

ON THE HERMITE-HADAMARD'S TYPE INEQUALITIES FOR  
CO-ORDINATED CONVEX FUNCTIONS

ABDULLAH AKKURT, MEHMET ZEKI SARIKAYA, HÜSEYİN BUDAK,  
AND HÜSEYİN YILDIRIM

ABSTRACT. In this paper, we obtain two identities for function of two variables and apply them to give new Hermite-Hadamard type integral inequalities for double integrals involving functions whose derivatives are bounded or co-ordinates convex function on  $\Delta := [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $a < b, c < d$ .

1. INTRODUCTION

The convexity property of a given function plays an important role in obtaining integral inequalities. Proving inequalities for convex functions has a long and rich history in mathematics. We refer the reader to a monograph written by Dragomir and Pearce [10]. Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex mapping defined on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ . The following double inequality:

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

is known in the literature as Hermite-Hadamard inequality for convex mappings. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping  $f$ . Both inequalities hold in the reversed direction if  $f$  is concave.

It is well known that the Hermite-Hadamard's inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, (see, [1]-[9]) and the references therein.

Let us now consider a bidimensional interval  $\Delta =: [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $a < b$  and  $c < d$ . A mapping  $f : \Delta \rightarrow \mathbb{R}$  is said to be convex on  $\Delta$  if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x, y) + (1-t)f(z, w)$$

holds, for all  $(x, y), (z, w) \in \Delta$  and  $t \in [0, 1]$ . A function  $f : \Delta \rightarrow \mathbb{R}$  is said to be on the co-ordinates on  $\Delta$  if the partial mappings  $f_y : [a, b] \rightarrow \mathbb{R}, f_y(u) = f(u, y)$  and  $f_x : [c, d] \rightarrow \mathbb{R}, f_x(v) = f(x, v)$  are convex where defined for all  $x \in [a, b]$  and  $y \in [c, d]$  (see [4]).

A formal definition for co-ordinated convex function may be stated as follows:

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*Key words and phrases.* Hadamard's type Inequalities, co-ordinated convex functions, Hölder's inequality.

**2010 Mathematics Subject Classification.** 26A33, 26D15, 41A55.

**Definition 1.** A function  $f : \Delta \rightarrow \mathbb{R}$  will be called co-ordinated convex on  $\Delta$ , for all  $t, s \in [0, 1]$  and  $(x, y), (u, w) \in \Delta$ , if the following inequality holds:

$$f(tx + (1 - t)y, su + (1 - s)w)$$

$$(1.2) \leq tsf(x, u) + s(1 - t)f(y, u) + t(1 - s)f(x, w) + (1 - t)(1 - s)f(y, w).$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex function which is not convex, (see, [10]). For several recent results concerning Hermite-Hadamard's inequality for some convex function on the co-ordinates on a rectangle from the plane  $\mathbb{R}^2$ , we refer the reader to ([10]-[18]).

In [10], Dragomir establish the following inequality of Hermite-Hadamard type for co-ordinated convex mapping on a rectangle from the plane  $\mathbb{R}^2$ . Later, in [11], Sarikaya and Yaldiz gave the another proof of a special version of the below theorem by using the definition of the co-ordinated convex function.

**Theorem 1.** Suppose that  $f : \Delta \rightarrow \mathbb{R}$  is co-ordinated convex on  $\Delta$ . Then one has the inequalities:

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\ & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ (1.3) \quad & \leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \\ & \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\ & \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \end{aligned}$$

The above inequalities are sharp.

Our goal in this paper is to state and prove the Hermite-Hadamard type inequality for co-ordinated convex mapping on a rectangle from the plane  $\mathbb{R}^2$ . In order to achieve our goal, we first give two important identities and then by using these identities we prove some integral inequalities. We have obtained some results which are a simpler proof of the results in [19].

## 2. MAIN RESULT

To establish our main results, we need the following first identity:

**Lemma 1.** Let  $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a partial differentiable mapping on  $\Delta := [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $a < b$  and  $c < d$  and  $f_{\sigma\tau} \in L(\Delta)$ . Then the following equality holds:

$$(2.1) \quad \begin{aligned} & \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ & - \frac{1}{2} \left[ \frac{1}{b-a} \int_c^d [f(x, c) + f(x, d)] dx + \frac{1}{d-c} \int_a^b [f(a, y) + f(b, y)] dy \right] \\ & = \frac{1}{4(b-a)(d-c)} \int_a^b \int_c^d I(x, y) dy dx \end{aligned}$$

where

$$\begin{aligned} I(x, y) &= \int_a^x \int_c^y f_{\sigma\tau}(\sigma, \tau) d\tau d\sigma + \int_a^x \int_d^y f_{\sigma\tau}(\sigma, \tau) d\tau d\sigma \\ &+ \int_b^x \int_c^y f_{\sigma\tau}(\sigma, \tau) d\tau d\sigma + \int_b^x \int_d^y f_{\sigma\tau}(\sigma, \tau) d\tau d\sigma. \end{aligned}$$

*Proof.* For any  $x, t \in [a, b]$  and  $s, y \in [c, d]$ ,  $x \neq t$ ,  $s \neq y$ , we have

$$(2.2) \quad \begin{aligned} \int_t^x \int_s^y f_{\sigma\tau}(\sigma, \tau) d\tau d\sigma &= \int_t^x [f_{\sigma}(\sigma, y) - f_{\sigma}(\sigma, s)] d\sigma \\ &= [f(\sigma, y) - f(\sigma, s)]_t^x \\ &= f(x, y) - f(x, s) - f(t, y) + f(t, s). \end{aligned}$$

Choose  $t = a$ ,  $s = c$ ;  $t = a$ ,  $s = d$ ;  $t = b$ ,  $s = c$ ;  $t = b$ ,  $s = d$  in (2.2), respectively, we get

$$\begin{aligned} I_1 &= \int_a^x \int_c^y f_{\sigma\tau}(\sigma, \tau) d\tau d\sigma = f(x, y) - f(x, c) - f(a, y) + f(a, c), \\ I_2 &= \int_a^x \int_d^y f_{\sigma\tau}(\sigma, \tau) d\tau d\sigma = f(x, y) - f(x, d) - f(a, y) + f(a, d), \\ I_3 &= \int_b^x \int_c^y f_{\sigma\tau}(\sigma, \tau) d\tau d\sigma = f(x, y) - f(x, c) - f(b, y) + f(b, c), \end{aligned}$$

and

$$I_4 = \int_b^x \int_d^y f_{\sigma\tau}(\sigma, \tau) d\tau d\sigma = f(x, y) - f(x, d) - f(b, y) + f(b, d).$$

Adding these four integrals side by side, we obtain

$$\begin{aligned}
 I(x, y) &= I_1 + I_2 + I_3 + I_4 \\
 (2.3) \quad &= 4f(x, y) - 2[f(x, c) + f(x, d)] - 2[f(a, y) + f(b, y)] \\
 &\quad + f(a, c) + f(a, d) + f(b, c) + f(b, d).
 \end{aligned}$$

By integrating (2.3) with respect to  $(x, y)$  on  $[a, b] \times [c, d]$ , we have

$$\begin{aligned}
 (2.4) \quad &\frac{1}{4} \int_a^b \int_c^d I(x, y) dy dx \\
 &= \int_a^b \int_c^d f(x, y) dy dx - \frac{d-c}{2} \int_a^b [f(x, c) + f(x, d)] dx \\
 &\quad - \frac{b-a}{2} \int_c^d [f(a, y) + f(b, y)] dy \\
 &\quad + \frac{(b-a)(d-c)}{4} \{f(a, c) + f(a, d) + f(b, c) + f(b, d)\}.
 \end{aligned}$$

Multiplying both sides of (2.4) by  $\frac{1}{(b-a)(d-c)}$ , we have

$$\begin{aligned}
 (2.5) \quad &\frac{1}{4(b-a)(d-c)} \int_a^b \int_c^d I(x, y) dy dx \\
 &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx - \frac{1}{2(b-a)} \int_a^b [f(x, c) + f(x, d)] dx \\
 &\quad - \frac{1}{2(d-c)} \int_c^d [f(a, y) + f(b, y)] dy + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}
 \end{aligned}$$

which completes the proof.  $\square$

**Theorem 2.** Let  $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a partial differentiable mapping on  $\Delta := [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $a < b$  and  $c < d$  and  $f_{\sigma\tau} \in L(\Delta)$ . If  $f_{\sigma\tau} \in L_\infty(\Delta)$ , i.e

$$\|f_{\sigma\tau}\|_\infty = \sup_{(\sigma, \tau) \in (a, b) \times (c, d)} \left| \frac{\partial^2 f(\sigma, \tau)}{\partial \sigma \partial \tau} \right| < \infty, \text{ then one has the inequality:}$$

$$\begin{aligned}
 (2.6) \quad &\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx - \frac{1}{2(b-a)} \int_c^d [f(x, c) + f(x, d)] dx \right. \\
 &\quad \left. - \frac{1}{2(d-c)} \int_a^b [f(a, y) + f(b, y)] dy + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right| \\
 &\leq \frac{\|f_{\sigma\tau}\|_\infty}{4} (b-a)(d-c).
 \end{aligned}$$

*Proof.* From Lemma 1, taking the modulus, it follows that

$$\begin{aligned} & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx - \frac{1}{2(b-a)} \int_c^d [f(x,c) + f(x,d)] dx \right. \\ & \left. - \frac{1}{2(d-c)} \int_a^b [f(a,y) + f(b,y)] dy + \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \right| \\ & \leq \frac{1}{4(b-a)(d-c)} \int_a^b \int_c^d \left\{ \int_a^x \int_c^y |f_{\sigma\tau}(\sigma,\tau)| d\tau d\sigma + \int_a^x \int_y^d |f_{\sigma\tau}(\sigma,\tau)| d\tau d\sigma \right. \\ & \left. + \int_x^b \int_c^y |f_{\sigma\tau}(\sigma,\tau)| d\tau d\sigma + \int_x^b \int_y^d |f_{\sigma\tau}(\sigma,\tau)| d\tau d\sigma \right\} dy dx. \end{aligned}$$

Since  $f_{\sigma\tau} \in L_\infty(\Delta)$ , we get

$$\begin{aligned} & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx - \frac{1}{2(b-a)} \int_c^d [f(x,c) + f(x,d)] dx \right. \\ & \left. - \frac{1}{2(d-c)} \int_a^b [f(a,y) + f(b,y)] dy + \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \right| \\ (2.7) \quad & \leq \frac{\|f_{\sigma\tau}\|_\infty}{4(b-a)(d-c)} \int_a^b \int_c^d \left\{ \int_a^x \int_c^y d\tau d\sigma \right\} dy dx \\ & = \frac{\|f_{\sigma\tau}\|_\infty}{4} (b-a)(d-c). \end{aligned}$$

This completes the proof.  $\square$

**Theorem 3.** Let  $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a partial differentiable mapping on  $\Delta := [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $a < b$  and  $c < d$  and  $f_{\sigma\tau} \in L(\Delta)$ . If  $|f_{\sigma\tau}|$  is a convex function on the co-ordinates on  $\Delta$ , then the following equality holds:

$$\begin{aligned} & \left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx \right. \\ (2.8) \quad & \left. - \frac{1}{2} \left[ \frac{1}{b-a} \int_c^d [f(x,c) + f(x,d)] dx + \frac{1}{d-c} \int_a^b [f(a,y) + f(b,y)] dy \right] \right| \\ & \leq \frac{(b-a)(d-c)}{4} \left\{ \frac{|f_{\sigma\tau}(a,c)| + |f_{\sigma\tau}(b,d)| + |f_{\sigma\tau}(a,d)| + |f_{\sigma\tau}(b,c)|}{4} \right\}. \end{aligned}$$

*Proof.* Since  $|f_{\sigma\tau}(\sigma, \tau)|$  is co-ordinates on  $\Delta$ , we know that  $x \in [a, b]$ ,  $y \in [c, d]$

$$\begin{aligned}
 |f_{\sigma\tau}(\sigma, \tau)| &= \left| f_{\sigma\tau} \left( \frac{b-\sigma}{b-a}a + \frac{\sigma-a}{b-a}b, \frac{d-\tau}{d-c}c + \frac{\tau-c}{d-c}d \right) \right| \\
 (2.9) \quad &\leq \frac{b-\sigma}{b-a} \frac{d-\tau}{d-c} |f_{\sigma\tau}(a, c)| + \frac{b-\sigma}{b-a} \frac{\tau-c}{d-c} |f_{\sigma\tau}(a, d)| \\
 &\quad + \frac{\sigma-a}{b-a} \frac{d-\tau}{d-c} |f_{\sigma\tau}(b, c)| + \frac{\sigma-a}{b-a} \frac{\tau-c}{d-c} |f_{\sigma\tau}(b, d)|.
 \end{aligned}$$

From Lemma 1, using co-ordinated convexity of  $|f_{\sigma\tau}|$ , we have

$$\begin{aligned}
 &\left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \right. \\
 &\quad \left. - \frac{1}{2} \left[ \frac{1}{b-a} \int_c^d [f(x, c) + f(x, d)] dx + \frac{1}{d-c} \int_a^b [f(a, y) + f(b, y)] dy \right] \right| \\
 &\leq \frac{1}{4(b-a)(d-c)} \int_a^b \int_c^d \left\{ \int_a^x \int_c^y |f_{\sigma\tau}(\sigma, \tau)| d\tau d\sigma + \int_a^x \int_y^d |f_{\sigma\tau}(\sigma, \tau)| d\tau d\sigma \right. \\
 &\quad \left. + \int_x^b \int_c^y |f_{\sigma\tau}(\sigma, \tau)| d\tau d\sigma + \int_x^b \int_y^d |f_{\sigma\tau}(\sigma, \tau)| d\tau d\sigma \right\} dy dx \\
 &\leq \frac{1}{4(b-a)(d-c)} \int_a^b \int_c^d \left\{ \int_a^x \int_c^y \left[ \frac{b-\sigma}{b-a} \frac{d-\tau}{d-c} |f_{\sigma\tau}(a, c)| + \frac{b-\sigma}{b-a} \frac{\tau-c}{d-c} |f_{\sigma\tau}(a, d)| \right. \right. \\
 &\quad \left. \left. + \frac{\sigma-a}{b-a} \frac{d-\tau}{d-c} |f_{\sigma\tau}(b, c)| + \frac{\sigma-a}{b-a} \frac{\tau-c}{d-c} |f_{\sigma\tau}(b, d)| \right] d\tau d\sigma \right. \\
 &\quad \left. + \int_a^x \int_y^d \left[ \frac{b-\sigma}{b-a} \frac{d-\tau}{d-c} |f_{\sigma\tau}(a, c)| + \frac{b-\sigma}{b-a} \frac{\tau-c}{d-c} |f_{\sigma\tau}(a, d)| \right. \right. \\
 &\quad \left. \left. + \frac{\sigma-a}{b-a} \frac{d-\tau}{d-c} |f_{\sigma\tau}(b, c)| + \frac{\sigma-a}{b-a} \frac{\tau-c}{d-c} |f_{\sigma\tau}(b, d)| \right] d\tau d\sigma \right\} dy dx
 \end{aligned}$$

$$\begin{aligned}
& + \int_x^b \int_c^y \left[ \frac{b-\sigma}{b-a} \frac{d-\tau}{d-c} |f_{\sigma\tau}(a,c)| + \frac{b-\sigma}{b-a} \frac{\tau-c}{d-c} |f_{\sigma\tau}(a,d)| \right. \\
& \left. + \frac{\sigma-a}{b-a} \frac{d-\tau}{d-c} |f_{\sigma\tau}(b,c)| + \frac{\sigma-a}{b-a} \frac{\tau-c}{d-c} |f_{\sigma\tau}(b,d)| \right] d\tau d\sigma \\
(2.10) \quad & + \int_x^b \int_y^d \left[ \frac{b-\sigma}{b-a} \frac{d-\tau}{d-c} |f_{\sigma\tau}(a,c)| + \frac{b-\sigma}{b-a} \frac{\tau-c}{d-c} |f_{\sigma\tau}(a,d)| \right. \\
& \left. + \frac{\sigma-a}{b-a} \frac{d-\tau}{d-c} |f_{\sigma\tau}(b,c)| + \frac{\sigma-a}{b-a} \frac{\tau-c}{d-c} |f_{\sigma\tau}(b,d)| \right] d\tau d\sigma \Big\} dy dx \\
& = A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

With a simple calculation, we have

$$\begin{aligned}
A_1 & = \frac{1}{4(b-a)^2(d-c)^2} \int_a^b \int_c^d \left\{ \int_a^x \int_c^y [(b-\sigma)(d-\tau) |f_{\sigma\tau}(a,c)| + (b-\sigma)(\tau-c) |f_{\sigma\tau}(a,d)| \right. \\
& \left. + (\sigma-a)(d-\tau) |f_{\sigma\tau}(b,c)| + (\sigma-a)(\tau-c) |f_{\sigma\tau}(b,d)|] d\tau d\sigma \right\} dy dx \\
& = \frac{1}{4(b-a)^2(d-c)^2} \int_a^b \int_c^d \left\{ \left[ \frac{(b-a)^2 - (b-x)^2}{2} \frac{(d-c)^2 - (d-y)^2}{2} |f_{\sigma\tau}(a,c)| \right. \right. \\
& \left. \left. \frac{(b-a)^2 - (b-x)^2}{2} \frac{(y-c)^2}{2} |f_{\sigma\tau}(a,d)| \right. \right. \\
& \left. \left. \frac{(x-a)^2}{2} \frac{(d-c)^2 - (d-y)^2}{2} |f_{\sigma\tau}(b,c)| + \frac{(x-a)^2}{2} \frac{(y-c)^2}{2} |f_{\sigma\tau}(b,d)| \right] \right\} dy dx \\
& = \frac{1}{4(b-a)^2(d-c)^2} \left\{ \left[ \frac{(b-a)^3}{3} \frac{(d-c)^3}{3} |f_{\sigma\tau}(a,c)| + \frac{(b-a)^3}{3} \frac{(d-c)^3}{6} |f_{\sigma\tau}(a,d)| \right. \right. \\
& \left. \left. + \frac{(b-a)^3}{6} \frac{(d-c)^3}{3} |f_{\sigma\tau}(b,c)| + \frac{(b-a)^3}{6} \frac{(d-c)^3}{6} |f_{\sigma\tau}(b,d)| \right] \right\} \\
& = \frac{(b-a)(d-c)}{4} \left\{ \frac{|f_{\sigma\tau}(a,c)|}{9} + \frac{|f_{\sigma\tau}(a,d)|}{18} + \frac{|f_{\sigma\tau}(b,c)|}{18} + \frac{|f_{\sigma\tau}(b,d)|}{36} \right\}.
\end{aligned}$$

Similarly, we also have the following equalities

$$\begin{aligned}
 A_2 &= \frac{1}{4(b-a)^2(d-c)^2} \int_a^b \int_c^d \left\{ \int_a^x \int_y^d [(b-\sigma)(d-\tau)|f_{\sigma\tau}(a,c)| + (b-\sigma)(\tau-c)|f_{\sigma\tau}(a,d)| \right. \\
 &\quad \left. + (\sigma-a)(d-\tau)|f_{\sigma\tau}(b,c)| + (\sigma-a)(\tau-c)|f_{\sigma\tau}(b,d)|] d\tau d\sigma \right\} dy dx \\
 &= \frac{1}{4(b-a)^2(d-c)^2} \int_a^b \int_c^d \left\{ \left[ \frac{(b-a)^2 - (b-x)^2}{2} \frac{(d-y)^2}{2} |f_{\sigma\tau}(a,c)| \right. \right. \\
 &\quad \left. \left. + \frac{(b-a)^2 - (b-x)^2}{2} \frac{(d-c)^2 - (y-c)^2}{2} |f_{\sigma\tau}(a,d)| + \frac{(x-a)^2}{2} \frac{(d-y)^2}{2} |f_{\sigma\tau}(b,c)| \right. \right. \\
 &\quad \left. \left. + \frac{(x-a)^2}{2} \frac{(d-c)^2 - (y-c)^2}{2} |f_{\sigma\tau}(b,d)| \right] \right\} dy dx \\
 &= \frac{1}{4(b-a)^2(d-c)^2} \left\{ \left[ \frac{(b-a)^3}{3} \frac{(d-c)^3}{6} |f_{\sigma\tau}(a,c)| + \frac{(b-a)^3}{3} \frac{(d-c)^3}{3} |f_{\sigma\tau}(a,d)| \right. \right. \\
 &\quad \left. \left. + \frac{(b-a)^3}{6} \frac{(d-c)^3}{6} |f_{\sigma\tau}(b,c)| + \frac{(b-a)^3}{6} \frac{(d-c)^3}{3} |f_{\sigma\tau}(b,d)| \right] \right\} \\
 &= \frac{(b-a)(d-c)}{4} \left\{ + \frac{|f_{\sigma\tau}(a,c)|}{18} + \frac{|f_{\sigma\tau}(a,d)|}{9} + \frac{|f_{\sigma\tau}(b,c)|}{36} + \frac{|f_{\sigma\tau}(b,d)|}{18} \right\}.
 \end{aligned}$$

and

$$\begin{aligned}
 A_3 &= \frac{1}{4(b-a)^2(d-c)^2} \int_a^b \int_c^d \left\{ \int_x^b \int_c^y [(b-\sigma)(d-\tau)|f_{\sigma\tau}(a,c)| + (b-\sigma)(\tau-c)|f_{\sigma\tau}(a,d)| \right. \\
 &\quad \left. + (\sigma-a)(d-\tau)|f_{\sigma\tau}(b,c)| + (\sigma-a)(\tau-c)|f_{\sigma\tau}(b,d)|] d\tau d\sigma \right\} dy dx \\
 &= \frac{1}{4(b-a)^2(d-c)^2} \int_a^b \int_c^d \left\{ \left[ \frac{(b-x)^2}{2} \frac{(d-c)^2 - (d-y)^2}{2} |f_{\sigma\tau}(a,c)| \right. \right. \\
 &\quad \left. \left. + \frac{(b-x)^2}{2} \frac{(y-c)^2}{2} |f_{\sigma\tau}(a,d)| + \frac{(b-a)^2 - (x-a)^2}{2} \frac{(d-c)^2 - (d-y)^2}{2} |f_{\sigma\tau}(b,c)| \right. \right. \\
 &\quad \left. \left. + \frac{(b-a)^2 - (x-a)^2}{2} \frac{(y-c)^2}{2} |f_{\sigma\tau}(b,d)| \right] \right\} dy dx
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4(b-a)^2(d-c)^2} \left\{ \left[ \frac{(b-a)^3(d-c)^3}{6} |f_{\sigma\tau}(a,c)| + \frac{(b-a)^3(d-c)^3}{6} |f_{\sigma\tau}(a,d)| \right. \right. \\
&\quad \left. \left. + \frac{(b-a)^3(d-c)^3}{3} |f_{\sigma\tau}(b,c)| + \frac{(b-a)^3(d-c)^3}{6} |f_{\sigma\tau}(b,d)| \right] \right\} \\
&= \frac{(b-a)(d-c)}{4} \left\{ \frac{|f_{\sigma\tau}(a,c)|}{18} + \frac{|f_{\sigma\tau}(a,d)|}{36} + \frac{|f_{\sigma\tau}(b,c)|}{9} + \frac{|f_{\sigma\tau}(b,d)|}{18} \right\}.
\end{aligned}$$

and

$$\begin{aligned}
A_4 &= \frac{1}{4(b-a)^2(d-c)^2} \int_a^b \int_c^d \left\{ \int_x^b \int_y^d [(b-\sigma)(d-\tau) |f_{\sigma\tau}(a,c)| + (b-\sigma)(\tau-c) |f_{\sigma\tau}(a,d)| \right. \\
&\quad \left. + (\sigma-a)(d-\tau) |f_{\sigma\tau}(b,c)| + (\sigma-a)(\tau-c) |f_{\sigma\tau}(b,d)|] d\tau d\sigma \right\} dy dx \\
&= \frac{1}{4(b-a)^2(d-c)^2} \int_a^b \int_c^d \left\{ \left[ \frac{(b-x)^2(d-y)^2}{2} |f_{\sigma\tau}(a,c)| \right. \right. \\
&\quad \left. \left. + \frac{(b-x)^2(d-c)^2 - (y-c)^2}{2} |f_{\sigma\tau}(a,d)| + \frac{(b-a)^2 - (x-a)^2(d-y)^2}{2} |f_{\sigma\tau}(b,c)| \right. \right. \\
&\quad \left. \left. + \frac{(b-a)^2 - (x-a)^2(d-c)^2 - (y-c)^2}{2} |f_{\sigma\tau}(b,d)| \right] \right\} dy dx \\
&= \frac{1}{4(b-a)^2(d-c)^2} \left\{ \left[ \frac{(b-a)^3(d-c)^3}{6} |f_{\sigma\tau}(a,c)| + \frac{(b-a)^3(d-c)^3}{6} |f_{\sigma\tau}(a,d)| \right. \right. \\
&\quad \left. \left. + \frac{(b-a)^3(d-c)^3}{3} |f_{\sigma\tau}(b,c)| + \frac{(b-a)^3(d-c)^3}{3} |f_{\sigma\tau}(b,d)| \right] \right\} \\
&= \frac{(b-a)(d-c)}{4} \left\{ \frac{|f_{\sigma\tau}(a,c)|}{36} + \frac{|f_{\sigma\tau}(a,d)|}{18} + \frac{|f_{\sigma\tau}(b,c)|}{18} + \frac{|f_{\sigma\tau}(b,d)|}{9} \right\}.
\end{aligned}$$

Thus, if we put the last four equalities in (2.10), we obtain (2.8). This completes the proof of the theorem.  $\square$

**Lemma 2.** *Let  $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a partial differentiable mapping on  $\Delta := [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $a < b$ ,  $c < d$  and  $f_{\sigma\tau} \in L(\Delta)$ . Then the following equality*

holds:

$$\begin{aligned}
 & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left\{ \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} f_{\sigma\tau}(\sigma, \tau) d\tau d\sigma \right\} ds dt \\
 (2.11) \quad & = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(d-c)} \int_c^d f\left(\frac{a+b}{2}, s\right) ds \\
 & - \frac{1}{(b-a)} \int_a^b f\left(t, \frac{c+d}{2}\right) dt + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) ds dt.
 \end{aligned}$$

*Proof.* Choose  $x = \frac{a+b}{2}$  and  $y = \frac{c+d}{2}$  in (2.2), we have

$$\begin{aligned}
 (2.12) \quad & \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} f_{\sigma\tau}(\sigma, \tau) d\tau d\sigma = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - f\left(\frac{a+b}{2}, s\right) - f\left(t, \frac{c+d}{2}\right) + f(t, s).
 \end{aligned}$$

By integrating (2.12) with respect to  $(s, t)$  on  $[a, b] \times [c, d]$ , we have

$$\begin{aligned}
 & \int_a^b \int_c^d \left\{ \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} f_{\sigma\tau}(\sigma, \tau) d\tau d\sigma \right\} ds dt \\
 (2.13) \quad & = (b-a)(d-c) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - (b-a) \int_c^d f\left(\frac{a+b}{2}, s\right) ds \\
 & - (d-c) \int_a^b f\left(t, \frac{c+d}{2}\right) dt + \int_a^b \int_c^d f(t, s) ds dt.
 \end{aligned}$$

Multiplying both sides of (2.13) by  $\frac{1}{(b-a)(d-c)}$ , we get the desired equality (2.11).  $\square$

**Theorem 4.** Let  $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a partial differentiable mapping on  $\Delta := [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $a < b$ ,  $c < d$  and  $f_{\sigma\tau} \in L(\Delta)$ . If  $f_{\sigma\tau} \in L_\infty(\Delta)$ , then the following equality holds:

$$\begin{aligned}
 (2.14) \quad & \left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(d-c)} \int_c^d f\left(\frac{a+b}{2}, s\right) ds \right. \\
 & \left. - \frac{1}{(b-a)} \int_a^b f\left(t, \frac{c+d}{2}\right) dt + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) ds dt \right| \\
 & \leq \frac{\|f_{\sigma\tau}\|_\infty}{16} (b-a)(d-c).
 \end{aligned}$$

*Proof.* In Lemma 2, taking the modulus, it follows that

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(d-c)} \int_c^d f\left(\frac{a+b}{2}, s\right) ds \right. \\ & \quad \left. - \frac{1}{(b-a)} \int_a^b f\left(t, \frac{c+d}{2}\right) dt + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) ds dt \right| \\ & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left\{ \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} |f_{\sigma\tau}(\sigma, \tau)| d\tau d\sigma \right\} ds dt. \end{aligned}$$

Since  $f_{\sigma\tau} \in L_\infty(\Delta)$ , we get

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(d-c)} \int_c^d f\left(\frac{a+b}{2}, s\right) ds \right. \\ & \quad \left. - \frac{1}{(b-a)} \int_a^b f\left(t, \frac{c+d}{2}\right) dt + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) ds dt \right| \\ (2.15) \quad & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left\{ \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} |f_{\sigma\tau}(\sigma, \tau)| d\tau d\sigma \right\} ds dt \\ & \leq \frac{\|f_{\sigma\tau}\|_\infty}{(b-a)(d-c)} \int_a^b \int_c^d \left\{ \left| \frac{a+b}{2} - t \right| \left| \frac{c+d}{2} - s \right| \right\} ds dt \\ & \leq \frac{\|f_{\sigma\tau}\|_\infty}{16} (b-a)(d-c). \end{aligned}$$

□

**Theorem 5.** Let  $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a partial differentiable mapping on  $\Delta := [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $a < b$ ,  $c < d$  and  $f_{\sigma\tau} \in L(\Delta)$ . If  $|f_{\sigma\tau}|$  is a convex function on the co-ordinates on  $\Delta$ , then the following equality holds:

$$\begin{aligned} (2.16) \quad & \left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(d-c)} \int_c^d f\left(\frac{a+b}{2}, s\right) ds \right. \\ & \quad \left. - \frac{1}{(b-a)} \int_a^b f\left(t, \frac{c+d}{2}\right) dt + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) ds dt \right| \\ & \leq \frac{25}{2^6 3^2} \frac{|f_{\sigma\tau}(a, c)| + |f_{\sigma\tau}(a, d)| + |f_{\sigma\tau}(b, c)| + |f_{\sigma\tau}(b, d)|}{4} (b-a)(d-c) \end{aligned}$$

*Proof.* Since  $|f_{\sigma\tau}(\sigma, \tau)|$  is co-ordinates on  $\Delta$ , we know that  $t \in [a, b]$ ,  $s \in [c, d]$

$$\begin{aligned}
 |f_{\sigma\tau}(\sigma, \tau)| &= \left| f_{\sigma\tau} \left( \frac{b-\sigma}{b-a}a + \frac{\sigma-a}{b-a}b, \frac{d-\tau}{d-c}c + \frac{\tau-c}{d-c}d \right) \right| \\
 (2.17) \quad &\leq \frac{b-\sigma}{b-a} \frac{d-\tau}{d-c} |f_{\sigma\tau}(a, c)| + \frac{b-\sigma}{b-a} \frac{\tau-c}{d-c} |f_{\sigma\tau}(a, d)| \\
 &\quad + \frac{\sigma-a}{b-a} \frac{d-\tau}{d-c} |f_{\sigma\tau}(b, c)| + \frac{\sigma-a}{b-a} \frac{\tau-c}{d-c} |f_{\sigma\tau}(b, d)|.
 \end{aligned}$$

From Lemma 2, using co-ordinated convexity of  $|f_{\sigma\tau}|$ , we have

$$\begin{aligned}
 &\left| f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) - \frac{1}{(d-c)} \int_c^d f \left( \frac{a+b}{2}, s \right) ds \right. \\
 &\quad \left. - \frac{1}{(b-a)} \int_a^b f \left( t, \frac{c+d}{2} \right) dt + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) ds dt \right| \\
 &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left| \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} |f_{\sigma\tau}(\sigma, \tau)| d\tau d\sigma \right| ds dt \\
 (2.18) \quad &\leq \frac{1}{(b-a)^2 (d-c)^2} \left[ \int_a^b \int_c^d |f_{\sigma\tau}(a, c)| \left| \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} (b-\sigma)(d-\tau) d\tau d\sigma \right| ds dt \right. \\
 &\quad \left. + \int_a^b \int_c^d |f_{\sigma\tau}(a, d)| \left| \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} (b-\sigma)(\tau-c) d\tau d\sigma \right| ds dt \right. \\
 &\quad \left. + \int_a^b \int_c^d |f_{\sigma\tau}(b, c)| \left| \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} (\sigma-a)(d-\tau) d\tau d\sigma \right| ds dt \right. \\
 &\quad \left. + \int_a^b \int_c^d |f_{\sigma\tau}(b, d)| \left| \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} (\sigma-a)(\tau-c) d\tau d\sigma \right| ds dt \right] \\
 &= J_1 + J_2 + J_3 + J_4.
 \end{aligned}$$

With a simple calculation, we have

$$\begin{aligned}
J_1 &= \frac{1}{(b-a)^2 (d-c)^2} \int_a^b \int_c^d |f_{\sigma\tau}(a, c)| \left| \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} (b-\sigma)(d-\tau) d\tau d\sigma \right| ds dt \\
&= \frac{|f_{\sigma\tau}(a, c)|}{(b-a)^2 (d-c)^2} \left[ \int_a^b \left| \int_t^{\frac{a+b}{2}} (b-\sigma) d\sigma \right| dt \right] \left[ \int_c^d \left| \int_s^{\frac{c+d}{2}} (d-\tau) d\tau \right| ds \right] \\
&= \frac{|f_{\sigma\tau}(a, c)|}{(b-a)^2 (d-c)^2} \left[ \int_a^{\frac{a+b}{2}} \int_t^{\frac{a+b}{2}} (b-\sigma) d\sigma dt + \int_{\frac{a+b}{2}}^b \int_t^t (b-\sigma) d\sigma dt \right] \\
&\quad \times \left[ \int_c^{\frac{c+d}{2}} \int_s^{\frac{c+d}{2}} (d-\tau) d\tau ds + \int_{\frac{c+d}{2}}^d \int_s^s (d-\tau) d\tau ds \right] \\
&= \frac{|f_{\sigma\tau}(a, c)|}{(b-a)^2 (d-c)^2} \left[ \frac{5}{48} (b-a)^3 \right] \left[ \frac{5}{48} (d-c)^3 \right] \\
&= \frac{25}{2832} (b-a) (d-c) |f_{\sigma\tau}(a, c)|.
\end{aligned}$$

Similarly, we also have the following equalities:

$$\begin{aligned}
J_2 &= \frac{25}{2832} (b-a) (d-c) |f_{\sigma\tau}(a, d)|, \\
J_3 &= \frac{25}{2832} (b-a) (d-c) |f_{\sigma\tau}(b, c)|
\end{aligned}$$

and

$$J_4 = \frac{25}{2832} (b-a) (d-c) |f_{\sigma\tau}(b, d)|.$$

Thus, if we put the last four equalities in (2.18), we obtain (2.16). This completes the proof of the theorem.  $\square$

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[DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, UNIVERSITY OF KAHRAMAN-MARAŞ SÜTÇÜ İMAM, 46000, KAHRAMANMARAŞ, TURKEY  
E-mail address: [abdullahmat@gmail.com](mailto:abdullahmat@gmail.com)

[DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, DÜZCE UNIVERSITY, DÜZCE, TURKEY  
E-mail address: [sarikayamz@gmail.com](mailto:sarikayamz@gmail.com)

[DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, DÜZCE UNIVERSITY, DÜZCE, TURKEY  
E-mail address: [hsyn.budak@gmail.com](mailto:hsyn.budak@gmail.com)

[DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, UNIVERSITY OF KAHRAMAN-MARAŞ SÜTÇÜ İMAM, 46000, KAHRAMANMARAŞ, TURKEY  
E-mail address: [hyildir@ksu.edu.tr](mailto:hyildir@ksu.edu.tr)