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SOME INEQUALITIES IN INNER PRODUCT SPACES RELATED TO BUZANO'S AND GRÜSS' RESULTS

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Abstract. Some inequalities in inner product spaces related to Buzano's and Grüss' results are given. Applications for discrete and integral inequalities are provided as well.

1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\| \|y\| \geq |\langle x, y \rangle| \text{ for any } x, y \in H. \tag{1.1}$$

The equality case holds in (1.1) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$.

In 1985 the author [4] (see also [19]) established the following refinement of (1.1):

$$\|x\| \|y\| \geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \geq |\langle x, y \rangle| \tag{1.2}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Using the triangle inequality for modulus we have

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \geq |\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|$$

and by (1.2) we get

$$\begin{aligned} \|x\| \|y\| &\geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \\ &\geq 2 |\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|, \end{aligned}$$

which implies the *Buzano inequality* [2]

$$\frac{1}{2} [\|x\| \|y\| + |\langle x, y \rangle|] \geq |\langle x, e \rangle \langle e, y \rangle| \tag{1.3}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

In [5], the author has proved the following Grüss' type inequality in real or complex inner product spaces.

Theorem 1. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and $e \in H$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0 \text{ and } \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0 \tag{1.4}$$

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hold, then we have the inequality

$$|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle| \le \frac{1}{4}|\Phi - \varphi| |\Gamma - \gamma| . \tag{1.5}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

For other Schwarz, Buzano and Grüss related inequalities in inner product spaces, see [1]-[3], [4]-[13], [17]-[20], [22]-[29], and the monographs [14], [15] and [16].

2. Main Results

The following results hold:

Theorem 2. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. If $x,y,e,f \in H$ with $\|e\| = \|f\| = 1$, then*

$$\begin{aligned} &\|x\|\|y\| - |\langle x,e\rangle \langle f,y\rangle| \\ &\ge |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle + \langle x,e\rangle \langle f,y\rangle \langle e,f\rangle| . \end{aligned} \tag{2.1}$$

Proof. Using Schwarz inequality we have

$$\|x - \langle x,e\rangle e\|^2 \|y - \langle y,f\rangle f\|^2 \ge |\langle x - \langle x,e\rangle e, y - \langle y,f\rangle f\rangle|^2 \tag{2.2}$$

for any $x,y,e,f \in H$ with $\|e\| = \|f\| = 1$.

Since

$$\|x - \langle x,e\rangle e\|^2 = \|x\|^2 - |\langle x,e\rangle|^2, \quad \|y - \langle y,f\rangle f\|^2 = \|y\|^2 - |\langle y,f\rangle|^2$$

and

$$\langle x - \langle x,e\rangle e, y - \langle y,f\rangle f\rangle = \langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle + \langle x,e\rangle \langle f,y\rangle \langle e,f\rangle ,$$

then by (2.2) we get

$$\begin{aligned} &\left(\|x\|^2 - |\langle x,e\rangle|^2\right)\left(\|y\|^2 - |\langle y,f\rangle|^2\right) \\ &\ge |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle + \langle x,e\rangle \langle f,y\rangle \langle e,f\rangle|^2 \end{aligned} \tag{2.3}$$

for any $x,y,e,f \in H$ with $\|e\| = \|f\| = 1$.

Using the elementary inequality

$$(ac - bd)^2 \ge \left(a^2 - b^2\right)\left(c^2 - d^2\right)$$

that holds for any real numbers $a,b,c,d \in \mathbb{R}$, we have

$$(\|x\|\|y\| - |\langle x,e\rangle| |\langle y,f\rangle|)^2 \ge \left(\|x\|^2 - |\langle x,e\rangle|^2\right)\left(\|y\|^2 - |\langle y,f\rangle|^2\right) \tag{2.4}$$

for any $x,y,e,f \in H$ with $\|e\| = \|f\| = 1$.

By Schwarz inequality for the pairs (x,e) and (y,f) we have

$$\|x\| \ge |\langle x,e\rangle| \text{ and } \|y\| \ge |\langle y,f\rangle| ,$$

which shows that

$$\|x\|\|y\| - |\langle x,e\rangle| |\langle y,f\rangle| \ge 0,$$

for any $x,y,e,f \in H$ with $\|e\| = \|f\| = 1$.

Making use of (2.3) and (2.4) we get

$$(\|x\|\|y\| - |\langle x,e\rangle| |\langle y,f\rangle|)^2 \geq |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle + \langle x,e\rangle \langle f,y\rangle \langle e,f\rangle|^2 \tag{2.5}$$

and by taking the square root in (2.5) we get the desired result. □

Corollary 1. *With the assumptions of Theorem 2 and if $e \perp f$, i.e. $\langle e,f\rangle = 0$, then we have the inequality*

$$\|x\|\|y\| - |\langle x,e\rangle \langle f,y\rangle| \geq |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle|. \tag{2.6}$$

Remark 1. *From the inequality (2.6) we have*

$$\begin{aligned}\|x\|\|y\| &\geq |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle| + |\langle x,e\rangle \langle f,y\rangle| \\ &\geq |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle \pm \langle x,e\rangle \langle f,y\rangle|\end{aligned} \tag{2.7}$$

By the triangle inequality we also have

$$|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle| \geq |\langle x,e\rangle \langle e,y\rangle + \langle x,f\rangle \langle f,y\rangle| - |\langle x,y\rangle|$$

and by the first inequality in (2.9) we get

$$\|x\|\|y\| \geq |\langle x,e\rangle \langle e,y\rangle + \langle x,f\rangle \langle f,y\rangle| - |\langle x,y\rangle| + |\langle x,e\rangle \langle f,y\rangle|,$$

which implies

$$\begin{aligned}\|x\|\|y\| + |\langle x,y\rangle| &\geq |\langle x,e\rangle \langle e,y\rangle + \langle x,f\rangle \langle f,y\rangle| + |\langle x,e\rangle \langle f,y\rangle| \\ &\geq |\langle x,e\rangle \langle e,y\rangle + \langle x,f\rangle \langle f,y\rangle + \langle x,e\rangle \langle f,y\rangle|\end{aligned} \tag{2.8}$$

for any $x,y,e,f \in H$ with $\|e\| = \|f\| = 1$ and $e \perp f$.

Corollary 2. *With the assumptions of Theorem 2 we have*

$$\|x\|\|y\| - |\langle x,e\rangle \langle f,y\rangle| (1 - |\langle e,f\rangle|) \geq |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle| \tag{2.9}$$

and

$$\|x\|\|y\| + |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle| \geq |\langle x,e\rangle \langle f,y\rangle| (|\langle e,f\rangle| + 1). \tag{2.10}$$

Indeed, by the triangle inequality we have

$$\begin{aligned}&|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle + \langle x,e\rangle \langle f,y\rangle \langle e,f\rangle| \\ &\geq |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle| - |\langle x,e\rangle \langle f,y\rangle \langle e,f\rangle|\end{aligned}$$

and by (2.1) we get (2.9).

By the triangle inequality we also have

$$\begin{aligned}&|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle + \langle x,e\rangle \langle f,y\rangle \langle e,f\rangle| \\ &\geq |\langle x,e\rangle \langle f,y\rangle \langle e,f\rangle| - |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle|\end{aligned}$$

and by (2.1) we get (2.10).

Remark 2. *With the assumptions of Theorem 2 and if $|\langle e,f\rangle| = 1$, then we have*

$$\|x\|\|y\| \geq |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle| \tag{2.11}$$

and

$$\frac{1}{2}\left[\|x\|\|y\| + |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle - \langle x,f\rangle \langle f,y\rangle|\right] \geq |\langle x,e\rangle \langle f,y\rangle|. \tag{2.12}$$

If we take $f = e$ in (2.11) and (2.12), then we get the inequalities

$$\|x\|\|y\| \geq |\langle x,y\rangle - 2\langle x,e\rangle \langle e,y\rangle| \tag{2.13}$$

and

$$\frac{1}{2}\left[\|x\|\|y\|+|\langle x,y\rangle-2\langle x,e\rangle\langle e,y\rangle|\right]\geq|\langle x,e\rangle\langle e,y\rangle| \tag{2.14}$$

for any $x,y,e\in H$ *with* $\|e\|=1.$

Using the triangle inequality we have

$$|\langle x,y\rangle-2\langle x,e\rangle\langle e,y\rangle|\geq 2|\langle x,e\rangle\langle e,y\rangle|-|\langle x,y\rangle|$$

and by (2.13) we get

$$\|x\|\|y\|\geq|\langle x,y\rangle-2\langle x,e\rangle\langle e,y\rangle|\geq 2|\langle x,e\rangle\langle e,y\rangle|-|\langle x,y\rangle|. \tag{2.15}$$

The inequality between the first and last term in (2.15) is equivalent to Buzano's inequality (1.3).

The following lemma holds, see [6]:

Lemma 1. *Let* a,x,A *be vectors in the inner product space* $(H,\langle\cdot,\cdot\rangle)$ *over* $\mathbb{K}$ *with* $a\neq A$. *Then*

$$\operatorname{Re}\langle A-x,x-a\rangle\geq 0 \tag{2.16}$$

if and only if

$$\left\|x-\frac{a+A}{2}\right\|\leq\frac{1}{2}\|A-a\|. \tag{2.17}$$

Proof. Define

$$I_1:=\operatorname{Re}\langle A-x,x-a\rangle \text{ and } I_2:=\frac{1}{4}\|A-a\|^2-\left\|x-\frac{a+A}{2}\right\|^2.$$

A simple calculation shows that

$$I_1=I_2=\operatorname{Re}\left[\langle x,a\rangle+\langle A,x\rangle\right]-\operatorname{Re}\langle A,a\rangle-\|x\|^2$$

and thus, obviously, $I_1\geq 0$ iff $I_2\geq 0$ showing the required equivalence. □

The following corollary is obvious:

Corollary 3. *Let* $x,e\in H$ *with* $\|e\|=1$ *and* $\delta,\Delta\in\mathbb{K}$ *with* $\delta\neq\Delta$. *Then*

$$\operatorname{Re}\langle\Delta e-x,x-\delta e\rangle\geq 0 \tag{2.18}$$

iff

$$\left\|x-\frac{\delta+\Delta}{2}\cdot e\right\|\leq\frac{1}{2}|\Delta-\delta|. \tag{2.19}$$

Remark 3. *If* $H=\mathbb{C}$, *then* $\operatorname{Re}\left[(A-x)(\bar{x}-\bar{a})\right]\geq 0$ *if and only if* $\left|x-\frac{a+A}{2}\right|\leq\frac{1}{2}|A-a|$, *where* $a,x,A\in\mathbb{C}$. *If* $H=\mathbb{R}$, *and* $A>a$ *then* $a\leq x\leq A$ *if and only if* $\left|x-\frac{a+A}{2}\right|\leq\frac{1}{2}(A-a)$.

The following lemma is of interest [6].

Lemma 2. *Let* $x,e\in H$ *with* $\|e\|=1$. *Then one has the following representation*

$$\|x\|^2-|\langle x,e\rangle|^2=\inf_{\lambda\in\mathbb{K}}\|x-\lambda e\|^2\geq 0. \tag{2.20}$$

Proof. Observe, for any $\lambda \in \mathbb{K}$, that

$$\begin{aligned}\langle x-\lambda e, x-\langle x,e\rangle e\rangle &= \|x\|^2 - |\langle x,e\rangle|^2 - \lambda\left[\langle e,x\rangle - \langle e,x\rangle \|e\|^2\right]\\ &= \|x\|^2 - |\langle x,e\rangle|^2 .\end{aligned}$$

Using Schwarz's inequality, we have

$$\begin{aligned}\left[\|x\|^2 - |\langle x,e\rangle|^2\right]^2 &= |\langle x-\lambda e, x-\langle x,e\rangle e\rangle|^2 \le \|x-\lambda e\|^2 \|x-\langle x,e\rangle e\|^2\\ &= \|x-\lambda e\|^2 \left[\|x\|^2 - |\langle x,e\rangle|^2\right],\end{aligned}$$

giving the bound

$$\|x\|^2 - |\langle x,e\rangle|^2 \le \|x-\lambda e\|^2, \qquad \lambda \in \mathbb{K}. \tag{2.21}$$

Taking the infimum in (2.21) over $\lambda \in \mathbb{K}$, we deduce

$$\|x\|^2 - |\langle x,e\rangle|^2 \le \inf_{\lambda\in\mathbb{K}} \|x-\lambda e\|^2 .$$

Since, for $\lambda_0 = \langle x,e\rangle$, we get $\|x-\lambda_0 e\|^2 = \|x\|^2 - |\langle x,e\rangle|^2$, then the representation (2.20) is proved. □

The following result also holds:

Theorem 3. *Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $e, f \in H$, $\|e\| = \|f\| = 1$. If $\varphi,\gamma,\Phi,\Gamma$ are real or complex numbers and x,y are vectors in H such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x-\varphi e\rangle \ge 0, \;\; \operatorname{Re}\langle \Gamma f - y, y - \gamma f\rangle \ge 0 \tag{2.22}$$

hold, or, equivalently, the following assumptions

$$\left\| x - \frac{\varphi+\Phi}{2} e\right\| \le \frac{1}{2}|\Phi-\varphi|, \quad \left\| y - \frac{\gamma+\Gamma}{2} f\right\| \le \frac{1}{2}|\Gamma-\gamma| \tag{2.23}$$

are valid, then one has the inequality

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle + \langle x,e\rangle\langle f,y\rangle\langle e,f\rangle| \le \frac{1}{4}|\Phi-\varphi||\Gamma-\gamma|. \tag{2.24}$$

Proof. Using the inequality (2.3) and Lemma 2 we have

$$\begin{aligned}&|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle + \langle x,e\rangle\langle f,y\rangle\langle e,f\rangle|^2\\ &\le \left(\|x\|^2 - |\langle x,e\rangle|^2\right)\left(\|y\|^2 - |\langle y,f\rangle|^2\right) = \inf_{\lambda\in\mathbb{K}}\|x-\lambda e\|^2 \inf_{\eta\in\mathbb{K}}\|y-\eta f\|^2\\ &\le \left\| x - \frac{\varphi+\Phi}{2} e\right\|^2 \left\| y - \frac{\gamma+\Gamma}{2} f\right\|^2 \le \frac{1}{4}|\Phi-\varphi|^2 \frac{1}{4}|\Gamma-\gamma|^2,\end{aligned} \tag{2.25}$$

which is equivalent to the desired inequality (2.24). □

Corollary 4. *With the assumptions of Theorem 3 and if $e \perp f$, then we have the simpler inequality*

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle| \le \frac{1}{4}|\Phi-\varphi||\Gamma-\gamma|. \tag{2.26}$$

Remark 4. *If we take $f = e$ in Theorem 3, then we get the result from Theorem 1.*

3. Applications

Consider the Hilbert space $\mathbb{C}^n$ endowed with the inner product $\langle\cdot,\cdot\rangle_{\mathbf{p}}:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$ defined by

$$\langle\mathbf{x},\mathbf{y}\rangle_{\mathbf{p}}:=\sum_{j=1}^{n}p_j x_j\overline{y}_j,$$

where $\mathbf{p}=(p_1,...,p_n)$ is a probability distribution, i.e. $p_j\geq 0,\ j\in\{1,...,n\}$ with $\sum_{j=1}^{n}p_j=1$ and

$$\mathbf{x}=(x_1,...,x_n)\,,\ \mathbf{y}=(y_1,...,y_n)\in\mathbb{C}^n.$$

Assume that $\mathbf{e}=(e_1,...,e_n)\,,\mathbf{f}=(f_1,...,f_n)\in\mathbb{C}^n$ with

$$\sum_{j=1}^{n}p_j\left|e_j\right|^2=\sum_{j=1}^{n}p_j\left|f_j\right|^2=1. \tag{3.1}$$

Then for any $\mathbf{x}=(x_1,...,x_n)\,,\ \mathbf{y}=(y_1,...,y_n)\in\mathbb{C}^n$ we have the inequality

$$\begin{aligned}&\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2\right)^{1/2}\left(\sum_{j=1}^{n}p_j\left|y_j\right|^2\right)^{1/2}-\left|\sum_{j=1}^{n}p_jx_j\overline{e}_j\sum_{j=1}^{n}p_jf_j\overline{y}_j\right|\\&\geq\left|\sum_{j=1}^{n}p_jx_j\overline{y}_j-\sum_{j=1}^{n}p_jx_j\overline{e}_j\sum_{j=1}^{n}p_je_j\overline{y}_j\right.\\&\left.-\sum_{j=1}^{n}p_jx_j\overline{f}_j\sum_{j=1}^{n}p_jf_j\overline{y}_j+\sum_{j=1}^{n}p_jx_j\overline{e}_j\sum_{j=1}^{n}p_jf_j\overline{y}_j\sum_{j=1}^{n}p_je_j\overline{f}_j\right|.\end{aligned}\tag{3.2}$$

Moreover, if $\mathbf{e}=(e_1,...,e_n)\,,\mathbf{f}=(f_1,...,f_n)\in\mathbb{C}^n$ satisfy the additional condition

$$\sum_{j=1}^{n}p_je_j\overline{f}_j=0, \tag{3.3}$$

then from (3.2) we get

$$\begin{aligned}&\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2\right)^{1/2}\left(\sum_{j=1}^{n}p_j\left|y_j\right|^2\right)^{1/2}-\left|\sum_{j=1}^{n}p_jx_j\overline{e}_j\sum_{j=1}^{n}p_jf_j\overline{y}_j\right|\\&\geq\left|\sum_{j=1}^{n}p_jx_j\overline{y}_j-\sum_{j=1}^{n}p_jx_j\overline{e}_j\sum_{j=1}^{n}p_je_j\overline{y}_j-\sum_{j=1}^{n}p_jx_j\overline{f}_j\sum_{j=1}^{n}p_jf_j\overline{y}_j\right|.\end{aligned}\tag{3.4}$$

If we denote by $\mathcal{C}(0,1)$ the unit circle of radius 1 in $\mathbb{C}$, namely $\mathcal{C}(0,1)=\{z\in\mathbb{C}|\ \ |z|=1\}$, then for $\mathbf{e}=(e_1,...,e_n)\,,\mathbf{f}=(f_1,...,f_n)\in\mathbb{C}^n$ with $e_j,f_j\in\mathcal{C}(0,1)$ for any $j\in\{1,...,n\}$ we have that the condition (3.1) holds true and therefore the inequality (3.2) is valid.

If we consider the nonnegative weights $w_j\geq 0, j\in\{1,...,n\}$ with $W_n=\sum_{k=1}^{n}w_k>0$ and if we assume that

$$\frac{1}{W_n}\sum_{j=1}^{n}w_j\left|e_j\right|^2=\frac{1}{W_n}\sum_{j=1}^{n}w_j\left|f_j\right|^2=1 \tag{3.5}$$

then by (3.2) we get

$$\begin{aligned}
(3.6)\qquad & \left(\frac{1}{W_n}\sum_{j=1}^{n} w_j\left|x_j\right|^2\right)^{1/2}\left(\frac{1}{W_n}\sum_{j=1}^{n} w_j\left|y_j\right|^2\right)^{1/2} \\
& -\left|\frac{1}{W_n}\sum_{j=1}^{n} w_j x_j \overline{e}_j \frac{1}{W_n}\sum_{j=1}^{n} w_j f_j \overline{y}_j\right| \\
& \geq \left|\frac{1}{W_n}\sum_{j=1}^{n} w_j x_j \overline{y}_j - \frac{1}{W_n}\sum_{j=1}^{n} w_j x_j \overline{e}_j \frac{1}{W_n}\sum_{j=1}^{n} w_j e_j \overline{y}_j\right. \\
& - \frac{1}{W_n}\sum_{j=1}^{n} w_j x_j \overline{f}_j \frac{1}{W_n}\sum_{j=1}^{n} w_j f_j \overline{y}_j \\
& \left. + \frac{1}{W_n}\sum_{j=1}^{n} w_j x_j \overline{e}_j \frac{1}{W_n}\sum_{j=1}^{n} w_j f_j \overline{y}_j \frac{1}{W_n}\sum_{j=1}^{n} w_j e_j \overline{f}_j\right|.
\end{aligned}$$

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$.

The most important power series with nonnegative coefficients that can be used to illustrate the above results are:

$$\begin{aligned}
(3.7)\qquad & \exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n,\ z\in\mathbb{C},\ \frac{1}{1-z}=\sum_{n=0}^{\infty} z^n,\ z\in D(0,1), \\
& \ln\frac{1}{1-z}=\sum_{n=1}^{\infty}\frac{1}{n}z^n,\ z\in D(0,1),\ \cosh z=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n},\ z\in\mathbb{C}, \\
& \sinh z=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1},\ z\in\mathbb{C}.
\end{aligned}$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\begin{aligned}
(3.8)\qquad & \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1), \\
& \sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!}z^{2n+1},\ z\in D(0,1), \\
& \tanh^{-1}(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1), \\
& {}_2F_1(\alpha,\beta,\gamma,z):=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)\,\Gamma(\gamma)}{n!\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0 \\
& z\in D(0,1),
\end{aligned}$$

where Γ is *Gamma function.*

Proposition 1. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $0<p<R$, $u,v\in\mathcal{C}(0,1)$ and $x,y\in\mathbb{C}$ with $p|x|^2, p|y|^2<R$ then we have the inequality*

$$\left(\frac{f\left(p|x|^2\right)}{f(p)}\right)^{1/2}\left(\frac{f\left(p|y|^2\right)}{f(p)}\right)^{1/2}-\left|\frac{f(px\overline{u})}{f(p)}\frac{f(pv\overline{y})}{f(p)}\right| \tag{3.9}$$
$$\geq\left|\frac{f(px\overline{y})}{f(p)}-\frac{f(px\overline{u})}{f(p)}\frac{f(pu\overline{y})}{f(p)}-\frac{f(px\overline{v})}{f(p)}\frac{f(pv\overline{y})}{f(p)}+\frac{f(px\overline{u})}{f(p)}\frac{f(pv\overline{y})}{f(p)}\frac{f(pu\overline{v})}{f(p)}\right|.$$

Proof. If $u,v\in\mathcal{C}(0,1)$ then for any $n\geq0$ we have $u^n,v^n\in\mathcal{C}(0,1)$. Observe that for any $m\geq1$ we have that

$$\frac{\sum_{n=0}^{m}a_np^n|u^n|^2}{\sum_{n=0}^{m}a_np^n}=\frac{\sum_{n=0}^{m}a_np^n|v^n|^2}{\sum_{n=0}^{m}a_np^n}=\frac{\sum_{n=0}^{m}a_np^n}{\sum_{n=0}^{m}a_np^n}=1.$$

Using the inequality (3.6) we have

$$\left(\frac{\sum_{n=0}^{m}a_np^n|x|^{2n}}{\sum_{n=0}^{m}a_np^n}\right)^{1/2}\left(\frac{\sum_{n=0}^{m}a_np^n|y|^{2n}}{\sum_{n=0}^{m}a_np^n}\right)^{1/2} \tag{3.10}$$
$$-\left|\frac{\sum_{n=0}^{m}a_np^n(x\overline{u})^n}{\sum_{n=0}^{m}a_np^n}\frac{\sum_{n=0}^{m}a_np^n(v\overline{y})^n}{\sum_{n=0}^{m}a_np^n}\right|$$
$$\geq\left|\frac{\sum_{n=0}^{m}a_np^n(x\overline{y})^n}{\sum_{n=0}^{m}a_np^n}-\frac{\sum_{n=0}^{m}a_np^n(x\overline{u})^n}{\sum_{n=0}^{m}a_np^n}\frac{\sum_{n=0}^{m}a_np^n(u\overline{y})^n}{\sum_{n=0}^{m}a_np^n}\right.$$
$$-\frac{\sum_{n=0}^{m}a_np^n(x\overline{v})^n}{\sum_{n=0}^{m}a_np^n}\frac{\sum_{n=0}^{m}a_np^n(v\overline{y})^n}{\sum_{n=0}^{m}a_np^n}$$
$$\left.+\frac{\sum_{n=0}^{m}a_np^n(x\overline{u})^n}{\sum_{n=0}^{m}a_np^n}\frac{\sum_{n=0}^{m}a_np^n(v\overline{y})^n}{\sum_{n=0}^{m}a_np^n}\frac{\sum_{n=0}^{m}a_np^n(u\overline{v})^n}{\sum_{n=0}^{m}a_np^n}\right|.$$

Since all the series whose partial sums are involved in inequality (3.10) are convergent, then by letting $m\to\infty$ in (3.10) we get the desired result (3.9). □

Remark 5. *The inequality (3.9) can provide some particular inequalities of interest. For instance, if we take $f(z)=\exp(z)$, $z\in\mathbb{C}$, then we get*

$$\exp\left[p\left(\frac{|x|^2+|y|^2}{2}-1\right)\right]-|\exp[p(x\overline{u}+v\overline{y}-2)]| \tag{3.11}$$
$$\geq|\exp[p(x\overline{y}-1)]-\exp[p(x\overline{u}+u\overline{y}-2)]-\exp[p(x\overline{v}+v\overline{y}-2)]$$
$$+\exp[p(x\overline{u}+v\overline{y}+u\overline{v}-3)]|$$

for any $p>0, u,v\in\mathcal{C}(0,1)$ and $x,y\in\mathbb{C}$.

If we take $u=v=1$, then from (3.11) we get

$$\exp\left[p\left(\frac{|x|^2+|y|^2}{2}-1\right)\right]-|\exp[p(x+\overline{y}-2)]| \tag{3.12}$$
$$\geq|\exp[p(x\overline{y}-1)]-\exp[p(x+\overline{y}-2)]|$$

for any $p>0$ and $x,y\in\mathbb{C}$.

Moreover, if we take in (3.12) $x=\overline{y}=z\in\mathbb{C}$, then we get

$$\exp\left[p\left(|z|^2-1\right)\right]-|\exp[2p(z-1)]|\geq\left|\exp\left[p\left(z^2-1\right)\right]-\exp[2p(z-1)]\right| \tag{3.13}$$

for any $p>0$ and $z\in\mathbb{C}$.

Consider $L^2[a,b]$ the Hilbert space of all complex valued functions f with $\int_a^b|f(t)|^2dt<\infty$. The inner product is given by

$$\langle f,g\rangle_2:=\int_a^b f(t)\overline{g(t)}dt.$$

Assume that $h,k\in L^2[a,b]$ with

$$\int_a^b|h(t)|^2dt=\int_a^b|k(t)|^2dt=1. \tag{3.14}$$

For instance, if $h(t)=\frac{1}{\sqrt{b-a}}\rho(t)$, $k(t)=\frac{1}{\sqrt{b-a}}\varphi(t)$ with $\rho(t),\varphi(t)\in\mathcal{C}(0,1)$ for almost any $t\in[a,b]$, then $h,k\in L^2[a,b]$ and the condition (3.14) is satisfied.

Proposition 2. *Assume that $h,k\in L^2[a,b]$ with the property (3.14). Then for any $f,g\in L^2[a,b]$ we have the inequality*

$$\begin{aligned}&\left(\int_a^b|f(t)|^2dt\right)^{1/2}\left(\int_a^b|g(t)|^2dt\right)^{1/2}-\left|\int_a^b f(t)\overline{h(t)}dt\int_a^b k(t)\overline{g(t)}dt\right|\\&\geq\left|\int_a^b f(t)\overline{g(t)}dt-\int_a^b f(t)\overline{h(t)}dt\int_a^b h(t)\overline{g(t)}dt\right.\\&-\int_a^b f(t)\overline{k(t)}dt\int_a^b k(t)\overline{g(t)}dt\\&\left.+\int_a^b f(t)\overline{h(t)}dt\int_a^b k(t)\overline{g(t)}dt\int_a^b h(t)\overline{k(t)}dt\right|.\end{aligned} \tag{3.15}$$

The proof follows by Theorem 2 for the inner product $\langle\cdot,\cdot\rangle_2$.

Remark 6. *If $\rho(t),\varphi(t)\in\mathcal{C}(0,1)$ for almost any $t\in[a,b]$, then we have the following inequalities for integral means*

$$\begin{aligned}&\left(\frac{1}{b-a}\int_a^b|f(t)|^2dt\right)^{1/2}\left(\frac{1}{b-a}\int_a^b|g(t)|^2dt\right)^{1/2}\\&-\left|\frac{1}{b-a}\int_a^b f(t)\overline{\rho(t)}dt\frac{1}{b-a}\int_a^b\varphi(t)\overline{g(t)}dt\right|\\&\geq\left|\frac{1}{b-a}\int_a^b f(t)\overline{g(t)}dt-\frac{1}{b-a}\int_a^b f(t)\overline{\rho(t)}dt\frac{1}{b-a}\int_a^b\rho(t)\overline{g(t)}dt\right.\\&-\frac{1}{b-a}\int_a^b f(t)\overline{\varphi(t)}dt\frac{1}{b-a}\int_a^b\varphi(t)\overline{g(t)}dt\\&\left.+\frac{1}{b-a}\int_a^b f(t)\overline{\rho(t)}dt\frac{1}{b-a}\int_a^b\varphi(t)\overline{g(t)}dt\frac{1}{b-a}\int_a^b\rho(t)\overline{\varphi(t)}dt\right|,\end{aligned} \tag{3.16}$$

for any $f,g\in L^2[a,b]$.

If we take $\rho(t) = 1$, $\varphi(t) = sgn\left(t - \frac{a+b}{2}\right)$, $t \in [a,b]$, *then* $\rho(t), \varphi(t) \in \mathcal{C}(0,1)$ *for almost any* $t \in [a,b]$ *and since*

$$\int_a^b \rho(t)\overline{\varphi(t)} = \int_a^b sgn\left(t - \frac{a+b}{2}\right)dt = 0,$$

then we get from (3.16)

$$\begin{aligned}(3.17)\quad & \left(\frac{1}{b-a}\int_a^b |f(t)|^2 dt\right)^{1/2}\left(\frac{1}{b-a}\int_a^b |g(t)|^2 dt\right)^{1/2} \\ & - \left|\frac{1}{b-a}\int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)\overline{g(t)}dt\right| \\ & \geq \left|\frac{1}{b-a}\int_a^b f(t)\overline{g(t)}dt - \frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b \overline{g(t)}dt\right. \\ & \left. -\frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)f(t)\,dt\frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)\overline{g(t)}dt\right|\end{aligned}$$

for any $f, g \in L^2[a,b]$.

On making use of Theorem 3 one can state similar discrete and integral inequalities. However the details are not presented here.

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