the condition (3.5) is valid and $P \in \mathcal{B}_1(H)$ with $P > 0$. Then for any $\nu \in [0, 1]$, we have for m, M as defined by (3.3) and (3.4) that*

$$m^r \frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PB^{\nu})}{\operatorname{tr}(P)} \leq \frac{\operatorname{tr}(P[(1-\nu)A + \nu B])}{\operatorname{tr}(P)} \leq M^R \frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PB^{\nu})}{\operatorname{tr}(P)}, \tag{3.12}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

In particular, we have

$$m^{1/2} \frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}\left(PB^{1/2}\right)}{\operatorname{tr}(P)} \leq \frac{\operatorname{tr}\left(P\left(\frac{A+B}{2}\right)\right)}{\operatorname{tr}(P)} \leq M^{1/2} \frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}\left(PB^{1/2}\right)}{\operatorname{tr}(P)}. \tag{3.13}$$

For $0 < m_1 < M_1$, $0 < m_2 < M_2$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we define

$$m_{p,q} := \begin{cases} K\left(\frac{M_1^p}{m_2^q}\right) > 1 \text{ if } \frac{M_1^p}{m_2^q} < 1, \\ 1 \text{ if } \frac{m_1^p}{M_2^q} \leq 1 \leq \frac{M_1^p}{m_2^q}, \\ K\left(\frac{M_2^q}{m_1^p}\right) > 1 \text{ if } 1 < \frac{m_1^p}{M_2^q} \end{cases} \tag{3.14}$$

and

$$M_{p,q} := \begin{cases} K\left(\frac{M_2^q}{m_1^p}\right) > 1 \text{ if } \frac{M_1^p}{m_2^q} < 1, \\ \max\left\{K\left(\frac{M_2^q}{m_1^p}\right), K\left(\frac{M_1^p}{m_2^q}\right)\right\} > 1 \text{ if } \frac{m_1^p}{M_2^q} \leq 1 \leq \frac{M_1^p}{m_2^q}, \\ K\left(\frac{M_1^p}{m_2^q}\right) > 1 \text{ if } 1 < \frac{m_1^p}{M_2^q}. \end{cases} \tag{3.15}$$

Corollary 3. *Let A, B be two operators such that (3.5) is valid and P, $Q \in \mathcal{B}_1(H)$ with P, $Q > 0$. Then for any p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have for $m_{p,q}$, $M_{p,q}$ as* *defined by (3.14) and (3.15) that*

$$m_{p,q}^{t}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}\leq\frac{1}{p}\frac{\operatorname{tr}(PA^{p})}{\operatorname{tr}(P)}+\frac{1}{q}\frac{\operatorname{tr}(QB^{q})}{\operatorname{tr}(Q)}\leq M_{p,q}^{T}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)},\tag{3.16}$$

where $t=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ *and* $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

Proof. From (3.5) we have

$$0<m_1^pI\leq A^p<M_1^pI,\ 0<m_2^qI\leq B^q\leq M_2^qI.$$

By replacing A by A^p, B by B^q and $\nu=\frac{1}{q}$ in (3.6) then we get the desired result (3.16). □

Remark 4. *If we take* $Q=P$ *in (3.16), then we get*

$$m_{p,q}^{t}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\leq\frac{\operatorname{tr}\left[P\left(\frac{1}{p}A^{p}+\frac{1}{q}B^{q}\right)\right]}{\operatorname{tr}(P)}\leq M_{p,q}^{T}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}.\tag{3.17}$$

For $p=q=2$ we consider

$$\tilde{m}_2:=\begin{cases}K\left[\left(\frac{M_1}{m_2}\right)^2\right]>1\text{ if }\frac{M_1}{m_2}<1,\\ 1\text{ if }\frac{m_1}{M_2}\leq1\leq\frac{M_1}{m_2},\\ K\left[\left(\frac{M_2}{m_1}\right)^2\right]>1\text{ if }1<\frac{m_1}{M_2}\end{cases}\tag{3.18}$$

and

$$\tilde{M}_2:=\begin{cases}K\left[\left(\frac{M_2}{m_1}\right)^2\right]>1\text{ if }\frac{M_1}{m_2}<1,\\ \max\left\{K\left[\left(\frac{M_2}{m_1}\right)^2\right],K\left[\left(\frac{M_1}{m_2}\right)^2\right]\right\}>1\text{ if }\frac{m_1}{M_2}\leq1\leq\frac{M_1}{m_2},\\ K\left[\left(\frac{M_1}{m_2}\right)^2\right]>1\text{ if }1<\frac{m_1}{M_2}.\end{cases}\tag{3.19}$$

Corollary 4. *Let* A, B *be two operators such that (3.5) is valid and* $P,Q\in\mathcal{B}_1(H)$ *with* $P,Q>0$. *Then for* $\tilde{m}_2$, $\tilde{M}_2$ *as defined by (3.18) and (3.19) we have that*

$$\tilde{m}_2^{1/2}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}\leq\frac{1}{p}\frac{\operatorname{tr}(PA^{2})}{\operatorname{tr}(P)}+\frac{1}{q}\frac{\operatorname{tr}(QB^{2})}{\operatorname{tr}(Q)}\leq\tilde{M}_2^{1/2}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}.\tag{3.20}$$

In particular,

$$\tilde{m}_2^{1/2}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)} \le \frac{\operatorname{tr}\left[P\left(\frac{A^2+B^2}{2}\right)\right]}{\operatorname{tr}(P)} \le \tilde{M}_2^{1/2}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}. \tag{3.21}$$

Corollary 5. *If P, Q, S, V are positive invertible operators with P, Q, S, $V \in \mathcal{B}_1(H)$ and for $0 < m_1 < M_1$, $0 < m_2 < M_2$,*

$$0 < m_1 P \le S \le M_1 P,\ 0 < m_2 Q \le V \le M_2 Q. \tag{3.22}$$

Then for any $\nu \in [0,1]$, we have for m, M as defined by (3.3) and (3.4) that

$$\begin{aligned} m^r\frac{\operatorname{tr}(P\sharp_{1-\nu}S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(Q\sharp_{\nu}V)}{\operatorname{tr}(Q)} &\le (1-\nu)\frac{\operatorname{tr}(S)}{\operatorname{tr}(P)} + \nu\frac{\operatorname{tr}(V)}{\operatorname{tr}(Q)} \\ &\le M^R\frac{\operatorname{tr}(P\sharp_{1-\nu}S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(Q\sharp_{\nu}V)}{\operatorname{tr}(Q)}, \end{aligned} \tag{3.23}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu,\nu\}$.

In particular, we have

$$\begin{aligned} m^{1/2}\frac{\operatorname{tr}(P\sharp S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(Q\sharp V)}{\operatorname{tr}(Q)} &\le \frac{1}{2}\left[\frac{\operatorname{tr}(S)}{\operatorname{tr}(P)} + \frac{\operatorname{tr}(V)}{\operatorname{tr}(Q)}\right] \\ &\le M^{1/2}\frac{\operatorname{tr}(P\sharp S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(Q\sharp V)}{\operatorname{tr}(Q)}. \end{aligned} \tag{3.24}$$

Proof. From (3.22) we have

$$0 < m_1 \le P^{-1/2}SP^{-1/2} \le M_1,\ 0 < m_2 \le Q^{-1/2}VQ^{-1/2} \le M_2.$$

If we use the inequality (3.6) for $A = P^{-1/2}SP^{-1/2}$ and $B = Q^{-1/2}VQ^{-1/2}$ then

$$\begin{aligned} & m^r\frac{\operatorname{tr}\left(P\left(P^{-1/2}SP^{-1/2}\right)^{1-\nu}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\left(Q^{-1/2}VQ^{-1/2}\right)^{\nu}\right)}{\operatorname{tr}(Q)} \\ & \le (1-\nu)\frac{\operatorname{tr}\left(PP^{-1/2}SP^{-1/2}\right)}{\operatorname{tr}(P)} + \nu\frac{\operatorname{tr}\left(QQ^{-1/2}VQ^{-1/2}\right)}{\operatorname{tr}(Q)} \\ & \le M^R\frac{\operatorname{tr}\left(P\left(P^{-1/2}SP^{-1/2}\right)^{1-\nu}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(Q\left(Q^{-1/2}VQ^{-1/2}\right)^{\nu}\right)}{\operatorname{tr}(Q)}, \end{aligned}$$

which, by the properties of trace, is equivalent to (3.23). □

Remark 5. *If P, S, V are positive invertible operators with P, S, $V \in \mathcal{B}_1(H)$ and for $0 < m_1 < M_1$, $0 < m_2 < M_2$,*

$$0 < m_1 P \le S \le M_1 P,\ 0 < m_2 P \le V \le M_2 P, \tag{3.25}$$

then for any $\nu \in [0,1]$, we have for m, M as defined by (3.3) and (3.4) that

$$\begin{aligned} m^r\frac{\operatorname{tr}(P\sharp_{1-\nu}S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\sharp_{\nu}V)}{\operatorname{tr}(P)} &\le \frac{\operatorname{tr}((1-\nu)S + \nu V)}{\operatorname{tr}(P)} \\ &\le M^R\frac{\operatorname{tr}(P\sharp_{1-\nu}S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\sharp_{\nu}V)}{\operatorname{tr}(P)}, \end{aligned} \tag{3.26}$$

where $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$.

In particular, we have

$$m^{1/2}\frac{\operatorname{tr}(P\sharp S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\sharp V)}{\operatorname{tr}(P)}\leq\frac{\operatorname{tr}\left(\frac{S+V}{2}\right)}{\operatorname{tr}(P)}\leq M^{1/2}\frac{\operatorname{tr}(P\sharp S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\sharp V)}{\operatorname{tr}(P)}. \tag{3.27}$$

4. Trace Inequalities Via Tominaga and Furuichi Results

We recall that *Specht's ratio* is defined by [27]

$$S(h):=\begin{cases}\frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty)\\ \\ 1 \text{ if } h=1.\end{cases} \tag{4.1}$$

It is well known that $\lim_{h\to 1} S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^r\right)a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{4.2}$$

where $a,\ b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$.

The second inequality in (4.2) is due to Tominaga [28] while the first one is due to Furuichi [11].

If $a\in[m_1,M_1]$ and $b\in[m_2,M_2]$ with $0<m_1<M_1$, $0<m_2<M_2$ then

$$\frac{m_1}{M_2}\leq\frac{a}{b}\leq\frac{M_1}{m_2}.$$

Denote, for $r\in(0,1)$

$$\breve{m}_r=:\min_{(a,b)\in[m_1,M_1]\times[m_2,M_2]}S\left(\left(\frac{a}{b}\right)^r\right) \text{ and } \breve{M}=:\max_{(a,b)\in[m_1,M_1]\times[m_2,M_2]}S\left(\frac{a}{b}\right).$$

Taking into account the properties of Specht's ratio we have

$$\breve{m}_r:=\begin{cases}S\left(\left(\frac{M_1}{m_2}\right)^r\right)>1 \text{ if } \frac{M_1}{m_2}<1,\\ \\ 1 \text{ if } \frac{m_1}{M_2}\leq 1\leq\frac{M_1}{m_2},\\ \\ S\left(\left(\frac{M_2}{m_1}\right)^r\right)>1 \text{ if } 1<\frac{m_1}{M_2},\end{cases} \tag{4.3}$$

and

$$\breve{M}:=\begin{cases}S\left(\frac{M_2}{m_1}\right)>1 \text{ if } \frac{M_1}{m_2}<1,\\ \\ \max\left\{S\left(\frac{M_2}{m_1}\right),S\left(\frac{M_1}{m_2}\right)\right\}>1 \text{ if } \frac{m_1}{M_2}\leq 1\leq\frac{M_1}{m_2},\\ \\ S\left(\frac{M_1}{m_2}\right)>1 \text{ if } 1<\frac{m_1}{M_2}.\end{cases} \tag{4.4}$$

We have the following result:

Theorem 3. *Let A, B be two operators such that*

$$0 < m_1 I \le A < M_1 I,\ 0 < m_2 I \le B \le M_2 I \tag{4.5}$$

and P, $Q \in \mathcal{B}_1(H)$ with P, $Q > 0$. Then for any $\nu \in [0,1]$, we have for $\breve{m}_r$, $\breve{M}$ as defined by (4.3) and (4.4) that

$$\breve{m}_r \frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)} \le (1-\nu)\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \nu \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} \le \breve{M} \frac{\operatorname{tr}\left(PA^{1-\nu}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(QB^{\nu})}{\operatorname{tr}(Q)}, \tag{4.6}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

In particular, we have

$$\breve{m}_{1/2} \frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)} \le \frac{1}{2}\left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}\right] \le \breve{M} \frac{\operatorname{tr}\left(PA^{1/2}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}\left(QB^{1/2}\right)}{\operatorname{tr}(Q)}. \tag{4.7}$$

Proof. From (3.2) we have

$$\breve{m}_r a^{1-\nu} b^{\nu} \le (1-\nu) a + \nu b \le \breve{M} a^{1-\nu} b^{\nu},$$

where $a \in [m_1, M_1]$, $b \in [m_2, M_2]$ and $\nu \in [0,1]$.

Now, on making use of a similar argument as in the proof of Theorem 2, we get the desired result (4.6). □

For $0 < m_1 < M_1$, $0 < m_2 < M_2$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we define for $r \in (0,1)$

$$\breve{m}_{r,p,q} := \begin{cases} S\left(\left(\frac{M_1^p}{m_2^q}\right)^r\right) > 1 \text{ if } \frac{M_1^p}{m_2^q} < 1, \\ 1 \text{ if } \frac{m_1^p}{M_2^q} \le 1 \le \frac{M_1^p}{m_2^q}, \\ S\left(\left(\frac{M_2^q}{m_1^p}\right)^r\right) > 1 \text{ if } 1 < \frac{m_1^p}{M_2^q} \end{cases} \tag{4.8}$$

and

$$\breve{M}_{p,q} := \begin{cases} S\left(\frac{M_2^q}{m_1^p}\right) > 1 \text{ if } \frac{M_1^p}{m_2^q} < 1, \\ \max\left\{S\left(\frac{M_2^q}{m_1^p}\right), S\left(\frac{M_1^p}{m_2^q}\right)\right\} > 1 \text{ if } \frac{m_1^p}{M_2^q} \le 1 \le \frac{M_1^p}{m_2^q}, \\ S\left(\frac{M_1^p}{m_2^q}\right) > 1 \text{ if } 1 < \frac{m_1^p}{M_2^q}. \end{cases} \tag{4.9}$$

Corollary 6. *Let A, B be two operators such that (3.5) is valid and P, $Q \in \mathcal{B}_1(H)$ with P, $Q > 0$. Then for any p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have for $\breve{m}_{t,p,q}$, $\breve{M}_{p,q}$ as* *defined by (4.8) and (4.9) that*

$$\breve{m}_{t,p,q}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)} \leq \frac{1}{p}\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} + \frac{1}{q}\frac{\operatorname{tr}(QB^q)}{\operatorname{tr}(Q)} \leq \breve{M}_{p,q}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB)}{\operatorname{tr}(Q)}, \tag{4.10}$$

where $t = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

The interested reader may write similar inequalities to those in the previous section, however we do not present them here.

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