Received 30/10/15

FURTHER INEQUALITIES FOR RELATIVE OPERATOR ENTROPY

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Abstract. In this paper, that is a continuation of recent work [S. S. Dragomir, Some inequalities for relative operator entropy, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 145. `http://rgmia.org/papers/v18/v18a145.pdf`], we obtain further inequalities for relative operator entropy of two positive invertible operators. Applications for the operator entropy are also given. Some trace inequalities are derived as well.

1. Introduction

Kamei and Fujii [7], [8] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}}\left(\ln A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}, \tag{1.1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [13].

In general, we can define for positive operators A, B

$$S(A|B) := s - \lim_{\varepsilon\to 0+} S(A+\varepsilon I|B)$$

if it exists, here I is the identity operator.

For the entropy function $\eta(t) = -t\ln t$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A = S(A|I) \geq 0$$

for positive contraction A. This shows that the relative operator entropy (1.1) is a relative version of the operator entropy.

Following [9, p. 149-p. 155], we recall some important properties of relative operator entropy for A and B positive invertible operators:

(i) We have the equalities

$$S(A|B) = -A^{1/2}\left(\ln A^{1/2}B^{-1}A^{1/2}\right)A^{1/2} = B^{1/2}\eta\left(B^{-1/2}AB^{-1/2}\right)B^{1/2}; \tag{1.2}$$

(ii) We have the inequalities

$$S(A|B) \leq A(\ln\|B\| - \ln A) \text{ and } S(A|B) \leq B - A;$$

(iii) For any C, D positive invertible operators we have that

$$S(A+B|C+D) \geq S(A|C) + S(B|D);$$

1991 *Mathematics Subject Classification.* 47A63, 47A30,

Key words and phrases. Young's inequality, Arithmetic mean-Geometric mean inequality,.Relative Operator entropy, Operator entropy.

(iv) If $B \leq C$ then

$$S(A|B) \leq S(A|C);$$

(v) If $B_n \downarrow B$ then

$$S(A|B_n) \downarrow S(A|B);$$

(vi) For $\alpha > 0$ we have

$$S(\alpha A|\alpha B) = \alpha S(A|B);$$

(vii) For every operator T we have

$$T^* S(A|B) T \leq S(T^* AT|T^* BT).$$

The relative operator entropy is jointly concave, namely, for any positive invertible operators A, B, C, D we have

$$S(tA + (1-t) B|tC + (1-t) D) \geq tS(A|C) + (1-t) S(B|D)$$

for any $t \in [0, 1]$.

For other results on the relative operator entropy see [5], [10], [11], [12] and [14].

In the recent paper [4] we have obtained amongst other the following result:

Theorem 1. *Let A, B be two positive invertible operators such that the condition*

$$mA \leq B \leq MA, \tag{1.3}$$

for some m, M with $0 < m < M$, is valid, then we have

$$\begin{aligned} &\frac{1}{2M^2}(B - mA) A^{-1}(MA - B) \\ &\leq S(A|B) - \frac{\ln m}{M - m}(MA - B) - \frac{\ln M}{M - m}(B - mA) \\ &\leq \frac{1}{2m^2}(B - mA) A^{-1}(MA - B). \end{aligned} \tag{1.4}$$

In particular, we have the following result for the operator entropy:

Corollary 1. *Assume that $pI \leq C \leq PI$ for some constants p, P with $0 < p < P$. Then we have for operator entropy $\eta(C) = -C \ln C$ that*

$$\begin{aligned} &\frac{p}{2P}(IP - C) C^{-1}(C - Ip) \\ &\leq \eta(C) + \frac{P \ln P}{P - p}(C - pI) + \frac{p \ln p}{P - p}(PI - C) \\ &\leq \frac{P}{2p}(IP - C) C^{-1}(C - Ip). \end{aligned} \tag{1.5}$$

Observe that, if we replace in (1.2) B with A, then we get

$$\begin{aligned} S(B|A) &= A^{1/2} \eta\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} \\ &= A^{1/2}\left(-A^{-1/2} B A^{-1/2} \ln\left(A^{-1/2} B A^{-1/2}\right)\right) A^{1/2}, \end{aligned}$$

therefore we have

$$A^{1/2}\left(A^{-1/2} B A^{-1/2} \ln\left(A^{-1/2} B A^{-1/2}\right)\right) A^{1/2} = -S(B|A) \tag{1.6}$$

for positive invertible operators A and B.

It is well know that, in general $S(A|B)$ is not equal to $S(B|A)$.

Motivated by the above results, we establish in this paper some bounds for the quantity $S(B|A)$ under the same assumptions (1.3) for the operators A and B. For this purpose, we use some scalar inequalities for convex functions from [1], [2] and [3]. Applications for the operator entropy $\eta(C) = -C\ln C = S(C|I)$ under the natural assumption $pI \le C \le PI$ for some constants p, P with $0 < p < P$, are also provided.

2. Absolute Value Upper and Lower Bounds

We have:

Theorem 2. *Let A, B be two positive invertible operators such that the condition (1.3) is valid, then we have*

$$\begin{aligned} &2\left(\frac{1}{2}A - \frac{1}{M-m}A^{1/2}\left|A^{-1/2}\left(B - \frac{m+M}{2}A\right)A^{-1/2}\right|A^{1/2}\right)K(m,M) \quad (2.1)\\ &\le \frac{m\ln m}{M-m}(MA-B) + \frac{M\ln M}{M-m}(B-mA) + S(B|A)\\ &\le 2\left(\frac{1}{2}A + \frac{1}{M-m}A^{1/2}\left|A^{-1/2}\left(B - \frac{m+M}{2}A\right)A^{-1/2}\right|A^{1/2}\right)K(m,M), \end{aligned}$$

where

$$\begin{aligned} K(m,M) &:= \left[\frac{m\ln m + M\ln M}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right]\\ &= \ln\left(\frac{G(m^m, M^M)}{[A(m,M)]^{A(m,M)}}\right) \end{aligned}$$

and $G(a,b) := \sqrt{ab}$ is the geometric mean while $A(a,b) := \frac{a+b}{2}$ is the arithmetic mean of positive numbers a, b.

Proof. Recall the following result obtained by the author in 2006 [1] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} &n\min_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right] \quad (2.2)\\ &\le \frac{1}{P_n}\sum_{j=1}^{n}p_j\Phi(x_j) - \Phi\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\right)\\ &\le n\max_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right], \end{aligned}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n}p_j > 0$. For $n = 2$, we deduce from (2.2) that

$$\begin{aligned} &2r\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \quad (2.3)\\ &\le \nu\Phi(x) + (1-\nu)\Phi(y) - \Phi[\nu x + (1-\nu)y]\\ &\le 2R\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \end{aligned}$$

for any x, $y \in \mathbb{R}$ and $\nu \in [0,1]$, where $r = \min\{\nu, 1-\nu\}$ and $R = \max\{\nu, 1-\nu\}$.

Now, if we take in (2.3) the convex function $\Phi(t) = t\ln t$, $t > 0$, then we get

$$
\begin{aligned}
(2.4)\qquad & 2r\left[\frac{x\ln x + y\ln y}{2} - \left(\frac{x+y}{2}\right)\ln\left(\frac{x+y}{2}\right)\right] \\
&\leq \nu x\ln x + (1-\nu)\,y\ln y - [\nu x + (1-\nu)\,y]\ln[\nu x + (1-\nu)\,y] \\
&\leq 2R\left[\frac{x\ln x + y\ln y}{2} - \left(\frac{x+y}{2}\right)\ln\left(\frac{x+y}{2}\right)\right]
\end{aligned}
$$

for any x, $y > 0$ and $\nu \in [0,1]$.

This is an inequality of interest in itself as well.

Now, if we take in (2.4) $x = m$, $y = M$ and $\nu = \frac{M-u}{M-m} \in [0,1]$ with $u \in [m,M]$ then we get

$$
\begin{aligned}
(2.5)\qquad & 2\min\left\{\frac{M-u}{M-m}, \frac{u-m}{M-m}\right\} \\
&\times\left[\frac{m\ln m + M\ln M}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right] \\
&\leq \frac{M-u}{M-m}m\ln m + \frac{u-m}{M-m}M\ln M - u\ln u \\
&\leq 2\max\left\{\frac{M-u}{M-m}, \frac{u-m}{M-m}\right\} \\
&\times\left[\frac{m\ln m + M\ln M}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right].
\end{aligned}
$$

Since

$$
\min\left\{\frac{M-u}{M-m}, \frac{u-m}{M-m}\right\} = \frac{1}{2} - \left|\frac{u - \frac{m+M}{2}}{M-m}\right|
$$

and

$$
\max\left\{\frac{M-u}{M-m}, \frac{u-m}{M-m}\right\} = \frac{1}{2} + \left|\frac{u - \frac{m+M}{2}}{M-m}\right|,
$$

then from (2.5) we have

$$
\begin{aligned}
(2.6)\qquad & 2\left(\frac{1}{2} - \frac{1}{M-m}\left|u - \frac{m+M}{2}\right|\right)K(m,M) \\
&\leq \frac{M-u}{M-m}m\ln m + \frac{u-m}{M-m}M\ln M - u\ln u \\
&\leq 2\left(\frac{1}{2} + \frac{1}{M-m}\left|u - \frac{m+M}{2}\right|\right)K(m,M)
\end{aligned}
$$

for any $u \in [m,M]$.

Using the continuous functional calculus we have from (2.6) that

$$
\begin{aligned}
(2.7)\qquad & 2\left(\frac{1}{2}I - \frac{1}{M-m}\left|X - \frac{m+M}{2}I\right|\right)K(m,M) \\
&\leq m\ln m\frac{MI - X}{M-m} + M\ln M\frac{X - mI}{M-m} - X\ln X \\
&\leq 2\left(\frac{1}{2}I + \frac{1}{M-m}\left|X - \frac{m+M}{2}I\right|\right)K(m,M)
\end{aligned}
$$

for any selfadjoint operator X with the property that $mI \leq X \leq MI$.

Multiplying both sides of (1.3) by $A^{-1/2}$ we get

$$mI \leq A^{-1/2}BA^{-1/2} \leq MI$$

and by replacing X by $A^{-1/2}BA^{-1/2}$ in (2.7) we obtain

$$\begin{aligned} & 2\left(\frac{1}{2}I - \frac{1}{M-m}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}I\right|\right)K(m,M) \\ & \leq m\ln m\frac{MI - A^{-1/2}BA^{-1/2}}{M-m} + M\ln M\frac{A^{-1/2}BA^{-1/2} - mI}{M-m} \\ & - A^{-1/2}BA^{-1/2}\ln(A^{-1/2}BA^{-1/2}) \\ & \leq 2\left(\frac{1}{2}I + \frac{1}{M-m}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}I\right|\right)K(m,M). \end{aligned} \tag{2.8}$$

Multiplying both sides of (2.8) by $A^{1/2}$ we get the desired result (2.1). □

Remark 1. *If A and B commute, then*

$$A^{1/2}\left|A^{-1/2}\left(B - \frac{m+M}{2}A\right)A^{-1/2}\right|A^{1/2} = \left|B - \frac{m+M}{2}A\right|,$$

$$S(B|A) = B(\ln A - \ln B)$$

and by (2.1) we have

$$\begin{aligned} & (0 \leq)\, 2\left(\frac{1}{2}A - \frac{1}{M-m}\left|B - \frac{m+M}{2}A\right|\right)K(m,M) \\ & \leq \frac{m\ln m}{M-m}(MA - B) + \frac{M\ln M}{M-m}(B - mA) + B(\ln A - \ln B) \\ & \leq 2\left(\frac{1}{2}A + \frac{1}{M-m}\left|B - \frac{m+M}{2}A\right|\right)K(m,M). \end{aligned} \tag{2.9}$$

The above result can be applied for the operator entropy

$$\eta(C) = -C\ln C = S(C|I)$$

as follows:

Corollary 2. *Assume that $pI \leq C \leq PI$ for some p, P with $0 < p < P$. Then we have that*

$$\begin{aligned} & (0 \leq)\, 2\left(\frac{1}{2}I - \frac{1}{P-p}\left|C - \frac{p+P}{2}I\right|\right)K(p,P) \\ & \leq \frac{p\ln p}{P-p}(PI - C) + \frac{P\ln P}{P-p}(C - pI) + \eta(C) \\ & \leq 2\left(\frac{1}{2}I + \frac{1}{P-p}\left|C - \frac{p+P}{2}I\right|\right)K(p,P). \end{aligned} \tag{2.10}$$

3. An Upper Bound in Terms of Logarithm

We have the following inequality of interest for convex functions, see for instance [2]:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I, $a, b \in \mathring{I}$, the interior of I and $\nu \in [0,1]$. Then*

$$0 \leq (1-\nu) f(a) + \nu f(b) - f((1-\nu) a + \nu b) \leq \nu(1-\nu)(b-a)\left[f'_-(b) - f'_+(a)\right]. \tag{3.1}$$

In particular, we have

$$0 \leq \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \leq \frac{1}{4}(b-a)\left[f'_-(b) - f'_+(a)\right]. \tag{3.2}$$

The constant $\frac{1}{4}$ is best possible in both inequalities from (3.2).

We can state the following result:

Theorem 3. *Let A, B be two positive invertible operators such that the condition (1.3) is valid, then we have*

$$\begin{aligned} &(0 \leq) \frac{m \ln m}{M-m}(MA - B) + \frac{M \ln M}{M-m}(B - mA) + S(B|A) \\ &\leq \frac{\ln M - \ln m}{M-m}(B - mA) A^{-1}(MA - B) \\ &\leq \frac{1}{4}(M-m)(\ln M - \ln m) A. \end{aligned} \tag{3.3}$$

Proof. If we consider the convex function $f(t) = t \ln t$, $t > 0$, then $f'(t) = \ln t + 1$ and by (3.1) we have

$$\begin{aligned} 0 &\leq (1-\nu) a \ln a + \nu b \ln b - ((1-\nu) a + \nu b) \ln((1-\nu) a + \nu b) \\ &\leq \nu(1-\nu)(b-a)(\ln b - \ln a) \end{aligned} \tag{3.4}$$

for any $a, b > 0$ and and $\nu \in [0,1]$.

On applying the inequality (3.4) on the interval $[m, M]$ and for $\nu = \frac{x-m}{M-m} \in [0,1]$ with $x \in [m, M]$ then we get

$$\begin{aligned} 0 &\leq m \ln m \frac{M-x}{M-m} + M \ln M \frac{x-m}{M-m} - x \ln x \\ &\leq \frac{(x-m)(M-x)}{M-m}(\ln M - \ln m) \leq \frac{1}{4}(M-m)(\ln M - \ln m). \end{aligned} \tag{3.5}$$

Using the continuous functional calculus we have from (3.5) that

$$\begin{aligned} 0 &\leq m \ln m \frac{MI - X}{M-m} + M \ln M \frac{X - mI}{M-m} - X \ln X \\ &\leq (\ln M - \ln m) \frac{(X - mI)(M - XI)}{M-m} \leq \frac{1}{4}(M-m)(\ln M - \ln m) I \end{aligned} \tag{3.6}$$

for any selfadjoint operator X with the property that $mI \leq X \leq MI$.

By replacing X by $A^{-1/2}BA^{-1/2}$ in (2.7) we get

$$0 \leq m \ln m \frac{MI - A^{-1/2}BA^{-1/2}}{M-m} + M \ln M \frac{A^{-1/2}BA^{-1/2} - mI}{M-m} - A^{-1/2}BA^{-1/2} \ln (A^{-1/2}BA^{-1/2}) \leq (\ln M - \ln m) \frac{\left(A^{-1/2}BA^{-1/2} - mI\right)\left(MI - A^{-1/2}BA^{-1/2}\right)}{M-m} \leq \frac{1}{4}(M-m)(\ln M - \ln m) I. \tag{3.7}$$

Multiplying both sides of (3.7) by $A^{1/2}$ we get the desired result (3.3). □

Corollary 3. *Assume that $pI \leq C \leq PI$ for some p, P with $0 < p < P$. Then we have that*

$$(0 \leq) \frac{p \ln p}{P-p}(PI - C) + \frac{P \ln P}{P-p}(C - pI) + \eta(C) \leq \frac{\ln P - \ln p}{P-p}(C-pI)(PI-C) \leq \frac{1}{4}(P-p)(\ln P - \ln p). \tag{3.8}$$

4. Further Lower and Upper Bounds

We have the following result, see for instance [3]:

Lemma 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$, the interior of I. If there exists the constants d, D such that*

$$d \leq f''(t) \leq D \text{ for any } t \in \mathring{I}, \tag{4.1}$$

then

$$\frac{1}{2}\nu(1-\nu) d (b-a)^2 \leq (1-\nu) f(a) + \nu f(b) - f((1-\nu)a + \nu b) \leq \frac{1}{2}\nu(1-\nu) D (b-a)^2 \tag{4.2}$$

for any a, $b \in \mathring{I}$ and $\nu \in [0,1]$.

In particular, we have

$$\frac{1}{8}(b-a)^2 d \leq \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}(b-a)^2 D, \tag{4.3}$$

for any a, $b \in \mathring{I}$.

The constant $\frac{1}{8}$ is best possible in both inequalities in (4.3).

If $D > 0$, the second inequality in (4.2) is better than the corresponding inequality obtained by Furuichi and Minculete in [6] by applying Lagrange's theorem two times. They had instead of $\frac{1}{2}$ the constant 1. Our method also allowed to obtain, for $d > 0$, a lower bound that can not be established by Lagrange's theorem method employed in [6].

We can state the following result:

Theorem 4. *Let A, B be two positive invertible operators such that the condition (1.3) is valid, then we have*

$$
\begin{aligned}
(4.4)\qquad & (0 \leq) \frac{1}{2M}(B - mA)A^{-1}(MA - B) \\
& \leq \frac{m\ln m}{M-m}(MA - B) + \frac{M\ln M}{M-m}(B - mA) + S(B|A) \\
& \leq \frac{1}{2m}(B - mA)A^{-1}(MA - B).
\end{aligned}
$$

Proof. If we consider the convex function $f(t) = t\ln t$, $t > 0$, then $f''(t) = \frac{1}{t}$ and by (3.1) we have

$$
\begin{aligned}
(4.5)\qquad & \frac{1}{2}\nu(1-\nu)\frac{1}{\max\{a,b\}}(b-a)^2 \\
& \leq (1-\nu)a\ln a + \nu b\ln b - ((1-\nu)a + \nu b)\ln((1-\nu)a + \nu b) \\
& \leq \frac{1}{2}\nu(1-\nu)\frac{1}{\min\{a,b\}}(b-a)^2
\end{aligned}
$$

for any a, $b > 0$ and $\nu \in [0,1]$.

On applying the inequality (4.5) on the interval $[m, M]$ and for $\nu = \frac{x-m}{M-m} \in [0,1]$ with $x \in [m, M]$ then we get

$$
\begin{aligned}
(4.6)\qquad \frac{1}{2M}(x-m)(M-x) & \leq \frac{M-x}{M-m}m\ln m + \frac{x-m}{M-m}M\ln M - x\ln x \\
& \leq \frac{1}{2m}(x-m)(M-x).
\end{aligned}
$$

Using the continuous functional calculus we have from (4.6) that

$$
\begin{aligned}
(4.7)\quad \frac{1}{2M}(X-mI)(M-XI) & \leq \frac{MI-X}{M-m}m\ln m + \frac{X-mI}{M-m}M\ln M - X\ln X \\
& \leq \frac{1}{2m}(X-mI)(M-XI)
\end{aligned}
$$

for any selfadjoint operator X with the property that $mI \leq X \leq MI$.

Now, on using a similar argument to the one in the proof of Theorem 3 we deduce the desired result (4.4). □

Finally, we have

Corollary 4. *Assume that $pI \leq C \leq PI$ for some p, P with $0 < p < P$. Then we have the inequalities*

$$
\begin{aligned}
(4.8)\quad (0 \leq) \frac{1}{2P}(C-pI)(PI-C) & \leq \frac{p\ln p}{P-p}(PI-C) + \frac{P\ln P}{P-p}(C-pI) + \eta(C) \\
& \leq \frac{1}{2p}(C-pI)(PI-C).
\end{aligned}
$$

5. Applications for Trace Inequalities

If $\{e_i\}_{i\in I}$ is an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* provided

$$
(5.1)\qquad \|A\|_1 := \sum_{i\in I}\langle |A| e_i, e_i\rangle < \infty.
$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following properties are also well known:

(i) We have

$$\|A\|_1 = \|A^*\|_1$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ is an *operator ideal* in $\mathcal{B}(H)$, i.e.

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) $(\mathcal{B}_1(H), \|\cdot\|_1)$ is a *Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i \rangle, \tag{5.2}$$

where $\{e_i\}_{i\in I}$ is an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.1) converges absolutely and it is independent from the choice of basis.

The following results collects some properties of the trace:

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)};$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then AT, $TA \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{5.3}$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\| = 1$;

(iv) $\mathcal{B}_{fin}(H)$, the space of operators of *finite rank*, is a dense subspace of $\mathcal{B}_1(H)$.

We recall that *Specht's ratio* is defined by [15]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{5.4}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \; h > 0. \tag{5.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

In the recent paper [4] we have showed amongst other that

$$(0 \le) S(A|B) - \frac{\ln m}{M-m}(MA-B) - \frac{\ln M}{M-m}(B-mA) \le \ln S\left(\frac{M}{m}\right) A, \tag{5.6}$$

$$\begin{aligned} &(0 \le) S(A|B) - \frac{\ln m}{M-m}(MA-B) - \frac{\ln M}{M-m}(B-mA) \\ &\le \frac{4}{(M-m)^2}\left(K\left(\frac{M}{m}\right) - 1\right)(B-mA)A^{-1}(MA-B) \end{aligned} \tag{5.7}$$

and

$$\begin{aligned}(5.8)\qquad &\frac{1}{2M^2}(B-mA)A^{-1}(MA-B)\\ &\le S(A|B)-\frac{\ln m}{M-m}(MA-B)-\frac{\ln M}{M-m}(B-mA)\\ &\le \frac{1}{2m^2}(B-mA)A^{-1}(MA-B)\end{aligned}$$

for positive invertible operators A and B that satisfy the condition (1.3).

Observe that, if $A,\ B\in\mathcal{B}_1(H)$ with $\operatorname{tr}(A)=\operatorname{tr}(B)=1$ and satisfy (1.3), then we must assume $m\le 1\le M$ and by trace properties we have

$$\begin{aligned}\operatorname{tr}\left[(B-mA)A^{-1}(MA-B)\right]&=\operatorname{tr}\left[(m+M)B-mMA-BA^{-1}B\right]\\ &=m+M-mM-\operatorname{tr}\left(A^{-1}B^2\right)\\ &=(M-1)(1-m)-\chi^2(B,A),\end{aligned}$$

where $\chi^2(B,A)=:\operatorname{tr}\left(A^{-1}B^2\right)-1\ge 0$.

We also have

$$\frac{\ln m}{M-m}(M-1)+\frac{\ln M}{M-m}(1-m)=\ln\left(m^{\frac{M-1}{M-m}}M^{\frac{1-m}{M-m}}\right).$$

We can state the following result:

Proposition 1. *Let $A,\ B\in\mathcal{B}_1(H)$ with $\operatorname{tr}(A)=\operatorname{tr}(B)=1$ that satisfy (1.3) for some $m,\ M$ with $0<m<1<M$. Then we have the inequalities*

$$(5.9)\qquad (0\le)\operatorname{tr}S(A|B)-\ln\left(m^{\frac{M-1}{M-m}}M^{\frac{1-m}{M-m}}\right)\le \ln S\left(\frac{M}{m}\right),$$

$$\begin{aligned}(5.10)\qquad &(0\le)\operatorname{tr}S(A|B)-\ln\left(m^{\frac{M-1}{M-m}}M^{\frac{1-m}{M-m}}\right)\\ &\le\frac{4}{(M-m)^2}\left(K\left(\frac{M}{m}\right)-1\right)\left[(M-1)(1-m)-\chi^2(B,A)\right]\end{aligned}$$

and

$$\begin{aligned}(5.11)\qquad \frac{1}{2M^2}\left[(M-1)(1-m)-\chi^2(B,A)\right]&\le \operatorname{tr}S(A|B)-\ln\left(m^{\frac{M-1}{M-m}}M^{\frac{1-m}{M-m}}\right)\\ &\le\frac{1}{2m^2}\left[(M-1)(1-m)-\chi^2(B,A)\right].\end{aligned}$$

Observe that

$$\frac{m\ln m}{M-m}(M-1)+\frac{M\ln M}{M-m}(1-m)=\ln\left(m^{\frac{m(M-1)}{M-m}}M^{\frac{M(1-m)}{M-m}}\right),$$

then by taking the trace in the inequalities (3.3) and (4.4) we can state the following result as well:

Proposition 2. *Let $A,\ B\in\mathcal{B}_1(H)$ with $\operatorname{tr}(A)=\operatorname{tr}(B)=1$ that satisfy (1.3) for some $m,\ M$ with $0<m<1<M$. Then we have the inequalities*

$$\begin{aligned}(5.12)\qquad &(0\le)\ln\left(m^{\frac{m(M-1)}{M-m}}M^{\frac{M(1-m)}{M-m}}\right)+\operatorname{tr}S(B|A)\\ &\le\frac{\ln M-\ln m}{M-m}\left[(M-1)(1-m)-\chi^2(B,A)\right]\end{aligned}$$

and

$$\frac{1}{2M}\left[(M-1)(1-m)-\chi^2(B,A)\right] \le \ln\left(m^{\frac{m(M-1)}{M-m}} M^{\frac{M(1-m)}{M-m}}\right) + \operatorname{tr} S(B|A) \le \frac{1}{2m}\left[(M-1)(1-m)-\chi^2(B,A)\right]. \tag{5.13}$$

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