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ON THE HADAMARD'S TYPE INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this paper, we obtain two identities for function of two variables and apply them to give new Hermite-Hadamard type Fractional integral inequalities for double Fractional integrals involving functions whose derivatives are bounded or co-ordinates convex function on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b,\ c < d$.

1. Introduction

The convexity property of a given function plays an important role in obtaining integral inequalities. Proving inequalities for convex functions has a long and rich history in mathamatics. We refer the reader to a monograph written by Dragomir and Pearce [11]. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

is known in the literature as Hermite-Hadamard inequality for convex mappings. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave.

It is well known that the Hermite-Hadamard's inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, (see, [1]-[10]) and the references therein.

Let us now consider a bidemensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x,y) + (1-t)f(z,w)$$

holds, for all $(x,y), (z,w) \in \Delta$ and $t \in [0,1]$.A function $f : \Delta \to \mathbb{R}$ is said to be on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R},\ f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R},\ f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b]$ and $y \in [c,d]$ (see [4]).

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A formal definition for co-ordinated convex function may be stated as follows:

Definition 1. *A function $f : \Delta \to \mathbb{R}$ will be called co-ordinated canvex on Δ, for all $t, s \in [0,1]$ and $(x,y),(u,w) \in \Delta$, if the following inequality holds:*

$$\begin{aligned} & f(tx + (1-t)\,y, su + (1-s)\,w) \\ (1.2) \quad \leq \quad & tsf(x,u) + s(1-t)f(y,u) + t(1-s)f(x,w) + (1-t)(1-s)f(y,w). \end{aligned}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex function which is not convex, (see, [11]). For several recent results concerning Hermite-Hadamard's inequality for some convex function on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$, we refer the reader to ([11]-[18]).

In [11], Dragomir establish the following inequality of Hermite–Hadamard type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$. Later, in [12], Sarikaya and Yaldiz gave the another proof of a special version of the below theorem by using the definition of the co-ordinated convex function.

Theorem 1. *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ & \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ (1.3) \quad & \leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx \right. \\ & \left. + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right] \\ & \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \end{aligned}$$

The above inequalities are sharp.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [22]-[25].

Definition 2. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a^+}^{0} f(x) = J_{b^-}^{\alpha} f(x) = f(x)$.

It is remarkable that Sarikaya et al. [21] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2}$$

with $\alpha > 0$.

Meanwhile, Sarikaya et al. [21] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L_1[a,b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned} & \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \\ = \ & \frac{b-a}{2} \int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(ta + (1-t)b) dt. \end{aligned}$$

Definition 3. *Let $f \in L_1([a,b] \times [c,d])$. The Riemann-Liouville integrals $J_{a^+,c^+}^{\alpha,\beta}$, $J_{a^+,d^-}^{\alpha,\beta}$, $J_{b^-,c^+}^{\alpha,\beta}$ and $J_{b^-,d^-}^{\alpha,\beta}$ of order $\alpha, \beta > 0$ with $a, c \geq 0$ are defined by*

$$J_{a^+,c^+}^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1} (y-s)^{\beta-1} f(t,s) ds dt, \ x > a, \ y > c$$

$$J_{a^+,d^-}^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x-t)^{\alpha-1} (s-y)^{\beta-1} f(t,s) ds dt, \ x > a, \ y < d$$

$$J_{b^-,c^+}^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t-x)^{\alpha-1} (y-s)^{\beta-1} f(t,s) ds dt, \ x < b, \ y > c$$

and

$$J_{b^-,d^-}^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (t-x)^{\alpha-1} (s-y)^{\beta-1} f(t,s) ds dt, \ x<b, \ y<d$$

respectively. Here, Γ *is the Gamma function,*

$$J_{a^+,c^+}^{0,0} f(x,y) = J_{a^+,d^-}^{0,0} f(x,y) = J_{b^-,c^+}^{0,0} f(x,y) = J_{b^-,d^-}^{0,0} f(x,y) = f(x,y)$$

and

$$J_{a^+,c^+}^{1,1} f(x,y) = \int_a^x \int_c^y f(t,s) ds dt.$$

Similar to Definition 2 and Definition 3, we introduce the following fractional integrals:

$$J_{a^+}^{\alpha} f(x, \frac{c+d}{2}) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t, \frac{c+d}{2}) dt, \ x>a,$$

$$J_{b^-}^{\alpha} f(x, \frac{c+d}{2}) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t, \frac{c+d}{2}) dt, \ x<b,$$

$$J_{c^+}^{\beta} f(\frac{a+b}{2}, y) = \frac{1}{\Gamma(\beta)} \int_c^y (y-s)^{\beta-1} f(\frac{a+b}{2}, s) ds, \ y>c,$$

and

$$J_{d^-}^{\beta} f(\frac{a+b}{2}, y) = \frac{1}{\Gamma(\beta)} \int_y^d (s-y)^{\beta-1} f(\frac{a+b}{2}, s) ds, \ y<d.$$

Our goal in this paper is to state and prove the Hermite-Hadamard type inequality for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$. In order to achieve our goal, we first give two important identities and then by using these identities we prove some integral inequalities. We have obtained some results which are a simpler proof of the results in [20].

2. Main Results

To establish our main results, we need the following first identity:

Lemma 2. *Let* $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a,b] \times [c,d]$ *in* $\mathbb{R}^2$ *with* $a<b$ *and* $c<d$ *and* $f_{\sigma\tau} \in L(\Delta)$*. Then the following*

equality holds:
(2.1)

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]$$

$$-\frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b,c)+J^{\alpha}_{a^+}f(b,d)+J^{\alpha}_{b^-}f(a,c)+J^{\alpha}_{b^-}f(a,d)\right]$$

$$-\frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J^{\beta}_{c^+}f(a,d)+J^{\beta}_{c^+}f(b,d)+J^{\beta}_{d^-}f(a,c)+J^{\beta}_{d^-}f(b,c)\right]+F$$

$$=\frac{\alpha\beta}{16(b-a)^{\alpha}(d-c)^{\beta}}\left\{\int_a^b\int_c^d\left[(b-x)^{\alpha-1}(d-y)^{\beta-1}\right.\right.$$

$$\left.+(b-x)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(d-y)^{\beta-1}\right]$$

$$\times I(x,y)dydx\}$$

where

$$\begin{aligned} I(x,y) \quad &= \int_a^x\int_c^y f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma+\int_a^x\int_d^y f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma \\ &+\int_b^x\int_c^y f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma+\int_b^x\int_d^y f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma, \end{aligned}$$

and

$$F=f(a,c)+f(a,d)+f(b,c)+f(b,d).$$

Proof. For any $x,t\in[a,b]$ and $s,y\in[c,d]\,,\ x\neq t,\ s\neq y$, we have

$$\begin{aligned} \int_t^x\int_s^y f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma \quad &= \int_t^x\left[f_{\sigma}(\sigma,y)-f_{\sigma}(\sigma,s)\right]d\sigma \\ &= \left[f(\sigma,y)-f(\sigma,s)\right]\Big|_t^x \\ &= f(x,y)-f(x,s)-f(t,y)+f(t,s). \end{aligned} \tag{2.2}$$

Choose $t=a,\ s=c;\ t=a,\ s=d;\ t=b,\ s=c;\ t=b,\ s=d$ in (2.2), respectively, we get

$$I_1=\int_a^x\int_c^y f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma=f(x,y)-f(x,c)-f(a,y)+f(a,c),$$

$$I_2=\int_a^x\int_d^y f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma=f(x,y)-f(x,d)-f(a,y)+f(a,d),$$

$$I_3=\int_b^x\int_c^y f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma=f(x,y)-f(x,c)-f(b,y)+f(b,c),$$

and

$$I_4=\int_b^x\int_d^y f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma=f(x,y)-f(x,d)-f(b,y)+f(b,d).$$

Adding these four integrals side by side, we obtain

$$\begin{aligned}I(x,y)&=I_1+I_2+I_3+I_4\\&=4f(x,y)-2\left[f(x,c)+f(x,d)\right]-2\left[f(a,y)+f(b,y)\right]\\&+f(a,c)+f(a,d)+f(b,c)+f(b,d).\end{aligned}\tag{2.3}$$

Multiplying (2.3) by $\frac{(b-x)^{\alpha-1}(d-y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$ and integrating the resulting equality with respect to (x,y) on $[a,b]\times[c,d]$, we have

$$\begin{aligned}&\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}I(x,y)dydx\\&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}f(x,y)dydx\\&-\frac{1}{2\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}\left[f(x,c)+f(x,d)\right]dx\\&-\frac{1}{2\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}\left[f(a,y)+f(b,y)\right]dydx\\&+\frac{F}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}dydx.\end{aligned}\tag{2.4}$$

Thus, in (2.4) by means of simple calculations, we have

$$\begin{aligned}&J_{a^+,c^+}^{\alpha,\beta}(b,d)-\frac{(d-c)^{\beta}}{2\Gamma(\beta+1)}\left[J_{a^+}^{\alpha}(b,c)+J_{a^+}^{\alpha}(b,d)\right]\\&-\frac{(b-a)^{\alpha}}{2\Gamma(\alpha+1)}\left[J_{c^+}^{\beta}(a,d)+J_{c^+}^{\beta}(b,d)\right]+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}F\\&=\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}I(x,y)dydx.\end{aligned}\tag{2.5}$$

Multiplying (2.3) by $\frac{(b-x)^{\alpha-1}(y-c)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$ and integrating the resulting equality with respect to (x,y) on $[a,b]\times[c,d]$, and by similar calculations, we have

$$\begin{aligned}
&J^{\alpha,\beta}_{a^+,d^-}f(b,c)-\frac{(d-c)^{\beta}}{2\Gamma(\beta+1)}\left[J^{\alpha}_{a^+}f(b,c)+J^{\alpha}_{a^+}f(b,d)\right]\\
&-\frac{(b-a)^{\alpha}}{2\Gamma(\alpha+1)}\left[J^{\beta}_{d^-}f(a,c)+J^{\beta}_{d^-}f(b,c)\right]+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}F \qquad (2.6)\\
&=\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-x)^{\alpha-1}(y-c)^{\beta-1}I(x,y)dydx.
\end{aligned}$$

Multiplying (2.3) by $\frac{(x-a)^{\alpha-1}(y-c)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$ and integrating the resulting equality with respect to (x,y) on $[a,b]\times[c,d]$, we have

$$\begin{aligned}
&J^{\alpha,\beta}_{b^-,c^+}f(a,d)-\frac{(d-c)^{\beta}}{2\Gamma(\beta+1)}\left[J^{\alpha}_{b^-}f(a,c)+J^{\alpha}_{b^-}f(a,d)\right]\\
&-\frac{(b-a)^{\alpha}}{2\Gamma(\alpha+1)}\left[J^{\beta}_{c^+}f(a,d)+J^{\beta}_{c^+}f(b,d)\right]+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}F \qquad (2.7)\\
&=\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(x-a)^{\alpha-1}(y-c)^{\beta-1}I(x,y)dydx.
\end{aligned}$$

Multiplying (2.3) by $\frac{(x-a)^{\alpha-1}(d-y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$ and integrating the resulting equality with respect to (x,y) on $[a,b]\times[c,d]$, we have

$$\begin{aligned}
&J^{\alpha,\beta}_{b^-,d^-}f(a,c)-\frac{(d-c)^{\beta}}{2\Gamma(\beta+1)}\left[J^{\alpha}_{b^-}f(a,c)+J^{\alpha}_{b^-}f(a,d)\right]\\
&-\frac{(b-a)^{\alpha}}{2\Gamma(\alpha+1)}\left[J^{\beta}_{d^-}f(a,c)+J^{\beta}_{d^-}f(b,c)\right]+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}F \qquad (2.8)\\
&=\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(x-a)^{\alpha-1}(d-y)^{\beta-1}I(x,y)dydx.
\end{aligned}$$

Adding these (2.5), (2.6), (2.7) and (2.8) side by side, which completes the proof. □

Corollary 1. *If we take $\alpha=\beta=1$ in Lemma 2, we get*

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-\frac{1}{2(b-a)}\int_a^b\left[f(x,c)+f(x,d)\right]dx\\&-\frac{1}{2(d-c)}\int_c^d\left[f(a,y)+f(b,y)\right]dy+F\\&=\frac{1}{16(b-a)(d-c)}\int_a^b\int_c^d I(x,y)dydx.\end{aligned}$$

Theorem 3. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$ and $f_{\sigma\tau}\in L(\Delta)$. If $f_{\sigma\tau}\in L_\infty(\Delta)$, i.e $\|f_{\sigma\tau}\|_\infty=\sup_{(\sigma,\tau)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(\sigma,\tau)}{\partial\sigma\partial\tau}\right|<\infty$, then one has the inequality:*

(2.9)
$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]\right.\\&-\frac{\Gamma(\alpha+1)}{4(b-a)^\alpha}\left[J^\alpha_{a^+}f(b,c)+J^\alpha_{a^+}f(b,d)+J^\alpha_{b^-}f(a,c)+J^\alpha_{b^-}f(a,d)\right]\\&\left.-\frac{\Gamma(\beta+1)}{4(d-c)^\beta}\left[J^\beta_{c^+}f(a,d)+J^\beta_{c^+}f(b,d)+J^\beta_{d^-}f(a,c)+J^\beta_{d^-}f(b,c)\right]+F\right|\\&\le\frac{\|f_{\sigma\tau}\|_\infty}{4}(b-a)(d-c).\end{aligned}$$

Proof. From Lemma 2, taking the modulus, it follows that

$$\begin{aligned}&|J|\\&=\left|\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]\right.\\&-\frac{\Gamma(\alpha+1)}{4(b-a)^\alpha}\left[J^\alpha_{a^+}f(b,c)+J^\alpha_{a^+}f(b,d)+J^\alpha_{b^-}f(a,c)+J^\alpha_{b^-}f(a,d)\right]\\&\left.-\frac{\Gamma(\beta+1)}{4(d-c)^\beta}\left[J^\beta_{c^+}f(a,d)+J^\beta_{c^+}f(b,d)+J^\beta_{d^-}f(a,c)+J^\beta_{d^-}f(b,c)\right]+F\right|\end{aligned}$$

$$\leq \frac{\alpha\beta}{16(b-a)^{\alpha}(d-c)^{\beta}}\left\{\int_a^b\int_c^d\left[(b-x)^{\alpha-1}(d-y)^{\beta-1}\right.\right.$$

$$\left.+(b-x)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(d-y)^{\beta-1}\right]$$

$$\times\left[\int_a^x\int_c^y|f_{\sigma\tau}(\sigma,\tau)|\,d\tau d\sigma+\int_a^x\int_y^d|f_{\sigma\tau}(\sigma,\tau)|\,d\tau d\sigma\right.$$

$$\left.\left.+\int_x^b\int_c^y|f_{\sigma\tau}(\sigma,\tau)|\,d\tau d\sigma+\int_x^b\int_y^d|f_{\sigma\tau}(\sigma,\tau)|\,d\tau d\sigma\right]dydx\right\}.$$

Since $f_{\sigma\tau}\in L_{\infty}(\Delta)$, we get

$$|J|$$

$$\leq \frac{\alpha\beta\,||f_{\sigma\tau}||_{\infty}}{16(b-a)^{\alpha}(d-c)^{\beta}}\left\{\int_a^b\int_c^d(b-x)^{\alpha-1}(d-y)^{\beta-1}\left[\int_a^b\int_c^d d\tau d\sigma\right]dydx\right.$$

$$+\int_a^b\int_c^d(b-x)^{\alpha-1}(y-c)^{\beta-1}\left[\int_a^b\int_c^d d\tau d\sigma\right]dydx$$

$$+\int_a^b\int_c^d(x-a)^{\alpha-1}(y-c)^{\beta-1}\left[\int_a^b\int_c^d d\tau d\sigma\right]dydx$$

$$\left.+\int_a^b\int_c^d(x-a)^{\alpha-1}(d-y)^{\beta-1}\left[\int_a^b\int_c^d d\tau d\sigma\right]dydx\right\}$$

$$=\frac{\alpha\beta\,||f_{\sigma\tau}||_{\infty}}{16(b-a)^{\alpha}(d-c)^{\beta}}\frac{4(b-a)^{\alpha+1}}{\alpha}\frac{(d-c)^{\beta+1}}{\beta}$$

$$=\frac{||f_{\sigma\tau}||_{\infty}}{4}(b-a)(d-c).$$

This completes the proof. □

Corollary 2. *If we take $\alpha = \beta = 1$ in Theorem 3, we get*

$$\begin{aligned}&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx - \frac{1}{2(b-a)}\int_a^b [f(x,c)+f(x,d)]\,dx\right.\\&\left.-\frac{1}{2(d-c)}\int_c^d [f(a,y)+f(b,y)]\,dy + F\right|\\\leq\ &\frac{\|f_{\sigma\tau}\|_\infty}{4}(b-a)(d-c).\end{aligned}$$

Theorem 4. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$ and $f_{\sigma\tau}\in L(\Delta)$. If $|f_{\sigma\tau}|$ is a convex function on the co-ordinates on Δ, then the following inequality holds:*

(2.10)
$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]\right.\\&-\frac{\Gamma(\alpha+1)}{4(b-a)^\alpha}\left[J^\alpha_{a^+}f(b,c)+J^\alpha_{a^+}f(b,d)+J^\alpha_{b^-}f(a,c)+J^\alpha_{b^-}f(a,d)\right]\\&\left.-\frac{\Gamma(\beta+1)}{4(d-c)^\beta}\left[J^\beta_{c^+}f(a,d)+J^\beta_{c^+}f(b,d)+J^\beta_{d^-}f(a,c)+J^\beta_{d^-}f(b,c)\right]+F\right|\\&\leq\frac{(b-a)(d-c)}{4}\left[\frac{|f_{\sigma\tau}(a,c)+|\,|f_{\sigma\tau}(a,d)|+|f_{\sigma\tau}(b,c)|+|f_{\sigma\tau}(b,d)|}{4}\right]\end{aligned}$$

Proof. Since $|f_{\sigma\tau}(\sigma,\tau)|$ is co-ordinates on Δ, we know that $x\in[a,b],\ y\in[c,d]$

$$\begin{aligned}|f_{\sigma\tau}(\sigma,\tau)| &= \left|f_{\sigma\tau}\left(\frac{b-\sigma}{b-a}a+\frac{\sigma-a}{b-a}b,\frac{d-\tau}{d-c}c+\frac{\tau-c}{d-c}d\right)\right|\\&\leq\frac{b-\sigma}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(a,c)|+\frac{b-\sigma}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(a,d)|\\&+\frac{\sigma-a}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(b,c)|+\frac{\sigma-a}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(b,d)|.\end{aligned}\tag{2.11}$$

From Lemma 2, we have

(2.12)

$$
\begin{aligned}
&|J| \\
&\leq \frac{\alpha\beta}{16(b-a)^{\alpha}(d-c)^{\beta}}\left\{\int_a^b\int_c^d \Big[(b-x)^{\alpha-1}(d-y)^{\beta-1}\right. \\
&+(b-x)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(d-y)^{\beta-1}\Big] \\
&\times\left[\int_a^x\int_c^y |f_{\sigma\tau}(\sigma,\tau)|\,d\tau d\sigma+\int_a^x\int_y^d |f_{\sigma\tau}(\sigma,\tau)|\,d\tau d\sigma\right. \\
&\left.\left.+\int_x^b\int_c^y |f_{\sigma\tau}(\sigma,\tau)|\,d\tau d\sigma+\int_x^b\int_y^d |f_{\sigma\tau}(\sigma,\tau)|\,d\tau d\sigma\right] dydx\right\}
\end{aligned}
$$

By using co-ordinated convexity of $|f_{\sigma\tau}|$, we get

$$
\begin{aligned}
|J| &\leq \frac{\alpha\beta}{16(b-a)^{\alpha}(d-c)^{\beta}}\left\{\int_a^b\int_c^d \Big[(b-x)^{\alpha-1}(d-y)^{\beta-1}\right. \\
&+(b-x)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(d-y)^{\beta-1}\Big] \\
&\times\left(\int_a^x\int_c^y \left[\frac{b-\sigma}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(a,c)|+\frac{b-\sigma}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(a,d)|\right.\right. \\
&\left.+\frac{\sigma-a}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(b,c)|+\frac{\sigma-a}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(b,d)|\right] d\tau d\sigma \\
&+\int_a^x\int_y^d \left[\frac{b-\sigma}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(a,c)|+\frac{b-\sigma}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(a,d)|\right. \\
&\left.+\frac{\sigma-a}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(b,c)|+\frac{\sigma-a}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(b,d)|\right] d\tau d\sigma
\end{aligned}
$$

$$+\int_{x}^{b}\int_{c}^{y}\left[\frac{b-\sigma}{b-a}\frac{d-\tau}{d-c}\left|f_{\sigma\tau}(a,c)\right|+\frac{b-\sigma}{b-a}\frac{\tau-c}{d-c}\left|f_{\sigma\tau}(a,d)\right|\right.$$

$$\left.+\frac{\sigma-a}{b-a}\frac{d-\tau}{d-c}\left|f_{\sigma\tau}(b,c)\right|+\frac{\sigma-a}{b-a}\frac{\tau-c}{d-c}\left|f_{\sigma\tau}(b,d)\right|\right]d\tau d\sigma$$

$$+\int_{x}^{b}\int_{y}^{d}\left[\frac{b-\sigma}{b-a}\frac{d-\tau}{d-c}\left|f_{\sigma\tau}(a,c)\right|+\frac{b-\sigma}{b-a}\frac{\tau-c}{d-c}\left|f_{\sigma\tau}(a,d)\right|\right.$$

$$\left.\left.\left.+\frac{\sigma-a}{b-a}\frac{d-\tau}{d-c}\left|f_{\sigma\tau}(b,c)\right|+\frac{\sigma-a}{b-a}\frac{\tau-c}{d-c}\left|f_{\sigma\tau}(b,d)\right|\right]d\tau d\sigma\right)dydx\right\}$$

(2.13)

$$=\frac{\alpha\beta}{16(b-a)^{\alpha}(d-c)^{\beta}}\left\{\int_{a}^{b}\int_{c}^{d}\left[(b-x)^{\alpha-1}(d-y)^{\beta-1}\right.\right.$$

$$\left.+(b-x)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(d-y)^{\beta-1}\right]$$

$$\times\left(\int_{a}^{b}\int_{c}^{d}\left[\frac{b-\sigma}{b-a}\frac{d-\tau}{d-c}\left|f_{\sigma\tau}(a,c)\right|+\frac{b-\sigma}{b-a}\frac{\tau-c}{d-c}\left|f_{\sigma\tau}(a,d)\right|\right.\right.$$

$$\left.\left.+\frac{\sigma-a}{b-a}\frac{d-\tau}{d-c}\left|f_{\sigma\tau}(b,c)\right|+\frac{\sigma-a}{b-a}\frac{\tau-c}{d-c}\left|f_{\sigma\tau}(b,d)\right|\right]d\tau d\sigma\right\}dydx$$

$$=A_1+A_2+A_3+A_4.$$

With a simple calculation, we have

$$A_1=\frac{\alpha\beta}{16(b-a)^{\alpha+1}(d-c)^{\beta+1}}\int_{a}^{b}\int_{c}^{d}(b-x)^{\alpha-1}(d-y)^{\beta-1}$$

$$\times\left\{\int_{a}^{b}\int_{c}^{d}\left[(b-\sigma)(d-\tau)\left|f_{\sigma\tau}(a,c)\right|+(b-\sigma)(\tau-c)\left|f_{\sigma\tau}(a,d)\right|\right.\right.$$

$$\left.\left.+(\sigma-a)(d-\tau)\left|f_{\sigma\tau}(b,c)\right|+(\sigma-a)(\tau-c)\left|f_{\sigma\tau}(b,d)\right|\right]d\tau d\sigma\right\}dydx$$

$$= \frac{\alpha\beta}{16\,(b-a)^{\alpha+1}\,(d-c)^{\beta+1}}$$

(2.14)
$$\times\left\{\frac{(b-a)^{\alpha+2}}{2\alpha}\frac{(d-c)^{\beta+2}}{2\beta}\left[|f_{\sigma\tau}(a,c)+|\,|f_{\sigma\tau}(a,d)|\right.\right.$$

$$+|f_{\sigma\tau}(b,c)|+|f_{\sigma\tau}(b,d)|]$$
$$= \frac{(b-a)(d-c)}{16}\left[\frac{|f_{\sigma\tau}(a,c)+|\,|f_{\sigma\tau}(a,d)|+|f_{\sigma\tau}(b,c)|+|f_{\sigma\tau}(b,d)|}{4}\right].$$

Similarly, we also have the following equalities
(2.15)
$$A_2 = \frac{\alpha\beta}{16\,(b-a)^{\alpha+1}\,(d-c)^{\beta+1}}\int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1}$$

$$\times\left\{\int_a^b\int_c^d\left[(b-\sigma)(d-\tau)|f_{\sigma\tau}(a,c)|+(b-\sigma)(\tau-c)|f_{\sigma\tau}(a,d)|\right.\right.$$

$$\left.\left.+(\sigma-a)(d-\tau)|f_{\sigma\tau}(b,c)|+(\sigma-a)(\tau-c)|f_{\sigma\tau}(b,d)|\right]d\tau d\sigma\right\}dydx$$

$$= \frac{(b-a)(d-c)}{16}\left[\frac{|f_{\sigma\tau}(a,c)+|\,|f_{\sigma\tau}(a,d)|+|f_{\sigma\tau}(b,c)|+|f_{\sigma\tau}(b,d)|}{4}\right],$$

(2.16)
$$A_3 = \frac{\alpha\beta}{16\,(b-a)^{\alpha+1}\,(d-c)^{\beta+1}}\int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1}$$

$$\times\left\{\int_a^b\int_c^d\left[(b-\sigma)(d-\tau)|f_{\sigma\tau}(a,c)|+(b-\sigma)(\tau-c)|f_{\sigma\tau}(a,d)|\right.\right.$$

$$\left.\left.+(\sigma-a)(d-\tau)|f_{\sigma\tau}(b,c)|+(\sigma-a)(\tau-c)|f_{\sigma\tau}(b,d)|\right]d\tau d\sigma\right\}dydx$$

$$= \frac{(b-a)(d-c)}{16}\left[\frac{|f_{\sigma\tau}(a,c)+|\,|f_{\sigma\tau}(a,d)|+|f_{\sigma\tau}(b,c)|+|f_{\sigma\tau}(b,d)|}{4}\right]$$

and
(2.17)

$$\begin{aligned}
A_4 &= \frac{\alpha\beta}{16\left(b-a\right)^{\alpha+1}\left(d-c\right)^{\beta+1}}\int_a^b\int_c^d \left(x-a\right)^{\alpha-1}\left(d-y\right)^{\beta-1} \\
&\times\left\{\int_a^b\int_c^d \left[\left(b-\sigma\right)\left(d-\tau\right)\left|f_{\sigma\tau}\left(a,c\right)\right| + \left(b-\sigma\right)\left(\tau-c\right)\left|f_{\sigma\tau}\left(a,d\right)\right|\right.\right. \\
&\left.\left. + \left(\sigma-a\right)\left(d-\tau\right)\left|f_{\sigma\tau}\left(b,c\right)\right| + \left(\sigma-a\right)\left(\tau-c\right)\left|f_{\sigma\tau}\left(b,d\right)\right|\right] d\tau d\sigma\right\} dydx \\
&= \frac{\left(b-a\right)\left(d-c\right)}{16}\left[\frac{\left|f_{\sigma\tau}\left(a,c\right)+\right|\left|f_{\sigma\tau}\left(a,d\right)\right| + \left|f_{\sigma\tau}\left(b,c\right)\right| + \left|f_{\sigma\tau}\left(b,d\right)\right|}{4}\right].
\end{aligned}$$

Adding these (2.14), (2.15), (2.16) and (2.17) side by side, if we put in (2.13), we obtain (2.10). This completes the proof of the theorem. □

Corollary 3. *If we take $\alpha=\beta=1$ in Theorem 4, we get*

$$\begin{aligned}
&\left|\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_c^d f(x,y)dydx - \frac{1}{2\left(b-a\right)}\int_a^b \left[f\left(x,c\right)+f\left(x,d\right)\right]dx\right. \\
&\left. - \frac{1}{2\left(d-c\right)}\int_c^d \left[f\left(a,y\right)+f\left(b,y\right)\right]dy + F\right| \\
\leq\ & \frac{\left(b-a\right)\left(d-c\right)}{4}\left[\frac{\left|f_{\sigma\tau}\left(a,c\right)+\right|\left|f_{\sigma\tau}\left(a,d\right)\right| + \left|f_{\sigma\tau}\left(b,c\right)\right| + \left|f_{\sigma\tau}\left(b,d\right)\right|}{4}\right].
\end{aligned}$$

Lemma 3. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$, $c<d$ and $f_{\sigma\tau}\in L(\Delta)$. Then the following equality*

holds:

(2.18)

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\left\{J^{\beta}_{c^+}f\left(\tfrac{a+b}{2},d\right)+J^{\beta}_{d^-}f\left(\tfrac{a+b}{2},c\right)\right\}$$

$$-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left\{J^{\alpha}_{b^-}f\left(a,\tfrac{c+d}{2}\right)+J^{\alpha}_{a^+}f\left(b,\tfrac{c+d}{2}\right)\right\}$$

$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]$$

$$=\frac{\alpha\beta}{4(b-a)^{\alpha}(d-c)^{\beta}}\int_a^b\int_c^d\left\{\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right]\right.$$

$$\left.\times\left(\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma\right)\right\}dsdt.$$

Proof. Choose $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (2.2), we have

(2.19)

$$\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma=f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-f\left(\tfrac{a+b}{2},s\right)-f\left(t,\tfrac{c+d}{2}\right)+f(t,s).$$

Multiplying (2.19) by $\dfrac{(b-t)^{\alpha-1}(d-s)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$ and integrating the resulting equality with respect to (s,t) on $[a,b]\times[c,d]$, we get

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-t)^{\alpha-1}(d-s)^{\beta-1}\left\{\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma\right\}dsdt$$

$$=\frac{f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-t)^{\alpha-1}(d-s)^{\beta-1}dsdt$$

$$-\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-t)^{\alpha-1}(d-s)^{\beta-1}f\left(\tfrac{a+b}{2},s\right)dsdt$$

$$-\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-t)^{\alpha-1}(d-s)^{\beta-1}f\left(t,\tfrac{c+d}{2}\right)dsdt$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-t)^{\alpha-1}(d-s)^{\beta-1}f(t,s)dsdt.$$

By simple calculations, we have

$$\begin{aligned}
&\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}J_{c^{+}}^{\beta}f\left(\tfrac{a+b}{2},d\right)\\
&-\frac{(d-c)^{\beta}}{\Gamma(\beta+1)}J_{a^{+}}^{\alpha}f\left(b,\tfrac{c+d}{2}\right)+J_{a^{+},c^{+}}^{\alpha,\beta}f(b,d)\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{a}^{b}\int_{c}^{d}(b-t)^{\alpha-1}(d-s)^{\beta-1}\left\{\int_{t}^{\frac{a+b}{2}}\int_{s}^{\frac{c+d}{2}}f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma\right\}dsdt.
\end{aligned}\tag{2.20}$$

Multiplying (2.19) by $\dfrac{(b-t)^{\alpha-1}(s-c)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$, integrating the resulting equality with respect to (s,t) on $[a,b]\times[c,d]$, and by similar methods above we have

$$\begin{aligned}
&\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}J_{d^{-}}^{\beta}f\left(\tfrac{a+b}{2},c\right)\\
&-\frac{(d-c)^{\beta}}{\Gamma(\beta+1)}J_{a^{+}}^{\alpha}f\left(b,\tfrac{c+d}{2}\right)+J_{a^{+},d^{-}}^{\alpha,\beta}f(b,c)\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{a}^{b}\int_{c}^{d}(b-t)^{\alpha-1}(d-s)^{\beta-1}\left\{\int_{t}^{\frac{a+b}{2}}\int_{s}^{\frac{c+d}{2}}f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma\right\}dsdt.
\end{aligned}\tag{2.21}$$

Multiplying (2.19) by $\dfrac{(t-a)^{\alpha-1}(d-s)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$ integrating the resulting equality with respect to (s,t) on $[a,b]\times[c,d]$, and by similar methods above we have

$$\begin{aligned}
&\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}J_{c^{+}}^{\beta}f\left(\tfrac{a+b}{2},d\right)\\
&-\frac{(d-c)^{\beta}}{\Gamma(\beta+1)}J_{b^{-}}^{\alpha}f\left(a,\tfrac{c+d}{2}\right)+J_{b^{-},c^{+}}^{\alpha,\beta}f(a,d)\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{a}^{b}\int_{c}^{d}(b-t)^{\alpha-1}(d-s)^{\beta-1}\left\{\int_{t}^{\frac{a+b}{2}}\int_{s}^{\frac{c+d}{2}}f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma\right\}dsdt.
\end{aligned}\tag{2.22}$$

Multiplying (2.19) by $\dfrac{(t-a)^{\alpha-1}(s-c)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$ integrating the resulting equality with respect to (s,t) on $[a,b]\times[c,d]$, and by similar methods above we have

$$\begin{aligned}
&\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} J_{d^-}^{\beta} f\left(\tfrac{a+b}{2},c\right)\\
&-\frac{(d-c)^{\beta}}{\Gamma(\beta+1)} J_{b^-}^{\alpha} f\left(a,\tfrac{c+d}{2}\right)+J_{b^-,d^-}^{\alpha,\beta} f(a,c) \qquad (2.23)\\
&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (b-t)^{\alpha-1}(d-s)^{\beta-1}\left\{\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}} f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma\right\} dsdt.
\end{aligned}$$

Adding these (2.20), (2.21), (2.22) and (2.23) side by side and multiplying both sides by $\dfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}$, we get the desired equality (2.18). □

Corollary 4. *If we take $\alpha=\beta=1$ in Lemma 3, we get*

$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\\
&-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\\
=\ & \frac{1}{16(b-a)(d-c)}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d\left\{\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}} f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma\right\} dsdt.
\end{aligned}$$

Theorem 5. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $f_{\sigma\tau}\in L_{\infty}(\Delta)$, then the following equality holds:*

(2.24)
$$\begin{aligned}
&\Bigg| f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\left\{J_{c^+}^{\beta} f\left(\tfrac{a+b}{2},d\right)+J_{d^-}^{\beta} f\left(\tfrac{a+b}{2},c\right)\right\}\\
&-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left\{J_{b^-}^{\alpha} f\left(a,\tfrac{c+d}{2}\right)+J_{a^+}^{\alpha} f\left(b,\tfrac{c+d}{2}\right)\right\}\\
&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[J_{a^+,c^+}^{\alpha,\beta} f(b,d)+J_{a^+,d^-}^{\alpha,\beta} f(b,c)+J_{b^-,c^+}^{\alpha,\beta} f(a,d)+J_{b^-,d^-}^{\alpha,\beta} f(a,c)\right]\Bigg|\\
&\le\frac{\|f_{\sigma\tau}\|_{\infty}(b-a)(d-c)}{4}\left[\frac{\left(2^{1-\alpha}+(\alpha-1)\right)}{\alpha+1}\frac{\left(2^{1-\beta}+(\beta-1)\right)}{\beta+1}\right].
\end{aligned}$$

Proof. In Lemma 3, taking the modulus, it follows that

$$\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\left\{J^{\beta}_{c^+} f\left(\tfrac{a+b}{2},d\right) + J^{\beta}_{d^-} f\left(\tfrac{a+b}{2},c\right)\right\}\right.$$

$$-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left\{J^{\alpha}_{b^-} f\left(a,\tfrac{c+d}{2}\right) + J^{\alpha}_{a^+} f\left(b,\tfrac{c+d}{2}\right)\right\}$$

$$\left.+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[J^{\alpha,\beta}_{a^+,c^+} f(b,d) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) + J^{\alpha,\beta}_{b^-,c^+} f(a,d) + J^{\alpha,\beta}_{b^-,d^-} f(a,c)\right]\right|$$

(2.25)

$$\leq \frac{\alpha\beta\,||f_{\sigma\tau}||_{\infty}}{4(b-a)^{\alpha}(d-c)^{\beta}}\int_a^b\int_c^d \left\{\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right]\right.$$

$$\left.\times\left|\frac{a+b}{2}-t\right|\left|\frac{c+d}{2}-s\right|\right\} dsdt$$

$$= \frac{||f_{\sigma\tau}||_{\infty}(b-a)(d-c)}{4}\left[\frac{\left(2^{1-\alpha}+(\alpha-1)\right)}{\alpha+1}\frac{\left(2^{1-\beta}+(\beta-1)\right)}{\beta+1}\right]$$

for $f_{\sigma\tau} \in L_{\infty}(\Delta)$. □

Remark 1. *If we take $\alpha=\beta=1$ in Theorem 5, we get*

$$\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\right.$$

$$\left.-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right|$$

$$\leq \frac{||f_{\sigma\tau}||_{\infty}}{16}(b-a)(d-c).$$

which is proved by Sarikaya in [20]*.*

Theorem 6. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $|f_{\sigma\tau}|$ is a convex function on the co-ordinates on Δ, then the following equality holds:*

$$\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\left\{J^{\beta}_{c^+} f\left(\tfrac{a+b}{2},d\right) + J^{\beta}_{d^-} f\left(\tfrac{a+b}{2},c\right)\right\}\right.$$

$$-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left\{J^{\alpha}_{b^-} f\left(a,\tfrac{c+d}{2}\right) + J^{\alpha}_{a^+} f\left(b,\tfrac{c+d}{2}\right)\right\}$$

$$\left.+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[J^{\alpha,\beta}_{a^+,c^+} f(b,d) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) + J^{\alpha,\beta}_{b^-,c^+} f(a,d) + J^{\alpha,\beta}_{b^-,d^-} f(a,c)\right]\right|$$

$$\leq (b-a)(d-c)\frac{\alpha 2^{\alpha}-(\alpha+1)2^{\alpha-1}+1}{2^{\alpha}(\alpha+1)}\frac{\beta 2^{\beta}-(\beta+1)2^{\beta-1}+1}{2^{\beta}(\beta+1)} \tag{2.26}$$
$$\times\frac{|f_{\sigma\tau}(a,c)|+|f_{\sigma\tau}(a,d)|+|f_{\sigma\tau}(b,c)|+|f_{\sigma\tau}(b,d)|}{4}.$$

Proof. Since $|f_{\sigma\tau}(\sigma,\tau)|$ is co-ordinates on Δ, we know that $t\in[a,b]$, $s\in[c,d]$

$$\begin{aligned}
|f_{\sigma\tau}(\sigma,\tau)| &= \left|f_{\sigma\tau}\left(\frac{b-\sigma}{b-a}a+\frac{\sigma-a}{b-a}b,\frac{d-\tau}{d-c}c+\frac{\tau-c}{d-c}d\right)\right| \\
&\leq \frac{b-\sigma}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(a,c)|+\frac{b-\sigma}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(a,d)| \\
&\quad+\frac{\sigma-a}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(b,c)|+\frac{\sigma-a}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(b,d)|.
\end{aligned} \tag{2.27}$$

From Lemma 3, using co-ordinated convexity of $|f_{\sigma\tau}|$, we have

$$\begin{aligned}
&\Bigg|f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)-\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\left\{J_{c^+}^{\beta}f\left(\tfrac{a+b}{2},d\right)+J_{d^-}^{\beta}f\left(\tfrac{a+b}{2},c\right)\right\} \\
&-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left\{J_{b^-}^{\alpha}f\left(a,\tfrac{c+d}{2}\right)+J_{a^+}^{\alpha}f\left(b,\tfrac{c+d}{2}\right)\right\} \\
&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[J_{a^+,c^+}^{\alpha,\beta}f(b,d)+J_{a^+,d^-}^{\alpha,\beta}f(b,c)+J_{b^-,c^+}^{\alpha,\beta}f(a,d)+J_{b^-,d^-}^{\alpha,\beta}f(a,c)\right]\Bigg| \\
&\leq \frac{\alpha\beta}{4(b-a)^{\alpha}(d-c)^{\beta}}\int_a^b\int_c^d\left\{\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right]\right. \\
&\left.\times\left|\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}|f_{\sigma\tau}(\sigma,\tau)|\,d\tau d\sigma\right|\right\}dsdt
\end{aligned} \tag{2.28}$$
$$\begin{aligned}
&\leq \frac{\alpha\beta}{4(b-a)^{\alpha+1}(d-c)^{\beta+1}}\int_a^b\int_c^d\left\{\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right]\right. \\
&\times\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}\left[(b-\sigma)(d-\tau)|f_{\sigma\tau}(a,c)|+(b-\sigma)(\tau-c)|f_{\sigma\tau}(a,d)|\right. \\
&\left.\left.+(\sigma-a)(d-\tau)|f_{\sigma\tau}(b,c)|+(\sigma-a)(\tau-c)|f_{\sigma\tau}(b,d)|\right]d\tau d\sigma\right\}dsdt \\
&= K_1+K_2+K_3+K_4.
\end{aligned}$$

With a simple calculation, we have

$$\begin{aligned}
K_1 &= \frac{\alpha\beta}{4(b-a)^{\alpha+1}(d-c)^{\beta+1}}\int_a^b\int_c^d \left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right] \\
&\quad\times\left|f_{\sigma\tau}(a,c)\right|\left|\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}(b-\sigma)(d-\tau)\,d\tau d\sigma\right| dsdt \\
&= \frac{\alpha\beta\left|f_{\sigma\tau}(a,c)\right|}{4(b-a)^{\alpha+1}(d-c)^{\beta+1}}\left[\int_a^b\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\left|\int_t^{\frac{a+b}{2}}(b-\sigma)\,d\sigma\right|dt\right] \\
&\quad\times\left[\int_c^d\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right]\left|\int_s^{\frac{c+d}{2}}(d-\tau)\,d\tau\right|ds\right] \\
&= \frac{\alpha\beta\left|f_{\sigma\tau}(a,c)\right|}{4(b-a)^{\alpha+1}(d-c)^{\beta+1}}\left[\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\int_t^{\frac{a+b}{2}}(b-\sigma)\,d\sigma dt\right. \\
&\quad\left.+\int_{\frac{a+b}{2}}^b\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\int_{\frac{a+b}{2}}^t(b-\sigma)\,d\sigma dt\right] \\
&\quad\times\left[\int_c^{\frac{c+d}{2}}\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right]\int_s^{\frac{c+d}{2}}(d-\tau)\,d\tau ds\right. \\
&\quad\left.+\int_{\frac{c+d}{2}}^d\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right]\int_{\frac{c+d}{2}}^s(d-\tau)\,d\tau ds\right] \\
&= \frac{\left|f_{\sigma\tau}(a,c)\right|}{4}\frac{\alpha 2^\alpha-(\alpha+1)2^{\alpha-1}+1}{2^\alpha(\alpha+1)}\frac{\beta 2^\beta-(\beta+1)2^{\beta-1}+1}{2^\beta(\beta+1)}(b-a)(d-c).
\end{aligned}$$

Similarly, we also have the following equalities

$$\begin{aligned}
K_2 &= \frac{\alpha\beta}{4(b-a)^{\alpha+1}(d-c)^{\beta+1}}\int_a^b\int_c^d \left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right] \\
&\quad\times\left|f_{\sigma\tau}(a,d)\right|\left|\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}(b-\sigma)(\tau-c)\,d\tau d\sigma\right| dsdt \\
&= \frac{\left|f_{\sigma\tau}(a,d)\right|}{4}\frac{\alpha 2^\alpha-(\alpha+1)2^{\alpha-1}+1}{2^\alpha(\alpha+1)}\frac{\beta 2^\beta-(\beta+1)2^{\beta-1}+1}{2^\beta(\beta+1)}(b-a)(d-c),
\end{aligned}$$

$$
\begin{aligned}
K_3 &= \frac{\alpha\beta}{4(b-a)^{\alpha+1}(d-c)^{\beta+1}} \int_a^b \int_c^d \left[(b-t)^{\alpha-1} + (t-a)^{\alpha-1}\right] \left[(d-s)^{\beta-1} + (s-c)^{\beta-1}\right] \\
&\quad \times |f_{\sigma\tau}(b,c)| \left| \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} (\sigma-a)(d-\tau)\, d\tau d\sigma \right| dsdt \\
&= \frac{|f_{\sigma\tau}(b,c)|}{4} \frac{\alpha 2^\alpha - (\alpha+1) 2^{\alpha-1} + 1}{2^\alpha(\alpha+1)} \frac{\beta 2^\beta - (\beta+1) 2^{\beta-1} + 1}{2^\beta(\beta+1)} (b-a)(d-c)
\end{aligned}
$$

and

$$
\begin{aligned}
K_4 &= \frac{\alpha\beta}{4(b-a)^{\alpha+1}(d-c)^{\beta+1}} \int_a^b \int_c^d \left[(b-t)^{\alpha-1} + (t-a)^{\alpha-1}\right] \left[(d-s)^{\beta-1} + (s-c)^{\beta-1}\right] \\
&\quad \times |f_{\sigma\tau}(b,d)| \left| \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} (\sigma-a)(\tau-c)\, d\tau d\sigma \right| dsdt \\
&= \frac{|f_{\sigma\tau}(b,d)|}{4} \frac{\alpha 2^\alpha - (\alpha+1) 2^{\alpha-1} + 1}{2^\alpha(\alpha+1)} \frac{\beta 2^\beta - (\beta+1) 2^{\beta-1} + 1}{2^\beta(\beta+1)} (b-a)(d-c).
\end{aligned}
$$

Thus, if we put the last four equalities in (2.28), we obtain (2.26). This completes the proof of the theorem. □

Corollary 5. *If we take $\alpha = \beta = 1$ in Theorem 6, we get*

$$
\begin{aligned}
&\left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right) dx \right. \\
&\left. - \frac{1}{(d-c)} \int_c^d f\left(\frac{a+b}{2}, y\right) dy + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dydx \right| \\
\leq\; & \frac{(b-a)(d-c)}{16} \left(\frac{|f_{\sigma\tau}(a,c)| + |f_{\sigma\tau}(a,d)| + |f_{\sigma\tau}(b,c)| + |f_{\sigma\tau}(b,d)|}{4} \right).
\end{aligned}
$$

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