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SOME INEQUALITIES FOR HEINZ OPERATOR MEAN

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Abstract. In this paper we obtain some new inequalities for Heinz operator mean.

1. Introduction

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators and $\nu \in [0,1]$

$$A \nabla_\nu B := (1-\nu) A + \nu B,$$

the *weighted operator arithmetic mean,* and

$$A \sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2},$$

the *weighted operator geometric mean.* When $\nu = \frac{1}{2}$ we write $A \nabla B$ and $A \sharp B$ for brevity, respectively.

Define the *Heinz operator mean* by

$$H_\nu (A,B) := \frac{1}{2} (A \sharp_\nu B + A \sharp_{1-\nu} B).$$

The following interpolatory inequality is obvious

$$A \sharp B \le H_\nu (A,B) \le A \nabla B \tag{1.1}$$

for any $\nu \in [0,1]$.

The famous *Young inequality* for scalars says that if a, $b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^\nu \le (1-\nu) a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [12]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln \left(h^{\frac{1}{h-1}} \right)} \text{ if } h \in (0,1) \cup (1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h \to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \ne 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality:

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$$S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.4}$$

where $a,\ b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$.

The second inequality in (1.4) is due to Tominaga [13] while the first one is due to Furuichi [4].

The operator version is as follows [4], [13] : For two positive operators A, B and positive real numbers $m,\ m',M,\ M'$ satisfying either of the following conditions:

(i) $0<mI\leq A\leq m'I<M'I\leq B\leq MI$,

(ii) $0<mI\leq B\leq m'I<M'I\leq A\leq MI$

we have

$$S\left((h')^{r}\right)A\sharp_{\nu}B\leq A\nabla_{\nu}B\leq S(h)\,A\sharp_{\nu}B, \tag{1.5}$$

where $h:=\frac{M}{m},\ h':=\frac{M'}{m'}$ and $\nu\in[0,1]$.

We observe that, if we write the inequality (1.5) for $1-\nu$ and add the obtained inequalities, then we get by division with 2 that

$$S\left((h')^{r}\right)H_{\nu}(A,B)\leq A\nabla B\leq S(h)\,H_{\nu}(A,B)$$

that is equivalent to

$$S^{-1}(h)\,A\nabla B\leq H_{\nu}(A,B)\leq S^{-1}\left((h')^{r}\right)A\nabla B, \tag{1.6}$$

where $h:=\frac{M}{m},\ h':=\frac{M'}{m'}$ and $\nu\in[0,1]$.

We consider the *Kantorovich's constant* defined by

$$K(h):=\frac{(h+1)^{2}}{4h},\ \ h>0. \tag{1.7}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\geq1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds:

$$K^{r}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b\leq K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.8}$$

where $a,b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.8) was obtained by Zou et al. in [14] while the second by Liao et al. [11].

The operator version is as follows [14], [11]: For two positive operators A, B and positive real numbers $m,\ m',M,\ M'$ satisfying either of the conditions (i) or (ii) above, we have

$$K^{r}(h')\,A\sharp_{\nu}B\leq A\nabla_{\nu}B\leq K^{R}(h)\,A\sharp_{\nu}B, \tag{1.9}$$

where $h:=\frac{M}{m},\ h':=\frac{M'}{m'},\ \nu\in[0,1]\ r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

We observe that, if we write the inequality (1.9) for $1-\nu$ and add the obtained inequalities, then we get by division with 2 that

$$K^{r}(h')\,H_{\nu}(A,B)\leq A\nabla B\leq K^{R}(h)\,H_{\nu}(A,B)$$

that is equivalent to

$$K^{-R}(h)\,A\nabla B\leq H_{\nu}(A,B)\leq K^{-r}(h')\,A\nabla B, \tag{1.10}$$

where $h:=\frac{M}{m},\ h':=\frac{M'}{m'}$ and $\nu\in[0,1]$.

The inequalities (1.10) have been obtained in [11] where other bounds in terms of the *weighted operator harmonic mean*

$$A!_\nu B := \left[(1-\nu)A^{-1}+\nu B^{-1}\right]^{-1}$$

were also given.

Motivated by the above results, we establish in this paper some new inequalities for the Heinz mean. Related inequalities are also provided.

2. Upper and Lower Bounds for Heinz Mean

We start with the following result that provides a generalization for the inequalities (1.5) and (1.9):

Theorem 1. *Assume that A, B are positive invertible operators and the constants $M>m>0$ are such that*

$$mA \le B \le MA \tag{2.1}$$

in the operator order. Let $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$. Then we have the inequalities

$$\varphi_r(m,M)\,A\sharp_\nu B \le A\nabla_\nu B \le \Phi(m,M)\,A\sharp_\nu B, \tag{2.2}$$

where

$$\Phi(m,M) := \begin{cases} S(m) \text{ if } M<1, \\ \max\{S(m),S(M)\} \text{ if } m\le 1\le M, \\ S(M) \text{ if } 1<m, \end{cases}, \tag{2.3}$$

$$\varphi_r(m,M) := \begin{cases} S(M^r) \text{ if } M<1, \\ 1 \text{ if } m\le 1\le M, \\ S(m^r) \text{ if } 1<m, \end{cases}$$

and

$$\psi_r(m,M)\,A\sharp_\nu B \le A\nabla_\nu B \le \Psi_R(m,M)\,A\sharp_\nu B, \tag{2.4}$$

where

$$\Psi_R(m,M) := \begin{cases} K^R(m) \text{ if } M<1, \\ \max\left\{K^R(m),K^R(M)\right\} \text{ if } m\le 1\le M, \\ K^R(M) \text{ if } 1<m, \end{cases}, \tag{2.5}$$

$$\psi_r(m,M) := \begin{cases} K^r(M) \text{ if } M<1, \\ 1 \text{ if } m\le 1\le M, \\ K^r(m) \text{ if } 1<m. \end{cases}$$

Proof. From the inequality (1.4) we have

$$x^{\nu}\min_{x\in[m,M]}S\left(x^{r}\right)\leq S\left(x^{r}\right)x^{\nu}\leq(1-\nu)+\nu x\leq S\left(x\right)x^{\nu}\leq x^{\nu}\max_{x\in[m,M]}S\left(x\right) \tag{2.6}$$

where $x\in[m,M]$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$.

Since, by the properties of Specht's ratio S, we have

$$\max_{x\in[m,M]}S\left(x\right)=\begin{cases}S\left(m\right)\ \text{ if } M<1,\\[2ex] \max\left\{S\left(m\right),S\left(M\right)\right\}\ \text{ if } m\leq1\leq M,\\[2ex] S\left(M\right)\ \text{ if } 1<m,\end{cases}=\Phi\left(m,M\right)$$

and

$$\min_{x\in[m,M]}S\left(x^{r}\right)=\begin{cases}S\left(M^{r}\right)\ \text{ if } M<1,\\[2ex] 1 \text{ if } m\leq1\leq M,\\[2ex] S\left(m^{r}\right)\ \text{ if } 1<m,\end{cases}=\varphi_{r}\left(m,M\right),$$

then by (2.6) we have

$$x^{\nu}\varphi_{r}\left(m,M\right)\leq(1-\nu)+\nu x\leq x^{\nu}\Phi\left(m,M\right) \tag{2.7}$$

for any $x\in[m,M]$ and $\nu\in[0,1]$.

Using the functional calculus for the operator X with $mI\leq X\leq MI$ we have from (2.7) that

$$X^{\nu}\varphi_{r}\left(m,M\right)\leq(1-\nu)I+\nu X\leq X^{\nu}\Phi\left(m,M\right) \tag{2.8}$$

for any $\nu\in[0,1]$.

If the condition (2.1) holds true, then by multiplying in both sides with $A^{-1/2}$ we get $mI\leq A^{-1/2}BA^{-1/2}\leq MI$ and by taking $X=A^{-1/2}BA^{-1/2}$ in (2.8) we get

$$\begin{aligned}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}\varphi_{r}\left(m,M\right)&\leq(1-\nu)I+\nu A^{-1/2}BA^{-1/2}\\&\leq\left(A^{-1/2}BA^{-1/2}\right)^{\nu}\Phi\left(m,M\right).\end{aligned} \tag{2.9}$$

Now, if we multiply (2.9) in both sides with $A^{1/2}$ we get the desired result (2.2).

The second part follows in a similar way by utilizing the inequality

$$x^{\nu}\min_{x\in[m,M]}K^{r}\left(x\right)\leq K^{r}\left(x\right)x^{\nu}\leq(1-\nu)+\nu x\leq K^{R}\left(x\right)x^{\nu}\leq x^{\nu}\max_{x\in[m,M]}K^{R}\left(x\right),$$

which follows from (1.8). The details are omitted. □

Remark 1. *If (i)* $0<mI\leq A\leq m'I<M'I\leq B\leq MI$, $h=\frac{M}{m}$ *and* $h'=\frac{M'}{m'}$ *then we have*

$$A\leq\frac{M'}{m'}A=h'A\leq B\leq hA=\frac{M}{m}A,$$

and by (2.2) we get

$$S\left(\left(h'\right)^{r}\right)A\sharp_{\nu}B\leq A\nabla_{\nu}B\leq S\left(h\right)A\sharp_{\nu}B. \tag{2.10}$$

If (ii) $0<mI\leq B\leq m'I<M'I\leq A\leq MI$, *then we have*

$$\frac{1}{h}A\leq B\leq\frac{1}{h'}A\leq A$$

and by (2.2) we get

$$S\left(\left(\frac{1}{h'}\right)^r\right) A\sharp_\nu B \le A\nabla_\nu B \le S\left(\frac{1}{h}\right) A\sharp_\nu B,$$

which is equivalent to (2.10).

If we use the inequality (2.4) for the operators A and B that satisfy either of the conditions (i) or (ii), then we recapture (1.9).

Remark 2. *From (2.2) we get for $\nu = \frac{1}{2}$ that*

$$\left\{\begin{array}{l} S\left(M^r\right) A\sharp B \text{ if } M<1, \\ \\ A\sharp B \text{ if } m \le 1 \le M, \\ \\ S\left(m^r\right) A\sharp B \text{ if } 1<m, \end{array}\right. \le A\nabla B \tag{2.11}$$

$$\le \left\{\begin{array}{l} S\left(m\right) A\sharp B \text{ if } M<1, \\ \\ \max\left\{S\left(m\right), S\left(M\right)\right\} A\sharp B \text{ if } m \le 1 \le M, \\ \\ S\left(M\right) A\sharp B \text{ if } 1<m. \end{array}\right.$$

The following result contains two upper and lower bounds for the Heinz operator mean in terms of the operator arithmetic mean $A\nabla B$:

Corollary 1. *With the assumptions of Theorem 1 we have the following upper and lower bounds for the Heinz operator mean*

$$\Phi^{-1}\left(m,M\right) A\nabla B \le H_\nu\left(A,B\right) \le \varphi_r^{-1}\left(m,M\right) A\nabla B \tag{2.12}$$

and

$$\Psi_R^{-1}\left(m,M\right) A\nabla B \le H_\nu\left(A,B\right) \le \psi_r^{-1}\left(m,M\right) A\nabla B. \tag{2.13}$$

Remark 3. *If the operators A and B satisfy either of the conditions (i) or (ii) from Remark 1, then we have the inequality*

$$S^{-1}\left(h\right) A\nabla B \le H_\nu\left(A,B\right) \le S^{-1}\left(\left(h'\right)^r\right) A\nabla B \tag{2.14}$$

and

$$K^{-R}\left(h\right) A\nabla B \le H_\nu\left(A,B\right) \le K^{-r}\left(h'\right) A\nabla B. \tag{2.15}$$

The following result provides an upper and lower bound for the Heinz mean in terms of the operator geometric mean $A\sharp B$:

Theorem 2. *With the assumptions of Theorem 1 we have*

$$\omega\left(m,M\right) A\sharp B \le H_\nu\left(A,B\right) \le \Omega\left(m,M\right) A\sharp B, \tag{2.16}$$

where

$$\Omega\left(m,M\right) := \left\{\begin{array}{l} S\left(m^{|2\nu-1|}\right) \text{ if } M<1, \\ \\ \max\left\{S\left(m^{|2\nu-1|}\right), S\left(M^{|2\nu-1|}\right)\right\} \text{ if } m \le 1 \le M, \\ \\ S\left(M^{|2\nu-1|}\right) \text{ if } 1<m, \end{array}\right. \tag{2.17}$$

and

$$\omega(m,M) := \begin{cases} S\left(M^{\left|\nu-\frac{1}{2}\right|}\right) \text{ if } M<1, \\ 1 \text{ if } m \leq 1 \leq M, \\ S\left(m^{\left|\nu-\frac{1}{2}\right|}\right) \text{ if } 1<m, \end{cases} \tag{2.18}$$

where $\nu \in [0,1]$.

Proof. From the inequality (1.4) we have for $\nu = \frac{1}{2}$

$$S\left(\sqrt{\frac{c}{d}}\right)\sqrt{cd} \leq \frac{c+d}{2} \leq S\left(\frac{c}{d}\right)\sqrt{cd}, \tag{2.19}$$

for any c, $d>0$.

If we take in (2.19) $c = a^{1-\nu}b^{\nu}$ and $d = a^{\nu}b^{1-\nu}$ then we get

$$S\left(\left(\frac{a}{b}\right)^{\frac{1}{2}-\nu}\right)\sqrt{ab} \leq \frac{a^{1-\nu}b^{\nu}+a^{\nu}b^{1-\nu}}{2} \leq S\left(\left(\frac{a}{b}\right)^{1-2\nu}\right)\sqrt{ab}, \tag{2.20}$$

for any a, $b>0$ for any $\nu \in [0,1]$.

This is an inequality of interest in itself.

If we take in (2.20) $a = x$ and $b = 1$, then we get

$$S\left(x^{\frac{1}{2}-\nu}\right)\sqrt{x} \leq \frac{x^{1-\nu}+x^{\nu}}{2} \leq S\left(x^{1-2\nu}\right)\sqrt{x}, \tag{2.21}$$

for any $x>0$.

Now, if $x \in [m,M] \subset (0,\infty)$ then by (2.21) we have

$$\sqrt{x}\min_{x\in[m,M]} S\left(x^{\frac{1}{2}-\nu}\right) \leq \frac{x^{1-\nu}+x^{\nu}}{2} \leq \sqrt{x}\max_{x\in[m,M]} S\left(x^{1-2\nu}\right), \tag{2.22}$$

for any $x \in [m,M]$.

If $\nu \in \left(0,\frac{1}{2}\right)$, then

$$\max_{x\in[m,M]} S\left(x^{1-2\nu}\right) = \begin{cases} S\left(m^{1-2\nu}\right) \text{ if } M<1, \\ \max\left\{S\left(m^{1-2\nu}\right), S\left(M^{1-2\nu}\right)\right\} \text{ if } m \leq 1 \leq M, \\ S\left(M^{1-2\nu}\right) \text{ if } 1<m, \end{cases}$$

and

$$\min_{x\in[m,M]} S\left(x^{\frac{1}{2}-\nu}\right) = \begin{cases} S\left(M^{\frac{1-2\nu}{2}}\right) \text{ if } M<1, \\ 1 \text{ if } m \leq 1 \leq M, \\ S\left(m^{\frac{1-2\nu}{2}}\right) \text{ if } 1<m. \end{cases}$$

If $\nu \in \left(\frac{1}{2}, 1\right)$, then

$$\begin{aligned}\max_{x\in[m,M]} S\left(x^{1-2\nu}\right) &= \max_{x\in[m,M]} S\left(x^{2\nu-1}\right) \\ &= \begin{cases} S\left(m^{2\nu-1}\right) \text{ if } M<1, \\ \max\left\{S\left(m^{2\nu-1}\right), S\left(M^{2\nu-1}\right)\right\} \text{ if } m \le 1 \le M, \\ S\left(M^{2\nu-1}\right) \text{ if } 1<m, \end{cases}\end{aligned}$$

and

$$\min_{x\in[m,M]} S\left(x^{\frac{1}{2}-\nu}\right) = \min_{x\in[m,M]} S\left(x^{\nu-\frac{1}{2}}\right) = \begin{cases} S\left(M^{\frac{2\nu-1}{2}}\right) \text{ if } M<1, \\ 1 \text{ if } m \le 1 \le M, \\ S\left(m^{\frac{2\nu-1}{2}}\right) \text{ if } 1<m. \end{cases}$$

Then by (2.22) we have

$$\omega(m,M)\sqrt{x} \le \frac{x^{1-\nu}+x^{\nu}}{2} \le \Omega(m,M)\sqrt{x}, \tag{2.23}$$

for any $x \in [m, M]$.

If X is an operator with $mI \le X \le MI$, then by (2.23) we have

$$\omega(m,M) X^{1/2} \le \frac{X^{1-\nu}+X^{\nu}}{2} \le \Omega(m,M) X^{1/2}. \tag{2.24}$$

If the condition (2.1) holds true, then by multiplying in both sides with $A^{-1/2}$ we get $mI \le A^{-1/2}BA^{-1/2} \le MI$ and by taking $X = A^{-1/2}BA^{-1/2}$ in (2.24) we get

$$\begin{aligned}&\omega(m,M)\left(A^{-1/2}BA^{-1/2}\right)^{1/2} \\ &\le \frac{1}{2}\left[\left(A^{-1/2}BA^{-1/2}\right)^{1-\nu} + \left(A^{-1/2}BA^{-1/2}\right)^{\nu}\right] \\ &\le \Omega(m,M)\left(A^{-1/2}BA^{-1/2}\right)^{1/2}.\end{aligned} \tag{2.25}$$

Now, if we multiply (2.25) in both sides with $A^{1/2}$ we get the desired result (2.16). □

Corollary 2. *For two positive operators A, B and positive real numbers m, m', M, M' satisfying either of the following conditions:*

(i) $0 < mI \le A \le m'I < M'I \le B \le MI$,

(ii) $0 < mI \le B \le m'I < M'I \le A \le MI$,

we have for $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$ that

$$S\left((h')^{\left|\nu-\frac{1}{2}\right|}\right) A\sharp B \le H_{\nu}(A,B) \le S\left(h^{|2\nu-1|}\right) A\sharp B, \tag{2.26}$$

where $\nu \in [0, 1]$.

3. Related Results

We call *Heron means*, the means defined by

$$F_{\alpha}(a,b) := (1-\alpha)\sqrt{ab} + \alpha\frac{a+b}{2},$$

where $a,\ b>0$ and $\alpha\in[0,1]$.

In [1], Bhatia obtained the following interesting inequality between the Heinz and Heron means

$$H_{\nu}(a,b) \le F_{(2\nu-1)^2}(a,b) \tag{3.1}$$

where $a,\ b>0$ and $\alpha\in[0,1]$.

This inequality can be written as

$$(0\le)\, H_{\nu}(a,b) - \sqrt{ab} \le (2\nu-1)^2\left(\frac{a+b}{2}-\sqrt{ab}\right), \tag{3.2}$$

where $a,\ b>0$ and $\alpha\in[0,1]$.

Making use of a similar argument to the one in the proof of Theorem 1 we can state the following result as well:

Theorem 3. *Assume that $A,\ B$ are positive invertible operators and $\nu\in[0,1]$. Then*

$$(0\le)\, H_{\nu}(A,B) - A\sharp B \le (2\nu-1)^2(A\nabla B - A\sharp B). \tag{3.3}$$

Moreover, if there exists the constants $M>m>0$ such that the condition (2.1) is true, then we have the simpler upper bound

$$(0\le)\, H_{\nu}(A,B) - A\sharp B \le \frac{1}{2}(2\nu-1)^2\left(\sqrt{M}-\sqrt{m}\right)^2. \tag{3.4}$$

Kittaneh and Manasrah [5], [6] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le R\left(\sqrt{a}-\sqrt{b}\right)^2 \tag{3.5}$$

where $a,b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

If we replace in (3.5) ν with $1-\nu$, add the obtained inequalities and divide by 2, then we get

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \le \frac{a+b}{2} - H_{\nu}(a,b) \le R\left(\sqrt{a}-\sqrt{b}\right)^2, \tag{3.6}$$

where $a,b>0,\ \nu\in[0,1]$.

We also have by (3.6) that, see [7] and [9]:

Theorem 4. *Assume that $A,\ B$ are positive invertible operators and $\nu\in[0,1]$. Then*

$$2r(A\nabla B - A\sharp B) \le H_{\nu}(A,B) - A\sharp B \le 2R(A\nabla B - A\sharp B). \tag{3.7}$$

Since $(2\nu-1)^2 \le 2\max\{1-\nu,\nu\}$ for any $\nu\in[0,1]$, it follows that the inequality (3.3) is better than the right side of (3.7).

In [2], by using the equality

$$\frac{a+b}{2} + \frac{2ab}{a+b} - 2\sqrt{ab} = \frac{\left(\sqrt{a}-\sqrt{b}\right)^4}{2(a+b)} \ge 0 \tag{3.8}$$

for $a,\, b>0$, the authors obtained the interesting inequality

$$\frac{1}{2}\left[A\left(a,b\right)+H\left(a,b\right)\right]\geq G\left(a,b\right), \tag{3.9}$$

where $A\left(a,b\right)$ is the arithmetic mean, $H\left(a,b\right)$ is the harmonic mean and $G\left(a,b\right)$ is the geometric mean of positive numbers $a,\, b$.

Now, if we replace a by $a^{1-\nu}b^{\nu}$ and b by $a^{\nu}b^{1-\nu}$ in (3.9) then we get the following result for Heinz means

$$\frac{1}{2}\left[H_{\nu}\left(a,b\right)+H_{\nu}^{-1}\left(a^{-1},b^{-1}\right)\right]\geq G\left(a,b\right) \tag{3.10}$$

for any for $a,\, b>0$ and $\nu\in\left[0,1\right]$.

Since

$$\frac{1}{2\max\left\{a,b\right\}}\leq\frac{1}{a+b}\leq\frac{1}{2\min\left\{a,b\right\}},$$

then by (3.8) we have

$$\frac{1}{4}\frac{\left(\sqrt{a}-\sqrt{b}\right)^{4}}{\max\left\{a,b\right\}}\leq\frac{1}{2}\left[A\left(a,b\right)+H\left(a,b\right)\right]-G\left(a,b\right)\leq\frac{1}{4}\frac{\left(\sqrt{a}-\sqrt{b}\right)^{4}}{\min\left\{a,b\right\}}, \tag{3.11}$$

for any for $a,\, b>0$.

Since $\left(\sqrt{a}-\sqrt{b}\right)^{2}=2\left[A\left(a,b\right)-G\left(a,b\right)\right]$,

$$\frac{\left(\sqrt{a}-\sqrt{b}\right)^{2}}{\max\left\{a,b\right\}}=\frac{\left(\sqrt{a}-\sqrt{b}\right)^{2}}{\left(\max\left\{\sqrt{a},\sqrt{b}\right\}\right)^{2}}=\left(1-\frac{\min\left\{\sqrt{a},\sqrt{b}\right\}}{\max\left\{\sqrt{a},\sqrt{b}\right\}}\right)^{2}$$

and

$$\frac{\left(\sqrt{a}-\sqrt{b}\right)^{2}}{\min\left\{a,b\right\}}=\left(\frac{\max\left\{\sqrt{a},\sqrt{b}\right\}}{\min\left\{\sqrt{a},\sqrt{b}\right\}}-1\right)^{2},$$

then the inequality (3.11) can be written as

$$\begin{aligned}&\frac{1}{2}\left(1-\frac{\min\left\{\sqrt{a},\sqrt{b}\right\}}{\max\left\{\sqrt{a},\sqrt{b}\right\}}\right)^{2}\left[A\left(a,b\right)-G\left(a,b\right)\right]\\&\leq\frac{1}{2}\left[A\left(a,b\right)+H\left(a,b\right)\right]-G\left(a,b\right)\\&\leq\frac{1}{2}\left(\frac{\max\left\{\sqrt{a},\sqrt{b}\right\}}{\min\left\{\sqrt{a},\sqrt{b}\right\}}-1\right)^{2}\left[A\left(a,b\right)-G\left(a,b\right)\right],\end{aligned} \tag{3.12}$$

for any for $a,\, b>0$.

If $a,\, b\in\left[m,M\right]\subset\left(0,\infty\right)$, then by (3.12) we get

$$\begin{aligned}\frac{1}{2}\left(1-\sqrt{\frac{m}{M}}\right)^{2}\left[A\left(a,b\right)-G\left(a,b\right)\right]&\leq\frac{1}{2}\left[A\left(a,b\right)+H\left(a,b\right)\right]-G\left(a,b\right)\\&\leq\frac{1}{2}\left(\sqrt{\frac{M}{m}}-1\right)^{2}\left[A\left(a,b\right)-G\left(a,b\right)\right].\end{aligned} \tag{3.13}$$

If we use the elementary inequality for logarithm [8]

$$\ln x \leq \frac{x^2-1}{2x},\ x>0, \tag{3.14}$$

then we have

$$2\ln\left(a^{1-\nu}b^{\nu}\right) \leq a^{1-\nu}b^{\nu} - \left(a^{1-\nu}b^{\nu}\right)^{-1}$$

and

$$2\ln\left(a^{\nu}b^{1-\nu}\right) \leq a^{\nu}b^{1-\nu} - \left(a^{\nu}b^{1-\nu}\right)^{-1}.$$

If we add these two inequalities and divide by 4, then we get

$$\frac{1}{2}\left[H_{\nu}(a,b) - H_{\nu}\left(a^{-1},b^{-1}\right)\right] \geq \ln\left[G(a,b)\right], \tag{3.15}$$

for any for a, $b>0$ and $\nu \in [0,1]$.

Similar results may be stated for the corresponding operator means, however the details are nor presented here.

References

[1] R. Bhatia, Interpolating the arithmetic–geometric mean inequality and its operator version, *Lin. Alg. Appl.* **413** (2006) 355–363.

[2] Y.-M. Chu and W.-F. Xia, Inequalities for generalized logarithmic means, *J. Inequal. Appl.* Volume **2009**, Article ID 763252, 7 pages doi:10.1155/2009/763252.

[3] S. Furuichi, On refined Young inequalities and reverse inequalities, *J. Math. Inequal.* **5** (2011), 21-31.

[4] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.* **20** (2012), 46–49.

[5] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.* **361** (2010), 262-269.

[6] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Lin. Multilin. Alg.,* **59** (2011), 1031-1037.

[7] F. Kittaneh, M. Krnić, N. Lovričević and J. Pečarić, Improved arithmetic-geometric and Heinz means inequalities for Hilbert space operators, *Publ. Math. Debrecen*, 2012, **80**(3-4), 465–478.

[8] L. Kozma, Useful inequalities cheat sheet, [Online, http://www.lkozma.net/inequalities_cheat_sheet/].

[9] M. Krnić and J. Pečarić, Improved Heinz inequalities via the Jensen functional, *Cent. Eur. J. Math.* **11** (9) 2013,1698-1710.

[10] F. Kubo and T. Ando, Means of positive operators, *Math. Ann.* **264** (1980), 205–224.

[11] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.

[12] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.* **74** (1960), pp. 91-98.

[13] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.H.

[14] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.

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