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REVERSES OF CALLEBAUT'S AND HÖLDER'S INEQUALITIES FOR ISOTONIC FUNCTIONALS WITH APPLICATIONS

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Abstract. In this paper, by the use of two new reverses of Young's inequality, we establish some new inequalities related to Callebaut's and Hölder's inequalities for isotonic linear functionals. Applications for general Lebesgue integral and discrete counting measure are provided as well.

1. Introduction

Let L be a *linear class* of real-valued functions $g : E \to \mathbb{R}$ having the properties:

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1$, $t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalised if*
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2], [23] and [24]). For other inequalities for isotonic functionals see [1], [4]-[22] and [25]-[28].

We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second ($p_k \geq 0$, $k \in E$).

The functional version of *Callebaut's inequality* states that

$$A^2(fg) \leq A\left(f^{2-\nu} g^{\nu}\right) A\left(f^{\nu} g^{2-\nu}\right) \leq A\left(f^2\right) A\left(g^2\right) \tag{1.1}$$

provided that f^2, g^2, $f^{2-\nu} g^{\nu}$, $f^{\nu} g^{2-\nu}$, $fg \in L$ for some $\nu \in [0, 2]$. For the discrete and integral of one real variable versions see [3].

In the recent paper [11] we have established amongst other the following refinements and reverses of Callebaut's inequality:

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Theorem 1. *Let A, $B : L \to \mathbb{R}$ be two normalised isotonic functionals. If f, $g : E \to \mathbb{R}$ are such that $f > 0$, $g > 0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L$ for some $\nu \in [0,1]$ and*

$$0 < m \leq \frac{f}{g} \leq M < \infty, \tag{1.2}$$

then

$$\begin{aligned} &\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{m^2}{M^2}\right)^2\right] A\left(f^{2(1-\nu)}g^{2\nu}\right) B\left(f^{2\nu}g^{2(1-\nu)}\right) \\ &\leq (1-\nu) A\left(f^2\right) B\left(g^2\right) + \nu A\left(g^2\right) B\left(f^2\right) \\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M^2}{m^2}-1\right)^2\right] A\left(f^{2(1-\nu)}g^{2\nu}\right) B\left(f^{2\nu}g^{2(1-\nu)}\right). \end{aligned} \tag{1.3}$$

In particular, we have

$$\begin{aligned} &\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{m^2}{M^2}\right)^2\right] A\left(f^{2(1-\nu)}g^{2\nu}\right) A\left(f^{2\nu}g^{2(1-\nu)}\right) \\ &\leq A\left(f^2\right) A\left(g^2\right) \\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M^2}{m^2}-1\right)^2\right] A\left(f^{2(1-\nu)}g^{2\nu}\right) A\left(f^{2\nu}g^{2(1-\nu)}\right). \end{aligned} \tag{1.4}$$

for any $\nu \in [0,1]$.

In particular, we have the following refinements and reverses of the Cauchy-Bunyakovsky-Schwarz inequality [11].

Corollary 1. *Let A, $B : L \to \mathbb{R}$ be two normalised isotonic functionals. If f, $g : E \to \mathbb{R}$ are such that $f > 0$, $g > 0$, f^2, g^2, $fg \in L$ and the condition (1.2) holds true, then*

$$\begin{aligned} &\exp\left[\frac{1}{8}\left(1-\frac{m^2}{M^2}\right)^2\right] A(fg) B(fg) \\ &\leq (1-\nu) A\left(f^2\right) B\left(g^2\right) + \nu A\left(g^2\right) B\left(f^2\right) \\ &\leq \exp\left[\frac{1}{8}\left(\frac{M^2}{m^2}-1\right)^2\right] A(fg) B(fg), \end{aligned} \tag{1.5}$$

for any $\nu \in [0,1]$.

In particular, we have

$$\exp\left[\frac{1}{8}\left(\frac{M^2-m^2}{M^2}\right)^2\right] \leq \frac{A\left(f^2\right) A\left(g^2\right)}{A^2(fg)} \leq \exp\left[\frac{1}{8}\left(\frac{M^2-m^2}{m^2}\right)^2\right]. \tag{1.6}$$

The corresponding results related to Hölder's inequality also hold [11]:

Theorem 2. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If f, $g : E \to \mathbb{R}$ are such that fg, f^p, $g^q \in L$ and*

$$0 < m_1 \leq f \leq M_1 < \infty,\ 0 < m_2 \leq g \leq M_2 < \infty, \tag{1.7}$$

for some constants m_1, M_1, m_2, M_2, *then by putting*

$$M_{p,q} := \max\left\{\left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q\right\}$$

we have

$$\exp\left[\frac{1}{2pq}\left(\frac{M_{p,q}^2 - 1}{M_{p,q}^2}\right)^2\right] \leq \frac{\left[A\left(f^p\right)\right]^{1/p}\left[A\left(g^q\right)\right]^{1/q}}{A\left(fg\right)} \leq \exp\left[\frac{1}{2pq}\left(M_{p,q}^2 - 1\right)^2\right]. \tag{1.8}$$

The particular case $p = q = 2$ produces the following refinement and reverse of the Cauchy-Bunyakovsky-Schwarz inequality [11]

Corollary 2. *Let* A, $B : L \to \mathbb{R}$ *be two normalised isotonic functionals. If* f, $g : E \to \mathbb{R}$ *are such that* $f > 0$, $g > 0$, f^2, g^2, $fg \in L$ *and the condition (1.7) holds true, then by putting*

$$M := \max\left\{\frac{M_1}{m_1}, \frac{M_2}{m_2}\right\}$$

we have

$$\exp\left[\frac{1}{4}\left(\frac{M^4 - 1}{M^4}\right)^2\right] \leq \frac{A\left(f^2\right)A\left(g^2\right)}{A^2\left(fg\right)} \leq \exp\left[\frac{1}{4}\left(M^4 - 1\right)^2\right]. \tag{1.9}$$

Motivated by the above results, in this paper we establish some new reverses of Callebaut's and Hölder's inequalities for isotonic linear functionals. Applications for integrals and sums are also given.

2. Reverses of Callebaut's Inequality

We have the following reverse additive version of Young's inequality [12]:

Lemma 1. *If* a, $b \in [m, M] \subset (0, \infty)$ *and* $\nu \in [0, 1]$, *then we have*

$$(0 \leq)\left(1 - \nu\right)a + \nu b - a^{1-\nu}b^{\nu} \leq \max\left\{\iota_{m,M}\left(\nu\right), \iota_{m,M}\left(1 - \nu\right)\right\} \tag{2.1}$$

where

$$\iota_{m,M}\left(\nu\right) := \left(1 - \nu\right)m + \nu M - m^{1-\nu}M^{\nu}. \tag{2.2}$$

We can state the following result providing an additive reverse of Callebaut's inequality:

Theorem 3. *Let* A, $B : L \to \mathbb{R}$ *be two normalised isotonic functionals. If* f, $g : E \to \mathbb{R}$ *are such that* $f \geq 0$, $g > 0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L$ *for some* $\nu \in [0, 1]$ *and*

$$0 < m \leq \frac{f}{g} \leq M < \infty \tag{2.3}$$

for some constants m, M, *then*

$$\begin{aligned}(1 - \nu)\,A\left(f^2\right)B\left(g^2\right) + \nu A\left(g^2\right)B\left(f^2\right) - A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right)\\ \leq \max\left\{\iota_{m^2,M^2}\left(\nu\right), \iota_{m^2,M^2}\left(1 - \nu\right)\right\}A\left(g^2\right)B\left(g^2\right)\end{aligned} \tag{2.4}$$

where $\iota_{m^2,M^2}\left(\cdot\right)$ *is defined by (2.2).*

In particular, we have

$$0 \le A(f^2)A(g^2) - A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right) \le \max\left\{\iota_{m^2,M^2}(\nu), \iota_{m^2,M^2}(1-\nu)\right\}A^2(g^2). \tag{2.5}$$

Proof. For any x, $y \in E$ we have

$$m^2 \le \frac{f^2(x)}{g^2(x)}, \frac{f^2(y)}{g^2(y)} \le M^2.$$

If we use the inequalities (2.1) for

$$a = \frac{f^2(x)}{g^2(x)},\ b = \frac{f^2(y)}{g^2(y)},$$

then we get

$$(1-\nu)\frac{f^2(x)}{g^2(x)} + \nu\frac{f^2(y)}{g^2(y)} - \left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu} \le \max\left\{\iota_{m^2,M^2}(\nu), \iota_{m^2,M^2}(1-\nu)\right\} \tag{2.6}$$

for any $x, y \in E$.

Now, if we multiply (2.6) by $g^2(x)g^2(y) > 0$ then we get

$$\begin{aligned} 0 &\le (1-\nu)f^2(x)g^2(y) + \nu g^2(x)f^2(y) \\ &- f^{2(1-\nu)}(x)g^{2\nu}(x)f^{2\nu}(y)g^{2(1-\nu)}(y) \\ &\le \max\left\{\iota_{m^2,M^2}(\nu), \iota_{m^2,M^2}(1-\nu)\right\}g^2(x)g^2(y) \end{aligned} \tag{2.7}$$

for any x, $y \in E$.

Fix $y \in E$. Then by (2.7) we have in the order of L that

$$\begin{aligned} 0 &\le (1-\nu)g^2(y)f^2 + \nu f^2(y)g^2 - f^{2\nu}(y)g^{2(1-\nu)}(y)f^{2(1-\nu)}g^{2\nu} \\ &\le \max\left\{\iota_{m^2,M^2}(\nu), \iota_{m^2,M^2}(1-\nu)\right\}g^2(y)g^2. \end{aligned} \tag{2.8}$$

If we take the functional A in (2.8) then we get for any $y \in E$.

This inequality can be written in the order of L as

$$\begin{aligned} 0 &\le (1-\nu)A(f^2)g^2 + \nu A(g^2)f^2 - A\left(f^{2(1-\nu)}g^{2\nu}\right)f^{2\nu}g^{2(1-\nu)} \\ &\le \max\left\{\iota_{m^2,M^2}(\nu), \iota_{m^2,M^2}(1-\nu)\right\}A(g^2)g^2. \end{aligned} \tag{2.9}$$

Now, if we take the functional B in (2.9), then we get the desired result (2.4). □

The following additive reverse of two functional Cauchy-Bunyakovsky-Schwarz inequality for isotonic functionals holds:

Corollary 3. *Let A, $B : L \to \mathbb{R}$ be two normalised isotonic functionals. If f, $g : E \to \mathbb{R}$ are such that $f \ge 0$, $g > 0$, f^2, g^2, $fg \in L$ and the condition (2.3) holds true, then*

$$\begin{aligned} 0 &\le \frac{1}{2}\left[A(f^2)B(g^2) + A(g^2)B(f^2)\right] - A(fg)B(fg) \\ &\le \frac{1}{2}(M-m)^2 A(g^2)B(g^2). \end{aligned} \tag{2.10}$$

We also have the following multiplicative reverse of Young's inequality [12]:

Lemma 2. *If* a, $b \in [m, M] \subset (0, \infty)$ *and* $\nu \in [0, 1]$, *then we have*

$$(1 \leq) \frac{(1-\nu) a + \nu b}{a^{1-\nu} b^{\nu}} \leq \max \{\kappa_{m,M}(\nu), \kappa_{m,M}(1-\nu)\} \tag{2.11}$$

where

$$\kappa_{m,M}(\nu) := \frac{(1-\nu) m + \nu M}{m^{1-\nu} M^{\nu}}. \tag{2.12}$$

We also have the following multiplicative reverse of Callebaut's inequality:

Theorem 4. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}(1-\nu) A(f^2) B(g^2) + \nu A(g^2) B(f^2) \\ \leq \max\{\kappa_{m^2,M^2}(\nu), \kappa_{m^2,M^2}(1-\nu)\} A\left(f^{2(1-\nu)} g^{2\nu}\right) B\left(f^{2\nu} g^{2(1-\nu)}\right)\end{aligned} \tag{2.13}$$

where $\kappa_{m^2,M^2}(\cdot)$ *is defined by (2.12).*

In particular, we have

$$\begin{aligned}A(f^2) A(g^2) \\ \leq \max\{\kappa_{m^2,M^2}(\nu), \kappa_{m^2,M^2}(1-\nu)\} A\left(f^{2(1-\nu)} g^{2\nu}\right) A\left(f^{2\nu} g^{2(1-\nu)}\right).\end{aligned} \tag{2.14}$$

Proof. For any x, $y \in E$ we have

$$m^2 \leq \frac{f^2(x)}{g^2(x)}, \frac{f^2(y)}{g^2(y)} \leq M^2.$$

If we use the inequality (2.11) for

$$a = \frac{f^2(x)}{g^2(x)},\ b = \frac{f^2(y)}{g^2(y)},$$

then we get

$$\begin{aligned}(1-\nu) \frac{f^2(x)}{g^2(x)} + \nu \frac{f^2(y)}{g^2(y)} \\ \leq \max\{\kappa_{m^2,M^2}(\nu), \kappa_{m^2,M^2}(1-\nu)\} \left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu} \left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}\end{aligned} \tag{2.15}$$

for any x, $y \in E$.

Now, if we multiply (2.15) by $g^2(x) g^2(y) > 0$ then we get

$$\begin{aligned}(1-\nu) f^2(x) g^2(y) + \nu g^2(x) f^2(y) \\ \leq \max\{\kappa_{m^2,M^2}(\nu), \kappa_{m^2,M^2}(1-\nu)\} f^{2(1-\nu)}(x) g^{2\nu}(x) f^{2\nu}(y) g^{2(1-\nu)}(y)\end{aligned} \tag{2.16}$$

for any $x, y \in E$.

Further, by applying a similar argument to the one in the proof of Theorem 3 we deduce the desired result (2.13). □

The following multiplicative reverse of two functional Cauchy-Bunyakovsky-Schwarz inequality for isotonic functionals holds:

Corollary 4. *With the assumptions of Corollary 3 we have*

$$\frac{1}{2}\left[A(f^2) B(g^2) + A(g^2) B(f^2)\right] \leq \frac{m^2 + M^2}{2mM} A(fg) B(fg). \tag{2.17}$$

3. Reverses of Hölder's Inequality

We have the following reverse of Hölder's inequality for isotonic functionals:

Theorem 5. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional and $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f,\ g : E \to \mathbb{R}$ are such that $fg,\ f^p,\ g^q \in L$ and*

$$0 < m_1 \le f \le M_1 < \infty,\ \ 0 < m_2 \le g \le M_2 < \infty, \tag{3.1}$$

for some constants $m_1,\ M_1,\ m_2,\ M_2$, then by putting

$$M_{p,q} := \max\left\{ \left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q \right\}$$

we have

$$0 \le 1 - \frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \le \max\left\{ \iota_{\frac{1}{M_{p,q}},M_{p,q}}\left(\frac{1}{q}\right), \iota_{\frac{1}{M_{p,q}},M_{p,q}}\left(\frac{1}{p}\right) \right\}, \tag{3.2}$$

where $\iota_{\frac{1}{M_{p,q}},M_{p,q}}(\cdot)$ is defined by (2.2).

Proof. Observe that, by (3.1) we have

$$m_1^p \le A(f^p) \le M_1^p \text{ and } m_2^q \le A(g^q) \le M_2^q.$$

Also

$$\left(\frac{m_1}{M_1}\right)^p \le \frac{f^p}{A(f^p)} \le \left(\frac{M_1}{m_1}\right)^p$$

and

$$\left(\frac{m_2}{M_2}\right)^q \le \frac{g^q}{A(g^q)} \le \left(\frac{M_2}{m_2}\right)^q$$

giving that

$$m_{p,q} \le \frac{f^p}{A(f^p)}, \frac{g^q}{A(g^q)} \le M_{p,q},$$

where

$$m_{p,q} := \min\left\{ \left(\frac{m_1}{M_1}\right)^p, \left(\frac{m_2}{M_2}\right)^q \right\} = \min\left\{ \frac{1}{\left(\frac{M_1}{m_1}\right)^p}, \frac{1}{\left(\frac{M_2}{m_2}\right)^q} \right\}$$
$$= \frac{1}{\max\left\{ \left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q \right\}} = \frac{1}{M_{p,q}}.$$

Using the inequality (2.1) for $\nu = \frac{1}{q}$, $a = \frac{f^p}{A(f^p)}$, $b = \frac{g^q}{A(g^q)}$, $m = m_{p,q} = \frac{1}{M_{p,q}}$ and $M = M_{p,q}$ we get

$$\begin{aligned} 0 &\le \frac{1}{p}\frac{f^p}{A(f^p)} + \frac{1}{q}\frac{g^q}{A(g^q)} - \frac{fg}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \\ &\le \max\left\{ \iota_{\frac{1}{M_{p,q}},M_{p,q}}\left(\frac{1}{q}\right), \iota_{\frac{1}{M_{p,q}},M_{p,q}}\left(\frac{1}{p}\right) \right\}. \end{aligned} \tag{3.3}$$

This proves the desired result (3.2). □

We also have:

Theorem 6. *With the assumptions of Theorem 5 we have*

$$1 \le \frac{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}{A(fg)} \le \max\left\{\kappa_{\frac{1}{M_{p,q}},M_{p,q}}\left(\frac{1}{q}\right), \kappa_{\frac{1}{M_{p,q}},M_{p,q}}\left(\frac{1}{p}\right)\right\}. \tag{3.4}$$

Proof. From (2.11) we have for $\nu = \frac{1}{q}$, $a = \frac{f^p}{A(f^p)}$, $b = \frac{g^q}{A(g^q)}$, $m = \frac{1}{M_{p,q}}$ and $M = M_{p,q}$ that

$$\begin{aligned}&\frac{1}{p}\frac{f^p}{A(f^p)} + \frac{1}{q}\frac{g^q}{A(g^q)} \\ &\le \max\left\{\kappa_{\frac{1}{M_{p,q}},M_{p,q}}\left(\frac{1}{q}\right), \kappa_{\frac{1}{M_{p,q}},M_{p,q}}\left(\frac{1}{p}\right)\right\}\frac{fg}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}.\end{aligned} \tag{3.5}$$

If we take the functional A in (3.5), then we get

$$\begin{aligned}1 &= \frac{1}{p}\frac{A(f^p)}{A(f^p)} + \frac{1}{q}\frac{A(g^q)}{A(g^q)} \\ &\le \max\left\{\kappa_{\frac{1}{M_{p,q}},M_{p,q}}\left(\frac{1}{q}\right), \kappa_{\frac{1}{M_{p,q}},M_{p,q}}\left(\frac{1}{p}\right)\right\}\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}},\end{aligned} \tag{3.6}$$

which is equivalent to (3.4). □

4. Integral Inequalities

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\ge 0$ for μ -a.e. (almost every) $x\in\Omega$, consider the *Lebesgue space*

$$L_w(\Omega,\mu) := \{f:\Omega\to\mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|\,w(x)\,d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$. The same for other integrals involved below. We assume that $\int_\Omega w d\mu = 1$.

Let f, g be μ-measurable functions with the property that there exists the constants M, $m > 0$ such that

$$0 < m \le \frac{f}{g} \le M < \infty \ \mu\text{-almost everywhere (a.e.) on } \Omega. \tag{4.1}$$

If f^2, $g^2 \in L_w(\Omega,\mu)$, then by (2.5) we have for any $s\in[0,1]$ that

$$\begin{aligned}0 &\le \int_\Omega wf^2d\mu\int_\Omega wg^2d\mu - \int_\Omega wf^{2(1-s)}g^{2s}d\mu\int_\Omega wf^{2s}g^{2(1-s)}d\mu \\ &\le \max\left\{\iota_{m^2,M^2}(s), \iota_{m^2,M^2}(1-s)\right\}\left(\int_\Omega wg^2d\mu\right)^2.\end{aligned} \tag{4.2}$$

Moreover, from (2.14) we also have

$$\begin{aligned}&\int_\Omega wf^2d\mu\int_\Omega wg^2d\mu \\ &\le \max\left\{\kappa_{m^2,M^2}(\nu), \kappa_{m^2,M^2}(1-\nu)\right\}\int_\Omega wf^{2(1-s)}g^{2s}d\mu\int_\Omega wf^{2s}g^{2(1-s)}d\mu\end{aligned} \tag{4.3}$$

Let f, g be μ-measurable functions with the property that there exists the constants M_1, M_2, m_1, $m_2 > 0$ such that

$$0 < m_1 \le f \le M_1 < \infty,\ 0 < m_2 \le g \le M_2 < \infty,\ \mu\text{-a.e. on } \Omega. \tag{4.4}$$

Then by putting

$$M_{p,q} := \max\left\{ \left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q \right\}$$

where p, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have from (3.2) that

$$\begin{aligned} 0 &\le 1 - \frac{\int_\Omega wfgd\mu}{\left(\int_\Omega wf^p d\mu\right)^{1/p} \left(\int_\Omega wg^q d\mu\right)^{1/q}} \\ &\le \max\left\{ \iota_{\frac{1}{M_{p,q}}, M_{p,q}}\left(\frac{1}{q}\right), \iota_{\frac{1}{M_{p,q}}, M_{p,q}}\left(\frac{1}{p}\right) \right\}, \end{aligned} \tag{4.5}$$

while from (3.4) that

$$\begin{aligned} 1 &\le \frac{\left(\int_\Omega wf^p d\mu\right)^{1/p} \left(\int_\Omega wg^q d\mu\right)^{1/q}}{\int_\Omega wfgd\mu} \\ &\le \max\left\{ \kappa_{\frac{1}{M_{p,q}}, M_{p,q}}\left(\frac{1}{q}\right), \kappa_{\frac{1}{M_{p,q}}, M_{p,q}}\left(\frac{1}{p}\right) \right\}. \end{aligned} \tag{4.6}$$

5. Inequalities for Real Numbers

We consider the n-tuples of positive numbers $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and the probability distribution $p = (p_1, ..., p_n)$, i.e. $p_i \ge 0$ for any $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$.

If there exist the constants m, $M > 0$ such that

$$0 < m \le \frac{a_i}{b_i} \le M < \infty \text{ for any } i \in \{1, ..., n\},$$

then by (4.2) for the *counting discrete measure*, we have for any $s \in [0, 1]$ that

$$\begin{aligned} 0 &\le \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \sum_{i=1}^n p_i a_i^{2(1-s)} b_i^{2s} \sum_{i=1}^n p_i a_i^{2s} b_i^{2(1-s)} \\ &\le \max\left\{ \iota_{m^2, M^2}(s), \iota_{m^2, M^2}(1-s) \right\} \left(\sum_{i=1}^n p_i b_i^2 \right)^2 \end{aligned} \tag{5.1}$$

while from (4.3)

$$\begin{aligned} &\sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 \\ &\le \max\left\{ \kappa_{m^2, M^2}(s), \kappa_{m^2, M^2}(1-s) \right\} \sum_{i=1}^n p_i a_i^{2(1-s)} b_i^{2s} \sum_{i=1}^n p_i a_i^{2s} b_i^{2(1-s)}. \end{aligned} \tag{5.2}$$

If there exist the constants m_1, M_1, m_2, M_2 such that

$$0 < m_1 \le a_i \le M_1 < \infty,\ 0 < m_2 \le b_i \le M_2 < \infty \text{ for any } i \in \{1, ..., n\} \tag{5.3}$$

and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by putting

$$M_{p,q} := \max\left\{ \left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q \right\}$$

where p, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have from (4.5) that

$$(5.4)\quad 0 \le 1 - \frac{\sum_{i=1}^n p_i a_i b_i}{\left(\sum_{i=1}^n p_i a_i^p\right)^{1/p} \left(\sum_{i=1}^n p_i b_i^q\right)^{1/q}} \le \max\left\{ \iota_{\frac{1}{M_{p,q}}, M_{p,q}}\left(\frac{1}{q}\right), \iota_{\frac{1}{M_{p,q}}, M_{p,q}}\left(\frac{1}{p}\right) \right\},$$

while from (4.6) that

$$(5.5)\quad 1 \le \frac{\left(\sum_{i=1}^n p_i a_i^p\right)^{1/p} \left(\sum_{i=1}^n p_i b_i^q\right)^{1/q}}{\sum_{i=1}^n p_i a_i b_i} \le \max\left\{ \kappa_{\frac{1}{M_{p,q}}, M_{p,q}}\left(\frac{1}{q}\right), \kappa_{\frac{1}{M_{p,q}}, M_{p,q}}\left(\frac{1}{p}\right) \right\}.$$

References

[1] D. Andrica and C. Badea, Grüss' inequality for positive linear functionals, *Periodica Math. Hung.,* **19** (1998), 155-167.

[2] P. R. Beesack and J. E. Pečarić, On Jessen's inequality for convex functions, *J. Math. Anal. Appl.,* **110** (1985), 536-552.

[3] D. K. Callebaut, Generalization of Cauchy-Schwarz inequality, *J. Math. Anal. Appl.* **12** (1965), 491-494.

[4] S. S. Dragomir, A refinement of Hadamard's inequality for isotonic linear functionals, *Tamkang J. Math.* (Taiwan), **24** (1992), 101-106.

[5] S. S. Dragomir, On a reverse of Jessen's inequality for isotonic linear functionals, *J. Ineq. Pure & Appl. Math.*, **2**(3)(2001), Article 36.

[6] S. S. Dragomir, On the Jessen's inequality for isotonic linear functionals, *Nonlinear Analysis Forum,* **7**(2)(2002), 139-151.

[7] S. S. Dragomir, On the Lupaş-Beesack-Pečarić inequality for isotonic linear functionals, *Nonlinear Funct. Anal. & Appl.,* **7**(2)(2002), 285-298.

[8] S. S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), 417-478.

[9] S. S. Dragomir, A note on Young's inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 126. [http://rgmia.org/papers/v18/v18a126.pdf].

[10] S. S. Dragomir, A note on new refinements and reverses of Young's inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 130. [http://rgmia.org/papers/v18/v18a130.pdf].

[11] S. S. Dragomir, Some results related to Callebaut's and Hölder's inequalities for isotonic functionals with application, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 133. [http://rgmia.org/papers/v18/v18a133.pdf].

[12] S. S. Dragomir, Two new reverses of Young's inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 158. [http://rgmia.org/papers/v18/v18a158.pdf].

[13] S. S. Dragomir and N. M. Ionescu, On some inequalities for convex-dominated functions, *L'Anal. Num. Théor. L'Approx.,* **19** (1) (1990), 21-27.

[14] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications,* RGMIA Monographs, Victoria University, 2000.

[15] S. S. Dragomir, C. E. M. Pearce and J. E. Pečarić, On Jessen's and related inequalities for isotonic sublinear functionals, *Acta. Sci. Math.* (Szeged), **61** (1995), 373-382.

[16] S. Furuichi, Refined Young inequalities with Specht's ratio, *Journal of the Egyptian Mathematical Society* **20**(2012) , 46–49.

[17] S. Furuichi, On refined Young inequalities and reverse inequalities, *J. Math. Inequal.*, **5** (2011), 21-31.

[18] S. Furuichi and N. Minculete, Alternative reverse inequalities for Young's inequality, *J. Math Inequal.* **5** (2011), Number 4, 595–600.
[19] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.,* **361** (2010), 262-269
[20] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra.,* **59** (2011), 1031-1037.
[21] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.
[22] A. Lupaş, A generalisation of Hadamard's inequalities for convex functions, *Univ. Beograd. Elek. Fak.,* 577-579 (1976), 115-121.
[23] J .E. Pečarić, On Jessen's inequality for convex functions (III), *J. Math. Anal. Appl.,* **156** (1991), 231-239.
[24] J. E. Pečarić and P. R. Beesack, On Jessen's inequality for convex functions (II), *J. Math. Anal. Appl.,* **156** (1991), 231-239.
[25] J. E. Pečarić and S. S. Dragomir, A generalisation of Hadamard's inequality for isotonic linear functionals, *Radovi Mat.* (Sarjevo), **7** (1991), 103-107.
[26] J. E. Pečarić and I. Raşa, On Jessen's inequality, *Acta. Sci. Math.* (Szeged), **56** (1992), 305-309.
[27] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.*, **74** (1960), pp. 91-98.
[28] G. Toader and S. S. Dragomir, Refinement of Jessen's inequality, *Demonstratio Mathematica,* **28** (1995), 329-334.
[29] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.
[30] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.

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